Fractional ultrametric Differentiability in Representation Theory

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Representation Theory

Continuous Galois representations

We let $\mathbf{F}$ be a p-adic number field and $\mathbf{E}$ a sufficiently large p-adic number field, serving as the field of coefficients of the vector spaces our groups over $\mathbf{F}$ will act on.

Definition. By a **p-adic Galois representation** we mean a continuous representation of the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})$ of a finite extension $\mathbf{F}$ over a finite dimensional $\mathbf{E}$-vector space.

Remark. This class of representations as a whole seems rather intractable at the moment and examples of those are very hard to construct. So one contents oneself wit studying various smaller subcategories of it. The smallest and best understood one those of **crystalline** Galois representations.

Crystalline Galois representations

Definition. Let $\mathbf{F} = \mathbb{Q}_p$. A filtered ϕ-module V over $\mathbf{E}$ is an n-dimensional $\mathbf{E}$-vector space with

- an automorphism ϕ,
- and a decreasing, exhaustive and separated filtration $\ldots \supseteq V_n \supseteq V_{n+1} \supset \ldots$ indexed over $n \in \mathbb{Z}$.

We will make neither the definition of an *admissible* filtered ϕ-module nor the following definition of a *crystalline* p-adic Galois representation precise. Their importance stems from the following theorem (due to Colmez and Fontaine).

Theorem. *We have an equivalence of categories*

$$\{\text{admissible fil. } \phi\text{-mod.}\} \leftrightarrow \quad \{\text{crystalline } p\text{-adic Galois rep.}\}$$

p-adic Langlands for crystalline Galois representations

Let us fix such an admissible filtered ϕ-module V attached to the continuous n-dimensional $\mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})$-representation ρ.

Aim (p-adic Langlands programme). There exists a suitable cont. $\mathrm{GL}_n(\mathbf{F})$-representation $\widehat{\Pi(V)}$ such that

$$\rho \leftrightarrow V \mapsto \widehat{\Pi(V)}$$

furnishes a distinguished correspondence

$$\{\text{cont. } n\text{-dim. } \mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})\text{-rep.}\} \leftrightarrow \{\text{certain cont. } \mathrm{GL}_n(\mathbf{F})\text{-rep.}\}.$$

The recipe for $\Pi(V)$

The representation $\widehat{\Pi(V)}$ will be won in a three step process, in line with the constituting ingredients of a ϕ-module:

- The automorphism ϕ,
- the *jumps* $k \in \mathbb{Z}$ of the filtration (including multiplicities),
- and the actual subspaces $V_k \subseteq V$ itself.

This recipe is due to Christophe Breuil and has recently been made precise in joint work with Peter Schneider.

A natural candidate: Step 1.

Step 1.: We will match ϕ first. By the local Langlands programme, we can readily match:

$$\begin{aligned}&\{\phi\text{-modules}\}\\ \leftrightarrow&\{\text{unramified Galois representations of } \mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})\}\\ \leftrightarrow&\{\text{unramified smooth representations of } \mathrm{GL}_n(\mathbf{F})\}\end{aligned}$$

Let us denote this continuous $\mathrm{GL}_n(\mathbf{F})$-representation by $\pi = \pi(\phi)$. (It is infinite dimensional. Smoothness refers to the fact that the group G_v fixing a given vector $v \in V$ is open (i.e. the group action is continuous for the discrete topology on V) and unramifiedness means that there is one $v_0 \in V$ such that G_{v_0} is a maximal compact subgroup.)

A natural candidate: Step 2.

Step 2.: We will match the filtration jumps. Recall that is an ascending n-tuple of integers $(a_{k_1}, \ldots, a_{k_n})$ where the filtration steps differ. By the theory of rational representations of algebraic groups over algebraically closed fields, we have:

$$\{\text{ascending } n\text{-tuples in } \mathbb{Z}\} \leftrightarrow \{\text{irr. rat. } \mathrm{GL}_n\text{-rep.}\}.$$

We will denote this rational $\mathrm{GL}_n(\mathbf{F})$-representation by $\psi = \psi((a_{k_1} \leq \ldots \leq a_{k_n}))$. (It is finite dimensional. In case $G = \mathrm{GL}_2$ it can up two twists be realized by polynomials in two variables of homogenous degree.)

A natural candidate: Combining Step 1. and 2.

Step 3.: Let us hold stock: Starting from V, we now have a constructed a continuous representation π and an algebraic representation ψ. We combine these to yield the so called *locally algebraic* representation

$$\Pi = \pi \otimes_{\mathbf{E}} \psi.$$

A natural candidate: The final conjectural Step 3.

The missing data, the actual set of subspaces $V_k \subseteq V$ of the filtration, are now expected to correspond to a G-invariant seminorm $\|\cdot\|$ on $\Pi(V)$. We let $\|\cdot\| = \|\cdot\|_{\mathcal{V}}$ denote the conjectural seminorm corresponding to the set $\mathcal{V} = \{V_k\}$ of subspaces of V. Then we finally put

$$\widehat{\Pi(V)} := \text{ the completion of } \Pi(V) \text{ w.r.t. } \|\cdot\| \ .$$

We therefore propose more precisely to look for a correspondence between the following two categories:

$$\{\text{cryst. } n\text{-dim. } \mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})\text{-rep.}\} \leftrightarrow \{\text{unit. } \mathrm{GL}_n(\mathbf{F})\text{-Banach rep.}\}$$

(Here a continuous Banach space representation is **unitary** if its topology can be defined by a G-invariant norm.)

The example $G = \mathrm{GL}_2(\mathbb{Q}_p)$

Describing $\widehat{\Pi}$ step by step

The only case when this correspondence can made be precise is in the case of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ by the work of Colmez et al. In this case, we will:

- Make the locally algebraic representation Π explicit.
- We will consider a certain subspace $\Pi(N_0) \subseteq \Pi$ which is stable under a submonoid $P^+ \subseteq G$. We will then find a norm $\|\cdot\|$ on $\Pi(N_0)$ invariant under P^+.
- We show that the completed space $\widehat{\Pi(N_0)}$ w.r.t. $\|\cdot\|$ can be described by r-times differentiable functions for a real number $r \geq 0$.

The groups

We consider the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let

- $T = \begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq G$ be the diagonal matrices,
- $N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \subseteq G$ resp. $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix} \subseteq N$ the unipotent resp. integral unipotent ones, and
- $P = TN = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ resp. $\bar{P} = \begin{pmatrix} * & \\ * & * \end{pmatrix}$ the upper resp. lower triangular ones.

The most natural way to construct interesting representations of GL_n is as follows: Let $M = \prod_{i=1,\dots,r} GL_{n_i} \subseteq \mathrm{GL}_n$ with $n_1 + \cdots + n_r = n$ and let V_i be a $\mathbf{E}[\mathrm{GL}_{n_i}]$-module for $i = 1, \dots, r$. Then $V = \bigotimes_{i=1,\dots,r} V_i$ is an $\mathbf{E}[M]$-module and we can build the $\mathbf{E}[G]$-module

$$W = V \otimes_{\mathbf{E}[M]} \mathbf{E}[G].$$

In case $n = 2 = 1 + 1$ this becomes particularly simple.

The representation

We define a character $\chi : T \to \mathbf{E}^*$ which will be a product of an algebraic character ψ and an continuous character θ.

- We let $\psi = \psi_1 \otimes \psi_2 : T \to \mathbf{E}^*$ be the algebraic character given by the characters $\psi_1 = \cdot^{l+k}$ and $\psi_2 = \cdot^{l}$ on $\mathbb{Q}_p^*$ with $l + k \geq l \in \mathbb{Z}$, and
- we let $\theta = \theta_1 \otimes \theta_2 : T \to \mathbf{E}^*$ be a character satisfying $\theta_1(\mathbb{Z}_p^*) = \theta_2(\mathbb{Z}_p^*) = 1$, so that they are determined by their values on p.

Let $\chi = \psi \cdot \theta : P \twoheadrightarrow T \to \mathbf{E}^*$. We then define the G-representation $\Pi = \mathrm{Ind}_{\bar{P}}^{G} \chi$ as the $\mathbf{E}$-vector space

$$\{f : G \to \mathbf{E} \text{ loc. rat.} : f(\bar{p}g) = \chi(\bar{p}) \cdot f(g) \text{ for all } \bar{p} \in \bar{P}, g \in G\}$$

with the action of G given by right translation, $f^g = f(\cdot g)$.

The subspace

Let us denote by $\Pi(N_0) = \{f \in \Pi : \sup f \subseteq N_0\bar{P}\}$ the subspace with functions having support in the open subset $N_0\bar{P} \subseteq G$ only. Then identifying $N_0 = \mathbb{Z}_p$, we obtain the following description.

Proposition. *The restriction mapping $f \mapsto f_{|N_0}$ gives an isomorphism of* $\mathbf{E}$*-vector spaces*

$$\begin{aligned}\Pi(N_0) &= \mathcal{C}^{\mathrm{loc.pol.}\leq k}(\mathbb{Z}_p, \mathbf{E}) \\ &:= \{f : \mathbb{Z}_p \to \mathbf{E} : f \text{ loc. pol. of degree } \leq k\}.\end{aligned}$$

The monoid action on the subspace

We will study the group action on $\mathcal{C}^{\mathrm{loc.pol.}\leq k}(\mathbb{Z}_p, \mathbf{E})$. Let us denote

$$T^+ = \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ & 1 \end{pmatrix} \quad \text{and} \quad P^+ = P^+ N_0 = \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ & 1 \end{pmatrix}.$$

The monoid P^+ stabilizes $\Pi(N_0)$ and, identifying $P^+ = (\mathbb{Z}_p - \{0\}) \times \mathbb{Z}_p$, acts as follows on this subspace:

Lemma. *The action of the monoid P^+ on $\mathcal{C}^{\mathrm{loc.pol.}\leq k}(\mathbb{Z}_p, \mathbf{E})$ under the above isomorphism is given by*

- $f^t = \chi_1(t) f(\cdot / t)$ *for all* $t \in \mathbb{Z}_p - \{0\}$, *and*
- $f^n = f(\cdot + n)$ *for all* $n \in \mathbb{Z}_p$.

Here $\chi = \chi_1 \otimes \chi_1 : T \to \mathbf{E}^*$ *with* $\chi_1, \chi_1 : \mathbb{Q}_p^* \to \mathbf{E}^*$.

The universal lattice

We are looking for a G-invariant norm on Π. Let us identify seminorms with lattices by their unit balls. Then in the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$ there is, up to equivalence, only one such seminorm, given by the *universal* lattice $\mathbf{L}$. Letting $\mathbf{o_E} \subseteq \mathbf{E}$ denote the valuation ring of $\mathbf{E}$, it can be described as the $\mathbf{o_E}[P]$-module spanned by any finite set of generators of the $\mathbf{E}[G]$-module Π.

Lemma. *We have*

$$\mathbf{L} \cap \Pi(N_0) = \mathbf{o_E}[P^+] \cdot 1_{\mathbb{Z}_p} x^k.$$

The abstract universal norm

Let $v_{\mathbf{E}}$ denote the valuation of $\mathbf{E}$, normalized by $v_{\mathbf{E}}(p) = 1$. To ensure $\|\cdot\|$ to be nonzero, one has to impose

$$r := v_{\mathbf{E}}(\chi_1(p)) \geq 0.$$

By the shape of the universal lattice $\mathbf{L}$, we are therefore looking for the *greatest* norm $\|\cdot\|$ on $\mathcal{C}^{\text{loc.pol.}\leq k}(\mathbb{Z}_p, \mathbf{E})$ such that:

- There exists a constant $C > 0$ such that $\|1_{p^n\mathbb{Z}_p} x^k\| \leq C \cdot p^{(r-k)n}$ for all $n \in \mathbb{N}$.
- It is invariant under translation.

For the rest of the lecture, we aim at describing its associated completion.

The Main Theorem

The theorem

Theorem (Amice-Velu, Vishik, Berger-Breuil)**.** *The completion of* $\mathcal{C}^{\mathrm{loc.pol.}\leq k}(\mathbb{Z}_p, \mathbf{E})$ *w.r.t to the lattice* $\mathbf{L} = \mathbf{o_E}[P^+] \cdot 1_{\mathbb{Z}_p} x^k$ *is given by* $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{E})$*.*

The original definition of $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{E})$ due to Berger and Breuil, inspired by an observation of Schikhof, is given as follows:

Definition. A function $f : \mathbb{Z}_p \to \mathbf{E}$ is $\mathcal{C}^r$ if its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ satisfy $|a_n|n^r \to 0$ as $n \to \infty$.

The plan of proof

We will do the following:

- Give a conceptual definition of r-fold differentiability.
- Show that $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{E}) = \widehat{\mathcal{C}}^{\mathrm{loc.pol.} \leq k}(\mathbb{Z}_p, \mathbf{E})$. For this:
  - Prove $\mathcal{C}^{\mathrm{loc.pol.} \leq k}(\mathbb{Z}_p, \mathbf{E}) \subseteq \mathcal{C}^r(\mathbb{Z}_p, \mathbf{E})$ densely.
  - Prove $\|\cdot\|_{\mathcal{C}^r} = \|\cdot\|$. For this:
    - Demonstrate $\|\cdot\|_{\mathcal{C}^r}$ to leave $1_{\mathbb{Z}_p} x^k$ invariant under P^+.
    - Demonstrate $\|\cdot\|_{\mathcal{C}^r}$ to be the greatest such norm. For this: Find an orthogonal base $\{e_s\} \subseteq \mathbf{L}$, the van der Put base.

If time permits, we will show how to recover the original definition above from the conceptual one yet to be introduced.

General fractional differentiability

Strict differentiability

We want to give a conceptual definition of r-fold differentiability. We fix a non-Archimedeanly non-trivially valued complete field $\mathbf{K}$. Let V be a (finite dimensional) $\mathbf{K}$-vector space and $\mathbf{E}$ a $\mathbf{K}$-Banach space.

Definition. Let $f : X \to \mathbf{E}$ be some mapping defined on an open subset $X \subseteq V$. Then f is called *differentiable* or $\mathcal{C}^1$ in the point $a \in X$ if there exists a linear map $D_a : V \to W$ such that for every $\varepsilon > 0$ there is a neighborhood $U \ni a$ in X with

$$\|f(x+h) - f(x) - D_a \cdot h\| \leq \varepsilon \|h\| \quad \text{for all } x+h, x \in U.$$

Coordinatewise formulation of strict differentiability

In order to iterate this notion, we have to choose coordinates. Let $V = \mathbf{K}^d$ and $\mathbf{E}$ be a $\mathbf{K}$-Banach space. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the canonical basis of V.

Definition. Let $X \subseteq V$ be open. Define for all $x+h, x \in X$ with $h \in {\mathbf{K}^*}^d$ the function

$$f^{]1[} : (x+h, x) \mapsto A \in \mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$$

by specifying, for each coordinate $k = 1, \ldots, d$, the linear map A through

$$Ah_k \cdot \boldsymbol{e}_k = f(h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\cdot\boldsymbol{e}_{k-1} + h_k\cdot\boldsymbol{e}_k) - f(h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\cdot\boldsymbol{e}_{k-1})$$

for $k = 1, \ldots, d$.
Then f is a $\mathcal{C}^1$-function if and only if $f^{]1[}$ extends to a continuous function $f^{[1]} : X \times X \to \mathbf{E}$.

Iterated differentiability

Then we can obtain a notion of ν-fold differentiability for $\nu \geq 0$ as follows: Let $f \in \mathcal{C}^1(X, \mathbf{E})$ and let us regard the function $f^{[1]}$. Its domain $X \times X$ is again a $\mathbf{K}$-vector space with canonical choice of basis and its range $\mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$ again $\mathbf{K}$-Banach space. We can therefore iterate this definition by applying it to $f^{[1]}$. I.e. we define $f \in \mathcal{C}^2(X, \mathbf{E})$ if and only if $f^{[1]}$ exists and

$$f^{]2[} = (f^{[1]})^{]1[} : (X^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E}), \mathbf{E})$$

extends to a continuous function $f^{[2]}$ on $X^{[2]}$. (Here $X^{]1[} = \{(x + h, x) \in X^2 : h \in {\mathbf{K}^*}^d\}$.)

Fractional differentiability

We aim to give a definition of r-fold differentiability. Write $r = \nu + \rho \geq 0$ with $\nu \in \mathbb{N}$ and $\rho \in [0, 1[$. Then we define ρ-fold differentiability by a strengthened Lipschitz-continuity condition as follows.

Definition. Let $A \subseteq X$ and $f : A \to \mathbf{E}$. Then f is $\mathcal{C}^\rho$ at a point $a \in X$ if for all $\varepsilon > 0$ there exits $U_\varepsilon \ni a$ such that

$$\|f(x) - f(y)\| \leq \varepsilon \cdot \|x - y\|^\rho \quad \text{for all } x, y \in U_\varepsilon \cap A.$$

(The more general setup allowing for a $\mathcal{C}^\rho$-point outside the function's domain will be needed to define pointwise fractional differentiability.) Then we can define r-fold differentiability by demanding the ν-th iterated difference quotient $f^{]\nu[}$ to be $\mathcal{C}^\rho$ everywhere.

Schikhof's observation

This definition can also be given pointwise and more concisely. Indeed, for a symmetric function we are brought down to checking solely partial differentiability in its first coordinate, reducing an exponential growth of parameters to a linear one.

Definition. Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{K}$ a mapping thereon. For $\nu \in \mathbb{N}$, we put $X^{[\nu]} = X^{\{0,\ldots,\nu\}}$ and

$$X^{]\nu[} := \triangledown X^{[\nu]} = \{(x_0, \ldots, x_\nu) : x_i = x_j \text{ only if } i = j\}.$$

The ν**-th difference quotient** $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ of a function $f : X \to \mathbf{K}$ is inductively given by $f^{]0[} := f$ and for $n \in \mathbb{N}$ and $(x_0, \ldots, x_\nu) \in X^{]\nu[}$ by

$$f^{]\nu[}(x_0, \ldots, x_\nu) := \frac{f^{]\nu-1[}(x_0, x_2, \ldots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \ldots, x_\nu)}{x_0 - x_1}.$$

The final definition (in one variable)

Having already defined $\mathcal{C}^\rho$-functions for $\rho \in [0,1[$, we add up our definitions to obtain our notion of fractional differentiability over (non-Archimedeanly valued) complete fields:

Definition. Fix $r = \nu + \rho \in \mathbb{R}_{\geq 0}$. Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{K}$ a mapping thereon.

- We will say that f is $\mathcal{C}^r$ (or r **times continuously differentiable**) at a point $a \in X$ if $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ is $\mathcal{C}^\rho$ at $\vec{a} = (a, \ldots, a) \in X^{[\nu]}$.
- Then f will be a $\mathcal{C}^r$**-function** (or an r**-times continuously differentiable function**) if f is $\mathcal{C}^r$ at all points $a \in X$. The set of all $\mathcal{C}^r$-functions $f : X \to \mathbf{K}$ will be denoted by $\mathcal{C}^r(X, \mathbf{K})$.

Density of locally polynomial functions

The statement

We will fix a nonempty compact subset without isolated points $X \subseteq \mathbf{K}$.

Definition. We call $f : X \to \mathbf{K}$ a **locally polynomial function of degree at most** $g \in \mathbb{N}$, if for every point $a \in X$, there exists a neighborhood $U \ni a$ such that $f_{|U} = \sum_{i=0,\dots,g} a_i *^i$ is a polynomial function of degree at most g.

Proposition. *The locally polynomial functions of degree at most ν are dense in $\mathcal{C}^r(X, \mathbf{K})$.*

We will here only consider the case $r = \nu \in \mathbb{N}$. For $n \in \mathbb{N}$ and $\delta > 0$, put

$$X^{[n]}_{\leq\delta} = \{x \in X^{[n]} : \mathrm{diameter}\{x_0, \dots, x_n\} \leq \delta\}.$$

The strategy of proof

Fix $\varepsilon > 0$. By uniform continuity of $f^{[\nu]}$, there exists a $\delta > 0$ such that $\|f^{[\nu]}\|_{X^{[\nu]}_{\leq\delta}} \leq \varepsilon$.

The proof is by downward induction: We will fix this δ and successively construct locally constant $g_\nu, \ldots, g_n$ such that

$$f_n = f - g_\nu *^\nu - \cdots - g_n *^n$$

satisfies

$$\|f_n^{[n]}\|_{X^{[n]}_{\leq\delta}} \leq \varepsilon\delta^{\nu-n}.$$

We show this to entail

$$\|f_0^{[n]}\|_{X^{[n]}} \leq \varepsilon\delta^{\nu-n}$$

for all $n = 0, \ldots, \nu$. In particular $\|f - g\|_{\mathcal{C}^\nu} \leq \varepsilon$ with $g = g_\nu *^\nu - \cdots - g_0 *^0$ locally polynomial.

The construction of g_n

Lemma. *Let $f \in \mathcal{C}^n(X, \mathbf{K})$. Fix $\delta, \varepsilon > 0$. If*

$$\|f^{[n]}(x_0, \dots, x_n) - f^{[n]}(\vec{a})\| \leq \varepsilon$$

for all $(x_0, \dots, x_n), \vec{a} \in X^{[n]}$ with $\|(x_0, \dots, x_n) - \vec{a}\| \leq \delta$, then there exists a δ-constant $g : X \to \mathbf{K}$ such that $\tilde{f} := f - g*^n$ satisfies

$$\|\tilde{f}^{[n]}\|_{X^{[n]}_{\leq\delta}} \leq \varepsilon.$$

Proof. Follows from the well known density of

$$\mathcal{C}^{\mathrm{loc.cst.}}(X, \mathbf{K}) \subseteq \mathcal{C}^0(X, \mathbf{K})$$

together with the fact that, if g is locally constant, then ${g*^n}^{[n]}$ is locally constant on $X^{[n]}_{\leq\delta}$. ∎

The whole set

Given f_n, we can therefore construct locally constant g_{n-1} such that $f_{n-1} = f_n - g_{n-1} *^{n-1}$ satisfies $\|f_{n-1}^{[n-1]}\|_{X_{\leq\delta}^{[n-1]}} \leq \varepsilon\delta^{\nu-(n-1)}$, yielding locally constant $g_\nu, \ldots, g_0$. It rests to prove that $f_0 = f - g_\nu *^\nu - \cdots - g_0 *^0$ satisfies

$$\|(f_0)^{[n]}\|_{\sup} \leq \varepsilon\delta^{\nu-n}$$

for $n = 0, \ldots, \nu$. If $n = 0$, there is nothing to prove. If $n > 1$, we split up:

$$\|f_0^{[n]}\|_{\sup} = \max\{\|f_0^{[n]}\|_{X_{\leq\delta}^{[n]}}, \|f_0^{[n]}\|_{\{(x_0,\ldots,x_n) \text{ s.t. } \|x_k - x_l\| > \delta \text{ for some } k,l\}}\}.$$

The first term

We firstly show $\|f_0^{[n]}\|_{X_{\leq\delta}^{[n}} \leq \varepsilon\delta^{\nu-n}$.

Let $i = 0, \ldots, n$. Since g_i is δ-constant, the function $(g_i *^i)^{[i]}$ is constant on $X_{\leq\delta}^{[i]}$ and thus, if $i < n$, the function $(g_i *^i)^{[n]}$ thus vanishes on $X_{\leq\delta}^{[n]}$. Therefore, on $X_{\leq\delta}^{[n]}$, we have

$$\begin{aligned} f_0^{[n]} &= (f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \cdots - g_0)^{[n]} \\ &= (f - g_\nu *^\nu - \cdots - g_n *^n)^{[n]} = f_n^{[n]}. \end{aligned}$$

By construction of $g_\nu, \ldots, g_0 : X \to \mathbf{K}$, we find $f_n = f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \cdots - g_n *^n$ to satisfy $\|f_n\|_{X_{\leq\delta}^{[n}} \leq \varepsilon\delta^{\nu-n}$. Therefore

$$\|f_0^{[n]}\|_{X_{\leq\delta}^{[n]}} = \|f_n^{[n]}\|_{X_{\leq\delta}^{[n]}} \leq \varepsilon\delta^{\nu-n}.$$

The second term

We finally show

$$\|f_0^{[n]}\|_{\{(x_0,\ldots,x_n) \text{ s.t. } \|x_k - x_l\| > \delta \text{ for some } k,l\}} \leq \varepsilon\delta^{\nu-n}.$$

This is carried out by *onward* induction on $n = 0, \ldots, \nu$. By symmetry of $f_0^{[n]}$, we are reduced to the case $|x_0 - x_1| > \delta$. Then by definition

$$\begin{aligned} &|f_0^{[n]}(x_0, x_1, \ldots, x_n)| \\ =&|f_0^{[n-1]}(x_0, x_2, \ldots, x_{n-1}) - f_0^{[n]}(x_1, x_2, \ldots, x_{n-1})|/|x_0 - x_1| \\ <&\|f_0^{[n-1]}\|/\delta \leq \varepsilon\delta^{\nu-(n-1)}/\delta = \varepsilon\delta^{\nu-n}; \end{aligned}$$

the last inequality by the induction hypothesis.

The density of polynomial functions

Lemma (Araujo, Schikhof). *Let $\mathbf{1}_{\mathrm{B}} : X \to \mathbf{K}$ be the characteristic function of a closed ball $\mathrm{B} \subseteq X$ of positive radius and $\varepsilon > 0$. Then there exists a polynomial function $p : X \to \mathbf{K}$ such that $\|\mathbf{1}_{\mathrm{B}} - p\|_{\mathcal{C}^{\nu+1}} \leq \varepsilon$.*

Corollary. *The polynomial functions are dense in $\mathcal{C}^r(X, \mathbf{K})$*

Proof. Approximate f by a locally polynomial function $g = \sum g_i *^i$ with g_i locally constant. Then by linearity, we infer from the above that there exist polynomial functions p_i such that

$$\|g_i - p_i\|_{\mathcal{C}^r} \leq \|g_i - p_i\|_{\mathcal{C}^{\nu+1}} \leq \varepsilon.$$

Then the polynomial function $p = \sum p_i *^i$ is close to f. ∎

The van der Put base

The needed conditions

We want to show that $\|\cdot\|_{\mathcal{C}^r} = \|\cdot\|$. For this:

- Demonstrate $\|\cdot\|_{\mathcal{C}^r}$ to leave $1_{\mathbb{Z}_p} x^k$ invariant under P^+.
- Demonstrate $\|\cdot\|_{\mathcal{C}^r}$ to be the greatest such norm. For this: Find an orthogonal base in $\mathbf{L}$.

The first condition

We turn to the first condition. We have to show:

- There exists a constant $C > 0$ such that $\|1_{p^n\mathbb{Z}_p} x^k\| \leq C \cdot p^{(r-k)n}$ for all $n \in \mathbb{N}$.
- It is invariant under translation.

We will only give an idea of the proof, as it will also follow from the second statement at once. Then again, it emphasizes a key point. So consider the case $r = 1$. We have to check $\|\cdot\|_{\mathcal{C}^1}$ against the above two properties. By definition, it is translation invariant. Moreover

$$\|1_{p^n\mathbb{Z}_p} x^k\|_{\mathcal{C}^1} \leq \left|\frac{p^{kn} - 0}{p^n - p^{n-1}}\right| = c \cdot p^{(1-k)n}$$

with $c := p^{-1} > 0$ for all $n \in \mathbb{N}$.

The generalized van der Put base

Let $\mathbf{F}$ be a local field, $\mathbf{o}$ its ring of integers, $\mathbf{o}_{\leq i} = \{x \in \mathbf{o} : v_{\mathbf{F}}(x) \geq i\}$ and $\pi \in \mathbf{o}$ a uniformizer.

- We choose for $i \in \mathbb{N}$ an *increasing* family of systems of representatives $(S_{\leq i})$ of $\mathbf{o}/\mathbf{o}_{\leq i}$, i.e. $S_{\leq i} \subseteq S_{\leq j}$ for $i \leq j$ and put
$$S = \bigcup_{i \in \mathbb{N}} S_{\leq i} \subseteq \mathbf{o}.$$
- We have a natural notion of level for the elements in S, namely we put
$$\ell(s) := \min\{i \in \mathbb{N} : s \in S_{\leq i}\}.$$

We define our generalized van der Put-basis $\{e_s : s \in S\} \subseteq \mathcal{C}^{\mathrm{loc.cst.}}(\mathbf{o}, \mathbf{E})$ as the family of characteristic functions $e_s := \mathbf{1}_{s + \pi^{\ell(s)}\mathbf{o}}$.

The locally polynomial van der Put basis.

Lemma. *The family* $\{e_s(* - s)^i : s \in S, i \in \{0, \ldots, d\}\}$ *constitutes a basis of the* $\mathbf{E}$*-vector space* $\mathcal{C}^{\text{loc.cst.}\leq d}(\mathbf{o}, \mathbf{E})$.

To compute their norms, we have the following general Lemma.

Lemma 1. *Let* $d \leq r$. *Then we have* $\|*^{\mathrm{d}}\|_{\mathcal{C}^r} = 1$ *and for* $n \geq 1$ *holds*

$$\|\mathbf{1}_{\pi^n \mathbf{o}} *^d\|_{\mathcal{C}^r} = |1/\pi|^{(n-1)(r-d)}.$$

The Key Lemma

Let $s \in S$ with $i = \ell(s) \geq 1$. Then its **preceding** element $s^- \in S$ is defined as the unique element in $S_{\leq i^-}$ such that $s^- \equiv s \mod \pi^{i^-}\mathbf{o}$ with $i^- = i - 1$.

Lemma 2. *Let $f \in \mathcal{C}^{\mathrm{loc.cst.}\leq d}(\mathbf{o}, \mathbf{E})$ be a locally polynomial function of degree $\leq d$. Write $f = f_0 + \breve{f}$ with $f_0 = \sum_{s \in S} \lambda_s e_s$ locally constant. Then we have*

$$\breve{f}(s) - \breve{f}(s^-) = \sum_{i=1,\ldots,d} \mathcal{D}_i f(s^-)(s - s^-)^i \text{ for all nonzero } s \in S.$$

Here $\mathcal{D}_i f$ denotes the i-th derivative of f in the sense of Schikhof, i.e. $\mathcal{D}_i f(a) = f^{[i]}(a, \ldots, a)$. (We have $i!\mathcal{D}_i f(a) = f^{(i)}(a)$.)

Proof: Observations

- Let $g \in \mathcal{C}^{\text{loc.cst.}\leq d}(\mathbf{o}, \mathbf{K})$ and $s_0 \in S$. Write
$$g = \sum_{i=0,\dots,d} \sum_{s \in S} \lambda_{s,i} e_s (* - s)^i$$
We define
$$g_{\leq s_0} := \sum_{i=0,\dots,d} \sum_{s \leq s_0} \lambda_{s,i} e_s (* - s)^i.$$
Then it holds by definition of $\{e_s : s \in S\}$ that $g(s_0) - g(s_0^-) = g_{\leq s_0}(s_0) - g_{\leq s_0}(s_0^-)$.
- Let $U \subseteq \mathbf{o}$ be open, $g \in \mathcal{C}(\mathbf{o}, \mathbf{K})$ such that $f_{|U}$ is a polynomial function of degree d. Let $x, x_0 \in U$. Then by the uniqueness of the Taylor polynomial expansion
$$g(x) - g(x_0) = \sum_{i=0,\dots,d} \mathcal{D}_i g(x_0)(x - x_0)^i.$$
- Let $g \in \mathcal{C}(\mathbf{o}, \mathbf{K})$ be locally constant. Then $\mathcal{D}_i g = 0$ for all $i \geq 1$.

Proof: Conclusion

Combining these observations we obtain:

$$\begin{aligned}
\breve{f}(s) - \breve{f}(s^-) &= \breve{f}_{\leq s}(s) - \breve{f}_{\leq s}(s^-) \\
&= \sum_{i=1,\dots,d} \mathcal{D}_i \breve{f}_{\leq s}(s)(s - s^-)^i \\
&= \sum_{i=1,\dots,d} \mathcal{D}_i \breve{f}(s)(s - s^-)^i \\
&= \sum_{i=1,\dots,d} \mathcal{D}_i f(s)(s - s^-)^i.
\end{aligned}$$

We deduce the following corollary, which shows how to compute the van der Put-coefficients by the functions' derivatives itself.

Computing coefficients

Corollary 3. *Let* $f = f_0 + \cdots + f_d \in \mathcal{C}^{\text{loc.cst.}\leq d}(\mathbf{o}, \mathbf{E})$ *with* $f_i = \sum_{s \in S} \lambda_{s,i} e_s (* - s)^i$ *for* $i = 0, \ldots, d$*. We have* $\lambda_{0,i} = \mathcal{D}_i f(0)$ *and, for all nonzero* $s \in S$ *holds*

$$\lambda_{s,i} = \mathcal{D}_i f(s) - \mathcal{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathcal{D}_{i+j} f(s^-)].$$

Proof. Put $\mathcal{D}_i f = \mathcal{D}_i f_i + \breve{\mathcal{D}_i} f$ with $\mathcal{D}_i f_i = \sum_{s \in S} \lambda_{s,i} e_s$ locally constant. By the preceding Lemma

$$\begin{aligned}\lambda_{s,i} &= \mathcal{D}_i f_i(s) - \mathcal{D}_i f_i(s^-) = (\mathcal{D}_i f - \breve{\mathcal{D}_i} f)(s) - (\mathcal{D}_i f - \breve{\mathcal{D}_i} f)(s^-)\\ &= \mathcal{D}_i f(s) - \mathcal{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \mathcal{D}_j \mathcal{D}_i f(s^-)]\\ &= \mathcal{D}_i f(s) - \mathcal{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathcal{D}_{j+i} f(s^-)].\end{aligned}$$

∎

Orthonormality

Proposition 4. *The van der Put-base* $\{e_s(* - s)^i : s \in S, i = 0, \ldots, \nu\}$ *is an orthogonal basis of* $\mathcal{C}^r(\mathbf{o}, \mathbf{E})$ *with* $\|e_s(* - s)^i\|_{\mathcal{C}^r} = |1/\pi|^{(r-i)(\ell(s)-1)}$. *For any* $f \in \mathcal{C}^r(\mathbf{o}, \mathbf{E})$, *the corresponding coefficients are given by*

$$\lambda_{0,i} = (\mathcal{D}_i f)(s) \text{ for } i = 0, \ldots, \nu$$

respectively for all nonzero $s \in S, i = 0, \ldots, \nu$ *by the Taylor expansion*

$$\lambda_{s,i} = \mathcal{D}_i f(s) - \mathcal{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathcal{D}_{i+j} f(s^-)].$$

Sketch of the proof

Proof. We already computed above that

$$\|e_s(* - s)^i\|_{\mathcal{C}^r} = |1/\pi|^{(r-i)(\ell(s)-1)}.$$

Since $\{e_s : s \in S\}$ is a basis of the $\mathbf{E}$-vector space $\mathcal{C}^{\text{loc.cst.}}(\mathbf{o}, \mathbf{E})$, given $f \in \mathcal{C}^{\text{loc.cst.}\leq\nu}(\mathbf{o}, \mathbf{E})$, there exists unique $\lambda_{s,i} \in \mathbf{E}$ such that $f = \sum_{i=0,\ldots,\nu} \sum_{s\in S} \lambda_{s,i} e_s(* - s)^i$. Therefore $\|f\|_{\mathcal{C}^r} \leq \max\{|\lambda_{s,i}| \|e_s(* - s)^i\|_{\mathcal{C}^r} : s \in S, i = 0, \ldots, \nu\}$. It rests to prove that

$$|\lambda_{s,i}| \|e_s(* - s)^i\|_{\mathcal{C}^r} \leq \|f\|_{\mathcal{C}^r} \text{ for any } i \in \{0, \ldots, \nu\} \text{ and } s \in S.$$

Therefore we have to find an element x either in $X^{[n]}$ for some $n \leq \nu$ or in $X^{[\nu+1]}$ such that either $|f^{[n]}(x)|$ or $|f^{[r]}|(x)$ is greater than or equal to the left hand side. Given that the Taylor polynomials can be expressed by the difference quotients $f^{[n]}$, this follows from the explicit shape of the $\lambda_{s,i}$ given in the above Corollary 3. ∎

Conclusion

We return to our viewpoint of $\mathcal{C}^{\mathrm{loc.cst.}\leq k}(\mathbb{Z}_p, \mathbf{E})$ acted upon by the monoid $(\mathbb{Z}_p - \{0\}) \times \mathbb{Z}_p$ by

- $f^t = \chi_1(t) f(\cdot/t)$ for all $t \in \mathbb{Z}_p - \{0\}$, and
- $f^n = f(\cdot + n)$ for all $n \in \mathbb{Z}_p$.

Recall that $r = \chi_1(p)$. Then $S = \mathbb{N}$ and we have $e_n = 1_{n+p^{\ell(n)}\mathbb{Z}_p}$ with $\ell(0) = 0$ and $\ell(n) = \lceil \log_p(n) \rceil + 1$. Then we can compute

$$(p^{\ell(n)}, n) \cdot 1_{\mathbb{Z}_p} *^i = \mu \cdot e_{n,i}$$

with the scalar $\mu_{n,i} \in \mathbf{E}^*$ such that $|\mu_{n,i}| = p^{n(i-r)}$. Therefore, up to equivalence, indeed

$$\|(p^{\ell(n)}, n) \cdot 1_{\mathbb{Z}_p} *^i\|_{\mathcal{C}^r} = 1.$$

Therefore, if we denote by $\tilde{e}_{n,i} = 1/\mu_{n,i} \cdot e_{n,i}$ the normalized van der Put basis, we have $\{\tilde{e}_{n,i}\} \subseteq \mathbf{L}$, i.e. we have found an orthogonal base inside $\mathbf{L}$.

That’s it. Thank you very much for your attention.