# Kedlaya's slope filtration

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# 0 Prerequisites

We fix some notations. Let $W = W(\mathbf{k})$ for a perfect field $\mathbf{k}$ of characteristic $p > 0$, let $K_0 = W(\mathbf{k})[1/p]$ be its complete unramifiedly valued fraction field.

For an interval $I \subset [0,1)$ set $D(I) = \{x \in \overline{\mathbf{K}} \text{ with } |x| \in I\}$ and accordingly

$$\mathcal{O}_I = \{\sum_{n\in\mathbb{Z}} a_n u^n \quad \text{with coefficients in } \mathbf{K}_0 \text{ s.t. } \sum_{n\in\mathbb{Z}} a_n x^n \text{ exists for all } x \in D(I)\};$$

the unknown in the following being denoted by $u$. We recall the *Robba ring*

$$\mathcal{R} = \mathcal{O}_{[1-\varepsilon,1)} := \cup_{\rho<1}\mathcal{O}_{[\rho,1)},$$

the ring of Laurent series converging on an unspecified annulus with open outer boundary 1.

We introduced on $\mathcal{O}_I$ the map

$$\phi_{\mathcal{O}_I/W} \colon O_I \to \mathcal{O}_{\sqrt[p]{I}}, \quad u \mapsto u^p;$$

here, if $I = [a,b)$, we let $\sqrt[p]{I} := [\sqrt[p]{a}, \sqrt[p]{b})$. Then the direct limit of $\phi_{\mathcal{O}_I/W}$ for $I = [\rho, 1)$ and $\rho \to 1$ induces an operator $\phi_{\mathcal{R}/W}$ on $\mathcal{R}$. We let

$$\phi = \phi_W \circ \phi_{\mathcal{R}/W} : \mathcal{R} \circlearrowleft \qquad \text{with } \phi_W : \mathcal{R} \circlearrowleft \text{ the Frobenius of } \mathbf{K}_0 \text{ on the coefficients.}$$

Recall that we had the subrings $\mathcal{R}^{\text{int}} \subset \mathcal{R}^{\text{bdd}} \subset \mathcal{R}$ given by those elements with integral respectively bounded coefficients and equipped these with the minimum valuation (correspondingly maximum norm) on their coefficients. This valuation is invariant under the Frobenius as it acts isometrically on the coefficients. In particular it must respect both subrings.

# 1 Motivation

## Goal

We let $K/K_0$ be a finite totally ramified extension with uniformizer $\pi$ and $E(u)$ the minimal polynomial of $\pi$ over $K_0$. We abbreviate $\mathcal{O}_{[0,1)} = \mathcal{O}$. Recall the already obtained equivalence given in the first line, that

$$\begin{array}{ccc} \{\text{effective K-filtered } \phi, \text{N-modules}/\mathbf{K}_0\} & \simeq & \{(\phi, N_\nabla)\text{-modules}/\mathcal{O} \text{ of finite E-height}\} \\ \cup & & \cup \\ \{\text{weakly admissible ones}\} & \overset{!}{\simeq} & \{\text{all } \mathcal{M} \text{ s.t. } \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R} \text{ pure of slope } 0\}. \end{array}$$

## Weak admissibility

We want to prove the one in the bottom row. Recall that, given any ring R equipped with an endomorphism $\phi\colon R \circlearrowleft$, we called a *$\phi$-module over* R a finite free $\mathcal{R}$-modules M with an isomorphism $\phi^* M \to M$, that is, a semilinear map $\phi$ with invertible representing matrices. Now we can give the notions of the Hodge- and Newton-number and then weak admissibility, already introduced in the second talk.

**Definition 1.1**. Let D be a filtered $\phi$, N-module. If D is one dimensional, we put $t_N D = v(\alpha)$ if $\phi v = \alpha v$ for some basis vector $v \in D$ and let $t_H D$ be the unique $k \in \mathbb{Z}$ for which $\mathrm{Gr}_k D \neq 0$. Then for $d$-dimensional D, we put $t_N = t_N \wedge^d D$ and $t_H \wedge^d D$. Then we call a filtered $\phi$, N-module D over $K_0$ *weakly admissible*, if $t_H D = t_N D$ and $t_H \tilde{D} \leq t_N \tilde{D}$ for any $\phi$, N-submodule $\tilde{D} \subset D$.

*Remark* 1.2. We find $t_N$ respectively $t_H$ to "average" over all $v(\phi \cdot d) - v(d)$ for $d \in D$ respectively all (nonvanishing) graduation step indices.

## Pure modules

We want to put this in correspondence with pure modules over $\mathcal{R}$: Recall that $\mathcal{R}$ is equipped with a Frobenius $\phi : \mathcal{R} \circlearrowleft$. Then we can denote by $\mathrm{Mod}^{\phi}_{/\mathcal{R}}$ the category of *$\phi$-modules over $\mathcal{R}$*. Recall the following:

**Definition**. For any $\phi$-module V over a discretely valued field $\mathbf{K}$ with uniformizer $\pi$ and *isometric* $\phi : \mathbf{K} \circlearrowleft$, we defined the spectral valuation (correspondingly norm), called the *first slope* of V, by

$$\mathrm{v}_{\mathrm{sp}}(\phi) := \lim_{n \to \infty} 1/n \cdot v(\phi^n) \quad \text{with } v(\phi^n) := \min_{x \in V} v(\phi^n \cdot x) - v(x)$$

for any $\mathbf{K}$-vector space valuation $v$ on V. Note that the limit process turns this into a numerical invariant *independent of $v$*, as all such valuations are equivalent.

Then we defined the average valuation, called the *slope* of V by

$$\mu(V) := \mathrm{v}_{\mathrm{av}}(\phi) := v(\det \phi)/h \quad \text{with } h = \dim_{\mathbf{K}} V$$

(Informed by $|\det \phi| = \lambda(\phi E)$ for a Haar-measure $\lambda$ on V and $E \subset V$ a subset with $\lambda(E) = 1$.). Ultimately V was said to be *pure = isocline = isoclinic of slope* $s/r := \mu(V)$ if not only $\mathrm{v}_{\mathrm{sp}}(\phi) \leq \mathrm{v}_{\mathrm{av}}(\phi)$, but

$$\mathrm{v}_{\mathrm{sp}}(\phi) = \mathrm{v}_{\mathrm{av}}(\phi).$$

*Remark* 1.3. Let V be a pure $\phi$-module of slope $s/r = \mathrm{v_{sp}}(\phi)$. This is equivalent to $v(\phi \cdot x) = v(x) + s/r$ or $v(\pi^{-s}\phi^r \cdot x) = v(x)$ for all $x \in \mathrm{V}$ and this in turn to $\tilde{\phi} := \pi^{-s}\phi^r$ being *etale*, that is, base extended from the $\phi$-module over $\mathbf{o_K}$ given by $\mathrm{L} := \{v \in \mathrm{V} : v(x) \geq 0\}$ and $\tilde{\phi}_{\mathrm{L}} := \tilde{\phi}_{|\mathrm{L}}$. We conclude that we could rephrase the above definition as follows:

*Definition.* We call a $\phi$-module M over a discretely valued field $\mathbf{K}$ *etale* if it is obtained by base extension from a $\phi$-module over $\mathbf{o_K}$. It is *pure = isocline = isoclinic of slope $s/r$* if $\pi^{-s}\phi^r$ is obtained by base extension from a $\phi$-module over $\mathbf{o_K}$.

Having defined the notion of pureness over a discretely valued field, recall that we have inclusions $\mathcal{R} \supset \mathcal{R}^{\mathrm{bdd}} \supset \mathcal{R}^{\mathrm{int}}$, with $\mathcal{R}^{\mathrm{bdd}}$ a discretely valued field and $\mathcal{R}^{\mathrm{int}}$ its valuation ring.

**Definition 1.4.** A $\phi$-module $\mathcal{M}$ over $\mathcal{R}$ is *pure* of slope $s/r$ if $\mathcal{M}$ is obtained by base extension from a $\phi$-module over $\mathcal{R}^{\mathrm{bdd}}$ which is pure of slope $s/r$. That is, the twisted $\phi$-module over $\mathcal{R}$ equipped with $\pi^{-s}\phi^r : \mathcal{M} \circlearrowleft$ is obtained by base extension from a $\phi$-module over $\mathcal{R}^{\mathrm{int}}$ (Concretely: the matrix of $\phi$ has coefficients in $\mathcal{R}^{\mathrm{int}}$ with respect to some basis).

## Outline of the proof of the restricted equivalence of categories

Let me quickly outline how we will determine this correspondence between weakly admissibility on the one side and vanishing slope on the other side in the next three talks. In this talk we will explain:

**Theorem 1.5** (Kedlaya's slope filtration). *For any $\mathcal{M} \in \mathrm{Mod}^{\phi}_{/\mathcal{R}}$ exists a unique filtration*

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subset \cdots \subsetneq \mathcal{M}_r = \mathcal{M}$$

*by $\phi$-submodules such that $\mathcal{M}_i/\mathcal{M}_{i-1}$ is finite free over $\mathcal{R}$ and pure of increasing slope $s_i$, that is, $s_1 < \cdots < s_r$.*

Recall that we want to prove that effective weakly admissible filtered $\phi, \mathrm{N}$-modules D give rise to $\phi, \mathrm{N}_\nabla$-modules $\mathcal{M}$ with $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ pure of slope 0 and vice versa. Let me quickly give the upcoming proof's strategy, illustrating the strength of Theorem 1.5:

1. Let D be one dimensional. By construction of $\mathcal{M}(\mathrm{D})$, it has slope $s = t_{\mathrm{N}}(\mathrm{D}) - t_{\mathrm{H}}(\mathrm{D}) = 0$, by weak admissibility of D.

2. For any filtered $\phi, \mathrm{N}$-module D of dimension $d$, we have $t_* \mathrm{D} := t_* \wedge^d \mathrm{D}$ and by the Tannakian property $\mathcal{M}(\wedge^d \mathrm{D}) = \wedge^d \mathcal{M}(\mathrm{D})$. If $\mathcal{M}(\mathrm{D})$ has slope $s$, then by definition $\wedge^d \mathcal{M}(\mathrm{D})$ has slope $ds$. Thus

$$t_\mathrm{N}\mathrm{D} - t_\mathrm{H}\mathrm{D} = t_\mathrm{N} \wedge^d \mathrm{D} - t_\mathrm{H} \wedge^d \mathrm{D} \overset{\text{by 1.}}{=} \text{ slope of } \wedge^d \mathcal{M}(\mathrm{D}) = ds.$$

3. Next two talks: Show that the slope filtration arises from one over $\phi, \mathrm{N}_\nabla$-modules over $\mathcal{O}$ (of finite E-height). Hence by the enclosing equivalence, the graduation steps arise from filtered $\phi, \mathrm{N}$-modules $\mathrm{D}_1, \ldots, \mathrm{D}_r \subset \mathrm{D}$ over $\mathbf{K}_0$. Since $t_\mathrm{N}$ and $t_\mathrm{H}$ are additive on exact sequences, we find by 2. that

$$0 = t_\mathrm{N}(\mathrm{D}) - t_\mathrm{H}(\mathrm{D}) = \sum_{i=1,\ldots,r} t_\mathrm{N}(\mathrm{D}_i) - t_\mathrm{H}(\mathrm{D}_i)$$
$$= \sum_{i=1,\ldots,r} d_i s_i \text{ with } d_i \text{ the rank and } s_i \text{ the slope of } \mathcal{M}(\mathrm{D}_i).$$

Since $s_i = t_\mathrm{N}(\mathrm{D}_i) - t_\mathrm{H}(\mathrm{D}_i) \geq 0$ by semistability of D, in particular $s_1 = 0$ and, as the slopes increase, necessarily $r = 1$. In other words $\mathcal{M}(\mathrm{D}) = \mathcal{M}_1(\mathrm{D})$ is pure of slope 0.
In the other direction, if $\mathcal{M}(\mathrm{D})$ is a pure $\phi, \mathrm{N}_\nabla$-module of slope 0, we find by 2. that $t_\mathrm{N}\mathrm{D} - t_\mathrm{H}\mathrm{D} = 0$. By semistability of $\mathcal{M}(\mathrm{D})$ (see Theorem 3.18), in particular $\mu(\wedge^{\tilde{d}} \mathcal{M}(\tilde{\mathrm{D}})) \geq 0$ with $\tilde{d} = \operatorname{rank} \tilde{\mathrm{D}}$. Hence, again by 2., $t_\mathrm{N}(\tilde{\mathrm{D}}) - t_\mathrm{H}(\tilde{\mathrm{D}}) = \mu(\wedge^{\tilde{d}} \mathcal{M}(\tilde{\mathrm{D}})) \geq 0$.

# 2 Construction of the slope filtration

## Outline of the construction

Let us return to the proof of Kedlaya's slope filtration theorem. Strategy for the construction of the slope filtration:

(i) Construct the Harder-Narasimhan( abbreviated HN )-filtration with *semistable* graduation steps.

(ii) Prove that semistability implies pureness.

*Remark* 2.1. The proof of Kedlaya's slope filtration theorem works in larger generality than considered here. Improvements:

- We can replace $\mathbf{K}_0$ by any complete discretely valued field $\mathbf{K}$.

- One can replace our $\phi : \mathscr{R} \circlearrowleft$ by a *Frobenius lift*, that is, a map $\phi : \mathscr{R} \circlearrowleft$ of the form
$$\sum_{n \in \mathbb{Z}} a_n u^n \mapsto \sum_{n \in \mathbb{Z}} \phi_{\mathrm{K}}(a_n) \tilde{u}^n$$
such that
  - the map $\phi_{\mathbf{K}} : \mathbf{K} \circlearrowleft$ is an isometry,
  - the element $\tilde{u} \in \mathscr{R}$ satisfies $v(\tilde{u} - u^q) > 0$ for $q > 1$ (demanding $\tilde{u} \in \mathscr{R}^{\mathrm{bdd}}$).

  Notice $\phi$ is in particular isometric on $\mathscr{R}^{\mathrm{bdd}}$. In our case $\tilde{u} = u^p$.

- The assumption made in [SnTdA, Paragraph 11 until Bemerkung 11.1], replacing $\mathscr{R}$ by an abstract ring sharing sufficiently many properties with $\mathscr{R}$, for example, Bezout, $\mathscr{R}^*$ being a discrete valuation ring and a certain (Bij)-property (see Lemma 3.6).

*Assumption.* From now on, we will assume that $\phi : \mathscr{R} \circlearrowleft$ is a Frobenius lift such that $\mathbf{K}$ is a complete discretely valued field with a $q$-power Frobenius lift $\phi_{\mathrm{K}} : \mathbf{K} \circlearrowleft$. (Because $\phi : \mathbf{k}_{\mathscr{R}^{\mathrm{bdd}}} \circlearrowleft$ is given by $x \mapsto x^q$, we call $\phi : \mathscr{R} \circlearrowleft$ an *absolute Frobenius lift.*)

### Review of the HN-filtration

We recall some ingredients of the construction of the Harder-Narasimhan filtration:

**Definition 2.2.** For $\mathrm{M} \in \mathrm{Mod}^{\phi}_{/\mathscr{R}}$ of rank $h$, let
$$\mu(\mathrm{M}) := \frac{v(\det \phi)}{h}$$
be the *slope* of M. Then M is *semistable* if $\mu \mathrm{N} \geq \mu \mathrm{M}$ for any nonzero $\phi$-submodule $\mathcal{N}$.

*Remark* 2.3. If M is defined over $\mathscr{R}^{\mathrm{bdd}}$, this is just the usual slope. Otherwise, since $\phi^* \mathrm{M} \to \mathrm{M}$ is an isomorphism, we must have $\det \phi \in \mathscr{R}^* = \mathscr{R}^{\mathrm{bdd}*}$. Therefore $v(\det \phi)$ is meaningful and moreover well-defined as $\phi : \mathscr{R}^{\mathrm{bdd}} \circlearrowleft$ is isometric.

The Harder Narasimhan-filtration establishes the existence of a filtration with semistable $\phi$-module graduation steps:

**Proposition 2.4.** *Every* $\phi$*-module* M *over* $\mathcal{R}$ *admits a unique filtration by* $\phi$*-submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

*such that each quotient* $M_i/M_{i-1}$ *is semistable. This is the* Harder-Narasimhan filtration *of* M.

*Remark* 2.5. We have already seen the following theorem, showing that the notions of semistability and pureness coincide in the case of one dimensional $\phi$-modules over $\mathcal{R}$.

*Lemma. Every* $\phi$*-module* M *over* $\mathcal{R}$ *of rank* 1 *is* stable*, that is,* $\mu N > \mu M$ *for any proper nonzero* $\phi$*-submodule* N.

To contrast with the situation over $\mathcal{R}^{\text{bdd}}$, we demonstrate that this can indeed happen, that is, we give examples of proper nonzero $\phi$-submodules of the trivial $\phi$-module $\mathcal{R}(0)$ with positive slope (a more general receipt given in Remark 3.12 below).

*Example.*

(i) Let $\phi u = (1+u)^q - 1$ and fixing **K**. Put $x := \log(1+u) = u - u^2/2 + u^3/3 - \cdots$. Then $\phi x = \log((1+u)^q) = q\log(1+u) = qx$.

(ii) Let $\phi u = u^q$ and fixing **K**. Let $E(u) \in \mathbf{K}[[u]]^{\text{bdd}}$ with constant coefficient 1. Then the element $x := \prod_{n\geq 0} E(u^p) \in \mathcal{O}$ exists. We see

$$\phi(x) = \prod_{n\geq 0} \phi^{n+1}E(u) = 1/E(u) \prod_{n\geq 0} \phi^n E(u) = 1/E(u) \cdot x.$$

*Moral*: Let $M = \mathcal{R} \cdot e$ be a one dimensional $\phi$-module over $\mathcal{R}$. Then $\phi(e) = c \cdot e$ and since $\mathcal{R}^* = \mathcal{R}^{\text{bdd}*}$, the eigenvalue has a *slope* $s = v(c)$. *But there are eigenvectors of different slopes in* M, in contrast to the situation over a discretely valued field **K** such as $\mathbf{K} = \mathcal{R}^{\text{bdd}}$. Note that because $\phi_{|\mathcal{R}^{\text{bdd}}}$ is an isometry, this solely occurs because there are *unbounded* $x \in \mathcal{R}$ with $\phi x/x \in \mathcal{R}^* = \mathcal{R}^{\text{bdd}*}$ (with $\phi(x)/x :=$ the unique $c \in \mathcal{R}^*$ with $\phi x = c \cdot x$).

# 3 Semistability implies Pureness

## Strategy

Let us turn to the proof of semistability implying pureness. For this:

(i) Construct a ring $\widetilde{\mathscr{R}} \supset \mathscr{R}$ and prove that semistability over $\widetilde{\mathscr{R}}$ implies pureness.

(ii) Use faithfully flat descent to prove that $\mathrm{M} \mapsto \mathrm{M} \otimes_{\mathscr{R}} \widetilde{\mathscr{R}}$ for a $\phi$-module M over $\mathscr{R}$ satisfies:

  (a) If M is semistable, so is $\mathrm{M} \otimes \tilde{\mathscr{R}}$.

  (b) If $\mathrm{M} \otimes \widetilde{\mathscr{R}}$ is pure, so is M.

## The big ring $\widetilde{\mathscr{R}} \supset \mathscr{R}$

Firstly, we turn to the construction of $\tilde{\mathscr{R}}$ and list some of its Properties.

(i) To prove the faithfully flat descent of semistability and pureness, we will replace $\mathbf{K}$ by its *difference closure*, the smallest field extension of $\mathbf{K}$ over which every etale $\phi$-module is trivial. Since $\phi_{\mathbf{K}}$ is a Frobenius lift, this holds by Dieudonne-Manin if $\mathbf{k}$ is replaced by its algebraic closure (but can also be constructed in the general framework of Remark 2.1, see [KdFrb, Prop. 3.2.4]).

(ii) The overring $\tilde{\mathscr{R}}$ will be a ring of formal series over $\mathbb{Q}$ with coefficients in $\mathbf{K}$. It is endowed with an endomorphism $\phi$, acting *bijectively* through the Frobenius on coefficients and through $t^i \mapsto t^{iq}$ on the formal variables $t^i$ for $i \in \mathbb{Q}$. Formally,
**Definition 3.1.** Let $\tilde{\mathscr{R}} = \{f = \sum_{i \in \mathbb{Q}} a_i t^i \quad \text{with } a_i \in \mathbf{K}\}$ such that there is a $0 < \alpha = \alpha(f) < 1$ such that:

  (a) The sum exists for $t \mapsto x \in \mathrm{D}([\alpha, 1))$.

  (b) The set of rational indices for which $|a_i| \geq c$ is well-ordered for any $c > 0$.

  The second postulate is needed to ensure the natural multiplication being meaningful. Then in our situation $\mathscr{R} \hookrightarrow \tilde{\mathscr{R}}$ is the subring cut out by the formal series over $\mathbb{Z} \subset \mathbb{Q}$ with the natural inclusion up application of $\phi_{\mathbf{K}}^{-1}$ on the coefficients (For a general Frobenius $\phi : \mathscr{R} \circlearrowleft$, one can still embed $\mathscr{R} \hookrightarrow \tilde{\mathscr{R}}$ Frobenius-equivariantly, see [KdFrb, Prop. 2.2.6]).
  Then $\tilde{\mathscr{R}}$ is a Bezout domain with $\tilde{\mathscr{R}}^* = \mathscr{R}^{\tilde{\mathrm{bdd}}^*}$ and satisfies the (Bij)-property given in Lemma 3.6 below (as $\mathscr{R}$ does).

(iii) Its subring $\mathscr{R}^{\tilde{\mathrm{bdd}}}$ of series with bounded coefficients is a *henselian* discretely valued field with residue field $\mathbf{k}((t^{\mathbb{Q}}))$.

*Remark* 3.2. The last item shows that $\tilde{\mathscr{R}}^{\text{int}}$ contrasts to $\mathscr{R}^{\text{int}}$ by having an algebraically closed residue field. Hence up to completeness of $\tilde{\mathscr{R}}^{\text{int}}$, the assumptions (DM) on $A = \mathscr{R}^{\text{int}}$ in [SnTdA, Paragraph 8] hold. Hence by Dieudonne'-Manin every $\phi$-module over the completion of $\tilde{\mathscr{R}}^{\text{int}}$ is trivial (and by the proof of Proposition 3.10, it indeed already holds over $\tilde{\mathscr{R}}^{\text{int}}$). Moral: The ring $\tilde{\mathscr{R}} \supset \mathscr{R}$ collects sufficiently many properties (bijective Frobenius lift and algebraically closed residue field) to ensure that pure $\phi$-modules over $\tilde{\mathscr{R}}$ admit a Dieudonne'-Manin classification.

## Semistability implies pureness over $\tilde{\mathscr{R}}$

*Interlude: The (Inj)- and (Bij)-property.*

**Definition**. For any $\phi$-module $\mathscr{M}$ over $R = \tilde{\mathscr{R}}^{\text{int}}, \tilde{\mathscr{R}}^{\text{bdd}}, \tilde{\mathscr{R}}$, define the R-module

$$H^0(\mathscr{M}) := \ker \phi - \mathrm{id}_{\mathscr{M}} = \{x \in \mathscr{M} : \phi x = x\}.$$

*Remark* 3.3. Recall that $\mathrm{Hom}(M, N) = M^\vee \otimes N$ for free $\mathscr{R}$-modules M, N. In the category of $\phi$-modules over $\mathscr{R}$, one checks instead

$$\mathrm{Hom}(M, N) = H^0(M^\vee \otimes N).$$

The reason being the elements of $M^\vee \otimes N \simeq \mathrm{Hom}(M, N)$ fixed by $\phi^\vee \otimes \phi$ corresponding to those homomorphisms which are $\phi$-equivariant (as $\phi^\vee$ acts by $f \mapsto f \circ \phi^{-1}$ on $M^\vee$ to ensure $\mathscr{R}_\phi \otimes_{\mathscr{R}} \mathrm{Hom}(M, \mathscr{R}) \overset{\phi^\vee}{\simeq} \mathrm{Hom}(M, \mathscr{R})$).

**Lemma 3.4** ((Inj)-Property). *Let* A *be any* $n \times n$*-matrix in* $\mathscr{R}^{\text{int}}$. *If* $v \in \mathscr{R}^n$ *satisfies* $v - A\phi v \in {\mathscr{R}^{\text{bdd}}}^n$, *then already* $v \in {\mathscr{R}^{\text{bdd}}}^n$.

*Proof:* This is an application of the fact that the Frobenius increases the outer boundary of convergence. We have $|x| = |x|_1 = \lim_{\rho \to 1} |x|_\rho$ for $x \in \mathscr{R}$, yielding:

(i) We have to prove boundedness of $\{|v|_\rho : \rho < 1\}$ or some cofinal subset thereof.

(ii) Since $w := v - A\phi v$ has coefficients in $\mathscr{R}^{\text{bdd}}$, we find a $\delta > 0$ such that $|w|_\rho \leq c$ for all $\delta \leq \rho < 1$ with some constant $c > 0$.

(iii) We find $x \in \mathscr{R}^{\text{int}}$ if and only if $u^i x$ for $i \gg 0$ is bounded by 1 on $D([\delta, 1))$ for some $\delta$. Replacing $v, A$ by $1/u^i \cdot v, \phi(u^i)/u^i \cdot A$ for $i \gg 0$, we can therefore — as it doesn't affect $v \in {\mathscr{R}^{\text{bdd}}}^n$ - assume $|A|_\rho \leq 1$ for all $\rho \geq \delta$ and some $\delta < 1$.

By definition, we find positive $\delta < 1$ $v$ such that $v$ converges on $\mathrm{D}([\delta^q, 1))$. We compute

$$\begin{aligned} |v|_\rho &\le \max(|\mathrm{A}\phi v|_\rho, |v - \mathrm{A}\phi v|_\rho) \\ &\le \max(|\mathrm{A}|_{\rho^q}|\phi v|_\rho, c) = \max(|\phi v|_\rho, c) = \max(|v|_{\rho^q}, c). \end{aligned}$$

Thus in particular, if $|v|_{\delta^q} \le \mathrm{C}$ with fixed $\mathrm{C} \ge c$, then $|v|_\delta \le \mathrm{C}$. Therefore the cofinal set $\{|v|_{\sqrt[q^n]{\delta}} : n \ge 0\} \subset \{|v|_\rho : \rho < 1\}$ is bounded by C, q.e.d. □

**Lemma 3.5.** *Let $\mathscr{M}$ be a $\phi$-module over $\mathscr{R}^{\text{int}}$. Then the natural map* $\mathrm{H}^0(\mathscr{M} \otimes \mathscr{R}^{\text{bdd}}) \to \mathrm{H}^0(\mathscr{M} \otimes \mathscr{R})$ *is bijective.*

*Proof:* It is injective by $\mathscr{R}^{\text{bdd}} \hookrightarrow \mathscr{R}$ and $\mathscr{M}$ being free. Let $\mathscr{M} \simeq {\mathscr{R}^{\text{int}}}^n$ and under this isomorphism $\phi \simeq \mathrm{A}\phi$. Let $x \in \mathscr{M} \otimes \mathrm{R}b \simeq \mathscr{R}^n$ with $x - \mathrm{A}\phi x = 0 \in {\mathscr{R}^{\text{bdd}}}^n$. Then (Inj) implies $x \in {\mathscr{R}^{\text{bdd}}}^n$, therefore surjectivity. □

**Corollary.** *The base change functor from pure $\phi$-modules over $\mathscr{R}^{\text{bdd}}$ to pure $\phi$-modules over $\mathscr{R}$ is an equivalence of categories.*

*Proof:* By definition of pureness, we only need to check faithfulness: For any $\mathscr{R}^{\text{bdd}}$-modules M, N, we have a bijection

$$\mathrm{Hom}_{\phi\text{-modules}/\mathscr{R}^{\text{bdd}}}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}_{\phi\text{-modules}/\mathscr{R}^{\text{bdd}}}(\mathrm{M} \otimes \mathscr{R}, \mathrm{N} \otimes \mathscr{R}).$$

This is Remark 3.3 plugged into Lemma 3.5. □

**Definition.** For any $\phi$-module $\mathscr{M}$ over $\mathrm{R} = \tilde{\mathscr{R}}^{\text{int}}, \tilde{\mathscr{R}}^{\text{bdd}}, \tilde{\mathscr{R}}$, define the R-module

$$\mathrm{H}^1(\mathscr{M}) := \operatorname{coker} \phi - \mathrm{id}_{\mathscr{M}}.$$

**Lemma 3.6** ((Bij)-property). *Let* A *be any $n \times n$-matrix in $\tilde{\mathscr{R}}^{\text{int}}$. Then* $\mathrm{id} - \mathrm{A}\phi$ *is bijective on* $\tilde{\mathscr{R}}^n / {\tilde{\mathscr{R}}^{\text{bdd}}}^n$.

*Proof:* See [KdFrb, Prop. 1.14]. □

*Remark* 3.7. One can check $\mathrm{H}^1(\mathrm{M}^\vee \otimes_{\tilde{\mathscr{R}}} \mathrm{N})$ to parameterize the extensions of M by N (see [HPFrob, Prop. 2.4]), that is,

$$\mathrm{H}^1(\mathrm{M}^\vee \otimes_{\tilde{\mathscr{R}}} \mathrm{N}) = \mathrm{Ext}^1(\mathrm{M}, \mathrm{N}).$$

**Example**. Assume that every irreducible $\phi$-module over a ring R is trivial. Then $H^1(M,N)$ splits for $\phi$-modules M, N over R if $1-\phi : R \circlearrowleft$ is surjective (for example, $\phi = \cdot^q$ and R is an separably closed field). Let

$$0 \to M' \to M \to M'' \to 0$$

be exact and $M' = R \cdot v'$ and $M'' =< \bar{v''} >$. Then $\phi v'' = v'' + \varepsilon \cdot v'$. Then there is $\delta \in R$ with $\phi(\delta) = \delta - \varepsilon$. Put $\tilde{v}'' = v'' + \delta v'$. Then

$$\phi(\tilde{v}'') = v'' + \varepsilon v' + \delta v' - \varepsilon v' = v'' + \delta v' = \tilde{v}''.$$

Therefore any Jordan-Hölder series splits.

**Corollary 3.8**. *Every extension* $0 \to M' \to M \to M'' \to 0$ *of etale* $\phi$*-modules over* $\mathcal{R}$ *descends (that is, if* $M', M''$ *are etale, then* $M$ *is etale).*

*Proof:* Let M be an $\mathcal{R}^{\text{int}}$-module, that is, $\phi$ is defined by a matrix with entries in $\mathcal{R}^{\text{int}}$. Put $\check{F} := 1 - A\phi$. Then (Bij) says :

$$M_{\tilde{\mathcal{R}}} \xrightarrow{\check{F}} M_{\tilde{\mathcal{R}}} \quad \text{is surjective up to a difference in } M_{\mathcal{R}^{\tilde{\text{bdd}}}},$$

and more exactly, it is unique in the following sense :

$$M_{\mathcal{R}^{\tilde{\text{bdd}}}}/\text{im}(M_{\mathcal{R}^{\tilde{\text{bdd}}}} \xrightarrow{\check{F}} M_{\mathcal{R}^{\tilde{\text{bdd}}}}) \to M_{\tilde{\mathcal{R}}}/\text{im}(M_{\tilde{\mathcal{R}}} \xrightarrow{\check{F}} M_{\tilde{\mathcal{R}}}) \quad \text{is even bijective.}$$

Hence the natural map $H^1(M \otimes_{\mathcal{R}^{\tilde{\text{int}}}} \mathcal{R}^{\tilde{\text{bdd}}}) \to H^1(M \otimes_{\mathcal{R}^{\tilde{\text{int}}}} \tilde{\mathcal{R}})$ on $\phi$-modules over $\mathcal{R}^{\tilde{\text{int}}}$ is bijective. Plugging in Remark 3.7, the natural map $\text{Ext}^1_{/\mathcal{R}^{\tilde{\text{int}}}}(M, N) \to \text{Ext}^1_{/\tilde{\mathcal{R}}}(M \otimes_{\mathcal{R}^{\tilde{\text{int}}}} \tilde{\mathcal{R}}, N \otimes_{\mathcal{R}^{\tilde{\text{int}}}} \tilde{\mathcal{R}})$ is a surjection for any two fixed $\phi$-modules M and N over $\mathcal{R}^{\tilde{\text{int}}}$. □

*Remark*. Moral: (Inj)-property yields

$$\{\text{pure } \phi\text{-modules}/\mathcal{R}^{\text{bdd}}\} \hookrightarrow \{\text{pure } \phi\text{-modules}/\mathcal{R}\},$$

(Surj)-property yields

$$\{\text{extensions of pure } \phi\text{-modules}/\mathcal{R}^{\text{bdd}}\} \twoheadrightarrow \{\text{extensions of pure } \phi\text{-modules}/\mathcal{R}\}.$$

*Pureness of semistable $\phi$-modules over $\tilde{\mathcal{R}}$.*

**Proposition 3.9.** *Every semistable $\phi$-module* M *over $\tilde{\mathcal{R}}$ is pure.*

*Proof:* We construct inductively a filtration on M with pure graduation steps of rank one. If we without loss of generality assume $\mu M = 0$, these must by semistability have nonnegative slope. We will then prove that the minimal slope of these graduation steps which we can choose from will not increase (stay zero).

Formally: Proceed by induction on rank M. Every $\phi$-module of rank one is pure by $\tilde{\mathcal{R}}^* = \tilde{\mathcal{R}}^{\mathrm{bdd}^*}$. Let rank M $= l + 1 > 1$. By twisting, that is, replacing $\phi$ on $\mathcal{R}$ and M trough $\phi^r$ and $\phi : M \circlearrowleft$ through $\pi^{-s}$ - neither changing pureness nor semistability — we may assume $\mu(M) = 0$.

It suffices to find an etale submodule N of rank 1: Then M/N is semistable and hence pure by the induction hypothesis. Thus N and M/N are etale and we have M is an extension of these, that is, we have exactness of

$$0 \to N \to M \to M/N \to 0.$$

By definition, the pure modules N and M/N descend to modules over $\mathcal{R}^{\mathrm{int}}$ and by Corollary 3.8, M descends to an extension of these. Therefore M is etale.

To find such etale $N \subset M$, we will:

(i) Construct a $\phi$-submodule $N \subset M$ of rank 1.

(ii) Prove that we can construct one of minimal slope 0.

Ad (i): For this, we have to find a vector $m \in M$ with $\phi m = \pi^k m$ for $k \gg 0$ (positive by semistability). Viewing M as a space of column vectors and action of $\phi$ given through $A\phi$ for a $l \times l$-matrix A with coefficients in $\tilde{\mathcal{O}}_{[\rho^q,1)}$, this holds if and only if we can find $v \in M$ with $\pi^{-k} A\phi v - v = 0$. Put

$$F := \pi^{-k} A\phi \quad \text{and} \quad \check{F} := F - \mathrm{id}.$$

We want to construct a sequence $v_n \to v \neq 0$ such that the error term $\check{F} v_n := w_n \to 0$.

We observe that $\phi : \tilde{\mathcal{R}} \circlearrowleft$ is norm decreasing on $\sum_{i \geq 0} a_i u^i$ with respect to $|\cdot|_\rho$ for $r \in ]0,1[$, whereas $\phi^{-1} : \tilde{\mathcal{R}} \circlearrowleft$ is so on $\sum_{i \leq 0} a_i u^i$. For $f \in \tilde{\mathcal{R}}$, let

$$f^+ := \sum_{i \geq d} a_i u^i \quad \text{and} \quad f^- := \sum_{i < d} a_i u^i$$

for $d \in \mathbb{Q}_{>0}$ whose choice we will make precise below. We apply this component-wise to vectors in $\tilde{\mathcal{R}}^n$. Start with nonzero $v_0 \in V$. Let $w_n := \check{F} v_n$ be the error term and $v_{n+1} := v_n + f(w_n)$ for $f$ yet to be constructed giving $|w_{n+1}|_\rho < |w_n|_\rho$. By definition of $v_{n+1}$, we obtain the new error term

$$w_{n+1} = \check{F}(v_{n+1}) = \check{F}(f(w_n)) + w_n.$$

Since $|w_{n+1}|_\rho < |w_n|_\rho$, the map $f$ should cancel $w_{n+1} = w_{n+1}^+ + w_{n+1}^-$. This motivates

$$f(w) := w^+ - F^{-1}(w^-),$$

to yield

$$w_{n+1} = F w_n^+ + F^{-1} w_n^- = g(w_n) \quad \text{with } g := F \cdot^+ + F^{-1} \cdot^- .$$

We obtain by definition

$$\|g\|_\rho \leq \max(|\pi|^{-k} \|A\|_\rho \rho^{d(q-1)}, |\pi|^k \|A^{-1}\|_{\rho^q} \rho^{d(q^{-1}-1)}). \tag{$*$}$$

Since $|\pi|^k < \varepsilon$ for $k \ll 0$ and $\rho^{q-1} < \rho^{1-q^{-1}} < 1$, we can find $k$ such that there is suitable $d \in \mathbb{Q}_{>0}$ with $\|g\|_\rho < 1$.

Since $w_n \to 0$, this proves $(v_n)$ to be Cauchy and $v = \lim_{n\to\infty} v_n$ to exist with respect to $|\cdot|_\rho$, fulfilling $\check{F} v = 0$. As $d$ fulfills ($*$), we can check to find nonzero $v \in \tilde{\mathcal{O}}^l_{[\rho,\rho]}$ for the choice of $v_0 = (u^d, 0, \ldots, 0)$.

Because $\phi v_n$ converges to $\phi v$ with respect to $|\cdot|_{\sqrt[q]{\rho}}$, we find $F v_n = \pi^{-k} A \phi v \to F v$ with respect to $|\cdot|_{\sqrt[q]{\rho}}$. Therefore $v = F v \in M_{[r, \sqrt[q]{\rho}]}$. Iteratively $v \in M_{[r, \sqrt[q^l]{\rho}]}$ for $l \geq 0$. Thus $v \in \tilde{\mathcal{R}}^l \simeq M$. This proves 1.

For (ii), we need a bit of preparatory theory. This statement is [KisCrs, Theorem 1.3.2]:

**Proposition 3.10.** *For any rational number s, we have an equivalence of categories*

$$\{\textit{pure } \phi\textit{-modules} / K\} \to \{\textit{pure } \phi\textit{-modules} / \tilde{\mathcal{R}}\}$$
$$M \mapsto M \otimes_{\mathbf{K}} \tilde{\mathcal{R}}$$

*Proof:* By twisting — preserving pureness — we are reduced to the case $s = 0$, that is, - by fully faithfulness of base extension to the quotient field (for example, Section 3) - an equivalence between modules over $\mathbf{o_K}$ and $\tilde{\mathcal{R}}^{\text{int}}$ (see [KdFrb, Proof of Prop. 2.1.6]). We have to prove full faithfulness and essential surjectivity.

(i) Full faithfulness: By Remark 3.3, we must check: Let M be a $\phi$-module over $\mathbf{o_K}$. Then

$$\mathrm{H}^0(\mathrm{M}) \sim \mathrm{H}^0(\mathrm{M} \otimes_{\mathbf{K}} \tilde{\mathscr{R}}^{\mathrm{int}}),$$

By [SnTdA, Satz 2.1], (up to isomorphism) M is trivial. Hence we must show $\tilde{\mathscr{R}}^{\mathrm{int}^{\phi}} = \mathbf{K}$: If $\phi x = x$ for nonzero $x \in \tilde{\mathscr{R}}^{\mathrm{int}}$, then $|a_i| = |a_{iq^n}|$ for all $n \in \mathbb{Z}$. This contradicts the assumption of $(a_i : |a_i| \geq c)$ being wellordered for any $c > 0$.

(ii) Essential surjectivity: Equivalently we must prove that every etale $\phi$-module over $\tilde{\mathscr{R}}^{\mathrm{int}}$ is trivial. Let $\mathbf{E} = \hat{\tilde{\mathscr{R}}}^{\mathrm{int}}$. Since $\mathbf{k_E} = \mathbf{k}((t^{\mathbb{Q}}))$ is algebraically closed, this holds by [SnTdA, Satz 8.20]. Then one can prove the "decompletion result" that $\phi v = v$ for $v \in \mathbf{E}^l$ already implies $v \in \tilde{\mathscr{R}}^{\mathrm{int}^l}$, see [KdFrb, Prop. 2.5.8]. □

**Corollary 3.11.** *Let* M *and* N *be pure* $\phi$*-modules over* $\tilde{\mathscr{R}}$*. If* $\mu(\mathrm{M}) > \mu(\mathrm{N})$*, then* Hom(M, N) *is nonzero.*

*Proof:* By Remark 3.3, that is, writing $\mathrm{Hom}(\mathrm{M},\mathrm{N}) = \mathrm{H}^0(\mathrm{M}^\vee \otimes \mathrm{N})$, it suffices to prove: If M is pure with $\mu(\mathrm{M}) < 0$, then $\mathrm{H}^0(\mathrm{M}) \neq 0$, that is, there is a fix vector $v \in \mathrm{M}$. By Proposition 3.10, we have $\mathrm{M} = \mathrm{M}_0 \otimes_{\mathbf{K}} \tilde{\mathscr{R}}$ for $\mathrm{M}_0$ pure with $\mu\mathrm{M}_0 < 0$. Let $w \in \mathrm{M}_0$ be nonzero. Put

$$v = \sum_{n \in \mathbb{Z}} \phi^n(uw).$$

It suffices to prove $v \in \tilde{\mathscr{R}}_\rho$ for some $\rho \in (0,1)$. Then obviously $\phi v = v$ and by this $v \in \tilde{\mathscr{R}}_{[\rho, \sqrt[q]{\rho}]}$. Inductively $v \in \cap_{n \geq 0} \tilde{\mathscr{R}}_{[\rho, \sqrt[q^n]{\rho})} \subset \tilde{\mathscr{R}}$. Let $\phi^r = \pi^s =: c > 1$ for $s \in \mathbb{Z}_{<0}$ and $r \in \mathbb{N}$. We must have for $n \geq 0$ that

$$|\phi^{rn}(uw)|_r = |\pi^s|^n \rho^{q(rn)} = (c\rho^q r)^n \longrightarrow 0 \quad \text{ as } n \to \infty$$

and for $n \leq 0$ that

$$|\phi^{rn}(uw)|_r = |\pi^s|^n \rho^{q^{-1}(rn)} = (c\rho^{r/q})^n \longrightarrow 0 \quad \text{ as } n \to -\infty.$$

If we thus choose $\rho \in ]\sqrt[r/q]{1/c}, \sqrt[qr]{1/c}[$ nonempty, we are fine. □

*Remark* 3.12 (Reoccurrence of Remark 2.5). Corollary 3.11 shows that as soon as we allow different slopes, the above Proposition 3.10 fails: Let $\mathrm{M} = \mathbf{K}(0) := \mathbf{K} \cdot e$ with $\phi(e) := e$. By $\phi : \mathbf{K} \circlearrowleft$ being an isometry, every submodule must have slope 0. But over $\mathrm{M} \otimes \tilde{\mathscr{R}} = \tilde{\mathscr{R}}(0)$, we find by Corollary 3.11 eigenvectors of any positive valuation.

Ad (ii): Recall that by 1., we can construct $\phi$-submodules of rank one. Let $N' \subset M' := [l]_* M$ be such a $\phi^l$-submodule, necessarily of integral slope. Let $\mu(N') =: c \in \mathbb{Z}_{\geq 0}$ be the minimal slope of all these $N'$. We want to show $c = 0$. We are hence in the following situation:

$$0 \to N' \to M' \to \bar{M}' := M'/N' \to 0.$$

Assume that instead $c = \mu(N') \geq 1$. We want to contradict this by finding a vector of slope 0 in $M'$. Firstly, we can twist this exact sequence by $\tilde{\mathcal{R}}(1-c)$ to resort to $c = \mu(N') = 1$: If then $E(1-c) \subset M'(1-c)$ is the one dimensional $\phi$-submodule spanned by this vector of slope 0, then $\mu(E) = c - 1 < c$, still contradicting minimality of $c$.

Let $c = 1$ and put $N := [n]^* N'$ (Recall: If we identify

$$\phi^n\text{-module} / \mathcal{R} = \mathcal{R}\{T^n\}\text{-module}$$

for the $\phi^n$-twisted polynomial ring $\mathcal{R}\{T^n\}$, then the $\mathcal{R}\{T\}$-module $[n]^* M$ is given by

$$[n]^* M := M \otimes_{\mathcal{R}\{T^n\}} \mathcal{R}\{T\},$$

the adjoint functor of $M \mapsto [n]_* M$). Therefore $N' \to M'$ corresponds to some (nonzero) map $N \xrightarrow{f} M$. Since $N \subset M$ is pure, hence semistable, we find $\mu(f(N)) \leq \mu(N) = 1/n$ (equivalent characterization of semistablity: submodules do not have smaller slope if and only if quotients do not have larger slope).

If $\mu(f(N)) < 1/n$, then by $\operatorname{rank} f(N) \leq n$ already $\mu(f(N)) \leq 0$. Then $H^0(f(N)) \neq 0$. Reason: Let $f(N)_1$ denote the first step of the slope filtration, given by induction. Either already $\mu(f(N)_1) = 0$ - in which case $f(N)_1$ comes from a trivial etale $\phi$-module over **K** by Proposition 3.10 - or $\mu(f(N)_1) < 0$, when we use Corollary 3.11 to obtain a fixed vector inside $f(N)_1$.

If $\mu(f(N)) = 1/n$, then $\operatorname{rank} f(N) = n$ and hence $N \overset{f}{\hookrightarrow} M$. We are thus in the following situation:

$$0 \to [n]^* N' \to M \to P \to 0$$

with $N'$ a one dimensional $\phi^n$-module over $\tilde{\mathcal{R}}$ of degree 1 and P a one dimensional $\phi$-module over $\tilde{\mathcal{R}}$ of degree $-1$. Concretely: There is a basis $v_1, \ldots, v_{l-1}, v_l$ of N such that

$$\phi v_i = v_{i+1} \text{ for } i = 1, \ldots, l-2, \quad \phi v_{l-1} = \pi v_1, \quad \phi v_l = \pi^{-1} v_l + c_1 v_1 + \cdots + c_{l-1} v_{l-1}.$$

If this holds, one can prove M to have a fixed vector (see [KdFrb, Prop. 2.1.7]). ◻

## Faithfully flat descent

Recall that we want to prove: Let M be a $\phi$-module over $\mathscr{R}$, then

(i) If M is semistable, so is $M \otimes \tilde{\mathscr{R}}$.

(ii) If $M \otimes \widetilde{\mathscr{R}}$ is pure, so is M.

*Interlude: Gluing.* Consider the following gluing data: $\{M_i \to X_i\}$ coherent modules over affine schemes $X_i \overset{\text{open}}{\to} X$, subschemes $X_{ij} \overset{\text{open}}{\subset} X_i$ and identifications $M_i|X_{ij} \overset{f_{ij}}{\simeq} M_j|X_{ji}$ satisfying the so called cocylce condition (arising by demanding the relation $\sim$ to be transitive, if $X = \coprod X_i/\sim$ with $m_j \sim m_i$ if $m_j = f_{ij} m_i$).

More concisely: A coherent module $M' := \coprod M_i$ over the affine scheme $X' := \coprod X_i$ with identifications $M''_1 \simeq M''_2$, where $M''$ are coherent modules over $X'' := X' \times_X \times X'$ through the precomposed projection $X'' \to X'$ onto the $i$-th factor.

*The descent.* By the anti-equivalence of the categories of affine schemes and rings, the interlude informs:

**Theorem 3.13** (Grothendieck). *Let* $R \to S$ *be a faithfully flat ring homomorphism and define*

$$\iota_1, \iota_2 \colon S \to S_2 := S \otimes_R S \quad \textit{by } \iota_1(s) = s \otimes 1, \iota_2(s) = 1 \otimes s$$

*plus certain* $\iota_{ij} \colon S_2 \to S_3 := S \otimes_R S \otimes_R S$ *for distinct* $i, j = 1, 2, 3$ *(to formulate the cocycle condition). Then a* descent datum *on an* S*-module* $M_S$ *is an isomorphism of* $S \otimes_R S$*-modules*

$$M_S \otimes_{\iota_1} S_2 \simeq M_S \otimes_{\iota_2} S_2$$

*plus certain compatibility conditions with respect to the* $\iota_{ij}$ *accounting for the cocycle condition. Then we have an equivalence of categories*

$$\{R\textit{-modules}\} \to \{S\textit{-modules with descent datum}\}$$
$$M \mapsto M_S := M \otimes_R S \quad \textit{with } M_S \otimes_{\iota_1} S_2 \simeq M_S \otimes_{\iota_2} S_2 \textit{ canonical}$$

*Remark.* The canonical descent datum stems from $\iota_1 \circ \iota \simeq \iota_2 \circ \iota$ with $\iota \colon S \to S_2$ diagonally and then using that $S_2$ is the fibre coproduct of S and S over R.

**Corollary 3.14.** *Let* $R \to S$ *and* $\iota_1, \iota_2 \colon S \to S_2$ *as above. Let* $M$ *be a* $\phi$*-module over* $R$*, put* $M_S = M \otimes_R S$ *and let* $N_S \subset M_S$ *be a* $\phi$*-submodule over* $S$*. Then* $N_S$ *descends to a* $\phi$*-module over* $R$ *if and only if*

$$N_S \otimes_{\iota_1} S_2 = N_S \otimes_{\iota_2} S_2,$$

*considered as embedded into* $M \otimes_R S_2$.

*Proof:* For $N_S$, we have the descent datum $N_S \otimes_{\iota_1} S_2 = N_S \otimes_{\iota_2} S_2$, as the cocycle condition trivially fulfilled by $M_S$ restricts to $N_S$. Hence by Theorem 3.13, the module $N_S$ descends and also $\phi$ by full faithfulness. □

We adapt this to our situation: Let

$$\begin{aligned} \mathscr{S} &:= \tilde{\mathscr{R}} \otimes_{\mathscr{R}} \tilde{\mathscr{R}}, \\ \mathscr{S}^{\mathrm{bdd}} &:= \tilde{\mathscr{R}}^{\mathrm{bdd}} \otimes_{\mathscr{R}^{\mathrm{bdd}}} \tilde{\mathscr{R}}^{\mathrm{bdd}}, \\ \mathscr{S}^{\mathrm{int}} &:= \tilde{\mathscr{R}}^{\mathrm{int}} \otimes_{\mathscr{R}^{\mathrm{int}}} \tilde{\mathscr{R}}^{\mathrm{int}}. \end{aligned}$$

To apply faithfully flat descent, we need a number of ring maps, such as $\mathscr{R} \to \tilde{\mathscr{R}}$, to be faithfully flat. To show this, we firstly simply quote [KdFrb, Prop. 3.4.1].

**Lemma 3.15.** *Let* $R \hookrightarrow S$ *be an inclusion of domains with* $R$ *Bezout. Then* $S$ *is faithfully flat if and only if* $S^* \cap R = R^*$.

*Proof:* Recall that $S$ is flat (respectively faithful) over $R$ if and only if for each finitely generated proper ideal $I$ of $R$, the multiplication map $I \otimes_R S \to S$ is injective (respectively not surjective). Since $R$ is Bezout, $I$ admits a single generator $r \in R^*$ and $I \otimes_R S = r \cdot R \otimes_R S = r \cdot S$, so the map $I \otimes_R S = S$ is injective (and it is surjective if and only if $r \in S^*$. This yields the claim. □

If $R = \mathscr{R}, \tilde{\mathscr{R}}$, recall that $R^* = R^{\mathrm{bdd}*}$ and $R$ is Bezout. By the above Lemma 3.15, we have faithfully flat inclusions $\mathscr{R} \to \tilde{\mathscr{R}}$, $\mathscr{R}^{\mathrm{bdd}} \to \tilde{\mathscr{R}}^{\mathrm{bdd}}$. Moreover $\mathscr{S}^{\mathrm{bdd}} \hookrightarrow \mathscr{S}$ (see [KdFrb, Remark 3.5.3]). To prove the two remaining descent statements, we need the following crucial

**Proposition 3.16.** *Let* $A$ *be a square matrix over* $\mathscr{S}^{\mathrm{int}}$ *and* $v$ *a column vector over* $\mathscr{S}$ *such that* $A\phi v = v$*. Then* $v$ *has even entries in* $\mathscr{S}^{\mathrm{bdd}}$.

**Theorem 3.17.** *Let* $M$ *be a* $\phi$*-module over* $\mathscr{R}$*, then*

(i) *If* $M$ *is semistable, so is* $M \otimes \tilde{\mathscr{R}}$.

(ii) *If* $\mathrm{M} \otimes \widetilde{\mathcal{R}}$ *is pure, so is* M.

*Proof:* Ad (i): See [KdFrb, Proof of Theorem 3.1.2].

Ad (ii): By assumption, we can descend $\mathrm{M}_{\tilde{\mathcal{R}}} := \mathrm{M} \otimes_{\mathcal{R}} \tilde{\mathcal{R}}$ to a $\phi$-module $\mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}}$ over $\mathcal{R}^{\tilde{\mathrm{bdd}}}$. We want to apply descent for the faithfully flat ring homomorphism $\mathcal{R}^{\mathrm{bdd}} \to \mathcal{R}^{\tilde{\mathrm{bdd}}}$ for $\mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}}$.

By twisting — preserving pureness over $\mathcal{R}$ and $\tilde{\mathcal{R}}$ - we may assume $\mu(\mathrm{M}) = 0$, so that $\mathrm{M}_{\tilde{\mathcal{R}}}$ is etale. So let A the invertible matrix with coefficients in $\mathcal{R}^{\tilde{\mathrm{int}}}$ describing $\phi : \mathrm{M}_{\tilde{\mathcal{R}}} \circlearrowleft$ with respect to a basis $v_1, \ldots, v_n \in \mathrm{M} \subset \mathrm{M}_{\tilde{\mathcal{R}}}$. Let U be the base change matrix with coefficients in $\mathcal{S}$ of the bases $\{v_i \otimes_{\iota_1} 1\}, \{v_i \otimes_{\iota_2} 1\} \subset \mathrm{M} \otimes_{\mathcal{R}} \mathcal{S} =: \mathrm{M}_{\mathcal{S}}$, that is,

$$\mathrm{A} \otimes_{\iota_1} 1 = \mathrm{U}^{-1}(\mathrm{A} \otimes_{\iota_2} 1)\phi(\mathrm{U}) \quad \text{or} \quad \mathrm{U} = (\mathrm{A} \otimes_{\iota_2} 1)\phi(\mathrm{U})(\mathrm{A} \otimes_{\iota_1} 1)^{-1}.$$

Therefore $\mathrm{U} \in \mathrm{End}(\mathrm{M}_{\mathcal{S}})$ is fixed by the $\phi$-linear map $f \mapsto (\mathrm{A} \otimes_{\iota_1} 1) \circ f \circ (\mathrm{A} \otimes_{\iota_2} 1)^{-1}$ on $\mathrm{Hom}(\mathrm{M}_{\mathcal{S}}, \mathrm{M}_{\mathcal{S}})$ defined over $\mathcal{S}^{\mathrm{int}}$. By Proposition 3.16, U has entries in $\mathcal{S}^{\mathrm{bdd}}$ and also its inverse by the same argument with M replaced by $\mathrm{M}^{\vee}$ (doesn't follow automatically as we can't say much about $\mathcal{S}^{\mathrm{bdd}}$ otherwise). Therefore

$$\mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}} \otimes_{\iota_1} \mathcal{S}^{\mathrm{bdd}} = \mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}} \otimes_{\iota_2} \mathcal{S}^{\mathrm{bdd}} \subset \mathrm{M}_{\mathcal{S}}^{\mathrm{bdd}} \subset \mathrm{M}_{\mathcal{S}}.$$

By Corollary 3.14, $\mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}}$ descends to a $\phi$-module $\mathrm{M}_{\mathcal{R}^{\mathrm{bdd}}}$ over $\mathcal{R}^{\mathrm{bdd}}$.

It is left to prove that it is etale: Let $\mathrm{L}_{\mathcal{R}^{\mathrm{int}}} \subset \mathrm{M}_{\mathcal{R}^{\mathrm{bdd}}}$ be the minimal $\phi$-stable $\mathcal{R}^{\mathrm{int}}$-span of some basis, given by

$$\mathrm{L}_{\mathcal{R}^{\mathrm{int}}} :=< \phi^n b_i : n \geq 0 >_{\mathcal{R}^{\mathrm{int}}} \quad \text{for some basis } \{b_1, \ldots, b_l\} \text{ of } \mathrm{M}_{\mathcal{R}^{\mathrm{bdd}}}.$$

By $\phi : \mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}} \circlearrowleft$ being pure of slope 0, for any valuation $v$ on $\mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}}$, some power of $\phi$ fulfills $v(\phi(x)) = v(x)$. Therefore $\mathrm{L}_{\mathcal{R}^{\mathrm{int}}} \subset \mathrm{M}_{\mathcal{R}^{\mathrm{bdd}}}$ is by definition bounded, hence some lattice inside $\mathrm{M}_{\mathcal{R}^{\mathrm{bdd}}}$. By [KdFrb, Lemma 1.5.7], the $\phi$-stable lattice $\mathrm{L}_{\mathcal{R}^{\mathrm{int}}} \otimes_{\mathcal{R}^{\mathrm{int}}} \mathcal{R}^{\tilde{\mathrm{int}}} \subset \mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}}$ of the etale module $\mathrm{M}_{\mathcal{R}^{\tilde{\mathrm{bdd}}}}$ is even a $\phi$-module over $\mathcal{R}^{\tilde{\mathrm{int}}}$. Then also L over $\mathcal{R}^{\mathrm{int}}$ (for example, by looking at invertibility of the matrix representing $\phi$). □

## Addendum: Pureness implies semistability

For the uniqueness part of the slope filtration theorem, one needs to know that converse also holds true: Pureness implies semistability. This is easier to prove than its converse, which we do here.

**Theorem 3.18** (Pureness → Semistability). *Every pure* $\phi$*-module over* $\mathcal{R}$ *is semistable.*

*Proof:* Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of $\phi$-submodules and let $s = \mu\mathcal{M}$ be the slope of $\mathcal{M}$. We want to prove $\mu\mathcal{N} \geq s$. We record:

(i) For any $\phi$-module of rank $\leq i$ holds $\mu \wedge^i \mathcal{M} = i \cdot \mu\mathcal{M}$,

(ii) For any $\phi$-module $\mathcal{M}$ of rank one holds $\mu[a]_*\mathcal{M} = a\mu\mathcal{M}$ with the $\phi$-module $[a]_*\mathcal{M}$ given by replacing $\phi : \mathcal{M} \circlearrowleft$ by $\phi^a : \mathcal{M} \circlearrowleft$.

We simplify:

(i) We replace $\mathcal{M}$ by $\wedge^{\operatorname{rank}\mathcal{N}}\mathcal{M}$ to gain $\operatorname{rank}\mathcal{N} = 1$.

(ii) If $s = a/b \in \mathbb{Q}$, we replace $\mathcal{M}$ by $[b]_*\mathcal{M}$ to obtain $s \in \mathbb{Z}$.

(iii) We replace $\phi : \mathcal{M} \circlearrowleft$ by $\pi^{-t}$ with $t = \mu\mathcal{N}$ to obtain $\mathcal{N} \simeq \mathcal{R} \cdot e$.

We observe that these operations do not affect the order of $\mu\mathcal{M}$ and $\mu\mathcal{N}$. We are hence reduced to $\mathcal{M} \supset \mathcal{N} \simeq \mathcal{R} \cdot e$.

Since $\mathcal{N}$ is trivial, we find

$$\mathrm{H}^0(\mathcal{M}) = \mathrm{H}^0(\mathcal{N}^* \otimes \mathcal{M}) = \operatorname{Hom}(\mathcal{N}, \mathcal{M}) \neq 0.$$

We want to show that this implies $\mu\mathcal{M} \leq 0 (= \mu\mathcal{N})$. Replacing $\phi : \mathcal{M} \circlearrowleft$ by $\phi^b : \mathcal{M} \circlearrowleft$ if $s = a/b$, we may assume $s \in \mathbb{Z}$. By definition of pureness, we find a $\phi$-module $\mathcal{M}_0$ over $\mathcal{R}^{\mathrm{bdd}}$ such that $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}$. By Lemma 3.5, we have $\mathrm{H}^0(\mathcal{M}_0) = \mathrm{H}^0(\mathcal{M})$. By pureness, we find a basis $\{b_i\}$ such $v\phi b_i = s$ for the valuation $v$ on $\mathcal{M}$ given by the lattice of these. If $x = \sum a_i b_i$, we compute

$$\begin{aligned} v(\phi x) &= v(\phi \sum a_i b_i) \\ &\geq s + \min v(\phi a_i \cdot b_i) = s + \min v(\phi a_i) = \min v(a_i) = \min v(a_i \cdot b_i) = v(x). \end{aligned}$$

If $\phi x = x$, therefore necessarily $s \leq 0$. □

## References


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