Locally Algebraic Representations

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0. Reminder: p-adic numbers

$|\cdot|_p$ on $\mathbb{Z}$ measures how many times p appears in the prime factor decomposition of an integer. We put:

Definition

$|a|_p = 1/p^e$ if $a = a'p^e$ with p not dividing a'.

It extends multiplicatively to all rational numbers $\mathbb{Q}$. In line with $\mathbb{R}$ consisting of all the limits in $\mathbb{Q}$ with respect to the usual absolute value $|\cdot|$, we declare analogously:

Definition

The **p-adic numbers** $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$.

As over the real numbers, we have a *p-adic expansion* and we can write them in the form

$$\sum_{i>>-\infty} a_i p^i \quad \text{with } a_i \in \{0, \dots, p-1\}.$$

In striking contrast to $\mathbb{R}$, the topology of $\mathbb{Q}_p$ is totally disconnected. This means that there are a lot of locally constant functions on $\mathbb{Q}_p$; in fact, every continuous function is a uniform limit of theirs.
This gives rise to many new phenomena: For example, if G is a matrix group with entries in $\mathbb{Q}_p$, then it acts continuously on a finite dimensional $\mathbb{C}$-vector space V if and only if an open subgroup of G fixes all of V.

1. Unitary Banach spaces

1.1. p-adic Langlands

Let $\mathbb{Q}_p \subseteq \mathbf{F} \subseteq \mathbf{E}$ be finite extensions of $\mathbb{Q}_p$. We will study the actions of groups with coefficients in $\mathbf{F}$ (the *field of definition of our group*) on vector spaces over $\mathbf{E}$ (the *coefficient field*).

Definition

A **p-adic Galois representation** is a continuous action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})$ of $\mathbf{F}$ on a finite dimensional $\mathbf{E}$-vector space.

Let $G = \mathrm{GL}_n(\mathbf{F})$ for some $n \in \mathbb{N}$. Let V be a Banach space. Then $g \in G$ operates on V continuously if and only if there is $C > 0$ such that $\|g \cdot v\| \leq C\|v\|$ for all $v \in V$. If this constant does not depend on $g \in G$, we obtain the following definition.

Definition

A **unitary G-Banach space representation** is an action of the group G on an E-Banach space V whose topology on V is given by a G-invariant norm.

The following gives a very rough idea of the p-adic Langlands programme which is still in its infancy. There is yet in general no precise formulation of it.

Conjecture (p-adic Langlands philosophy)

There is a natural bijection $\rho \mapsto \Pi(\rho)$ between the following categories:

$$\{p\text{-adic representations of } \mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F}) \text{ of dimension } n\}$$
$$\updownarrow$$
$$\{\text{unitary Banach space representations of } \mathrm{GL}_n(\mathbf{F})\}.$$

- If $n = 2$ and $\mathbf{F} = \mathbb{Q}_p$, then this conjecture has a precise meaning, the bijection is even functorial and has been proved by Colmez, Berger, Breuil and others.
- In all other cases, be $n > 2$ or $\mathbf{F} \supset \mathbb{Q}_p$, very little is known.

1.2. Crystalline Galois ↔ Unitary Locally Algebraic

In [BS07] the authors attach to a certain family of *crystalline* Galois representations $\{\rho\}$ a certain *locally algebraic* G-representation, that is,

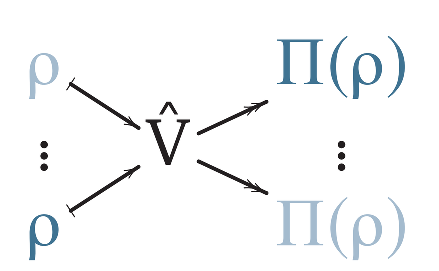
$$\{\rho\} \mapsto V = \text{ locally algebraic representation }.$$

Up to special obvious cases, one knows little about the corresponding Banach space representations.
The following is a key step in establishing such a p-adic Langlands correspondence for the subfamily of crystalline Galois representations.

Conjecture 1.1 (Breuil and Schneider)

If certain natural necessary conditions are satisfied, then V has a nonzero unitary completion $V \hookrightarrow \hat{V}$.

The vague hope is that all the corresponding unitary Banach space representations $\Pi(\rho)$ factorize through $\hat{V}$. That is



Remark

In what follows, all definitions and results still hold true, suitably generalized, for (the rational points of) a split reductive group **G** over a p-adic number field F. For the sake of exposition, we continue to work with GL_n (and eventually GL_2).

2. Induced locally algebraic representations

In the following we will work with the following subgroups of G. Let us recall that $\mathbf{F}$ defines the coefficients of G. We let $\mathbf{o_F}$ be the ring of integers of $\mathbf{F}$.

- Let $T \subseteq G$ be the torus in G, the subgroup of diagonal matrices with values in $\mathbf{F}^*$ and let $T_0 \subseteq T$ be its maximal compact subgroup with entries in $\mathbf{o}_\mathbf{F}^*$.
- Let P respectively $\bar{P}$ be the subgroup of upper respectively lower triangular matrices in G, and
- N respectively $\bar{N}$ the subgroup of P respectively $\bar{P}$ of matrices with diagonal entries in 1. Let $N_0 \subseteq N$ be the compact open subgroup with entries in $\mathbf{o_F}$.

The torus T operates on N by conjugation and we will denote these right respectively left actions by ${}^t n = tnt^{-1}$ respectively $n^t = t^{-1}nt$.

2.1. Induction of Representations

The most evident method to construct representations is by induction from smaller subgroups. Here we can pick

- copies $M_1 = \mathrm{GL}_{n_1}(\mathbf{F}), \dots, M_d = \mathrm{GL}_{n_d}(\mathbf{F})$ com $n_1 + \cdots + n_d = n$, and
- representations $\chi_1, \dots, \chi_d$ of these.

Let $M = M_1 \times \cdots \times M_d \subseteq G$ be their product and write $\chi = \chi_1 \otimes \cdots \otimes \chi_d$ for the representation of T given by their tensor product.

Definition

The **induced** $\mathbf{E}$-linear representation of G (or equivalently $\mathbf{E}[G]$-module) $\mathrm{ind}_M^G \chi$ of χ is defined as the $\mathbf{E}[G]$-module

$$\mathrm{ind}_M^G \chi = \chi \otimes_{\mathbf{E}[M]} \mathbf{E}[G].$$

We will in the following study the most elementary example:

- Let $n_1 = \ldots = n_d = 1$, so $T_1 = \ldots = T_d = \mathbf{F}^*$, and
- $\chi_1, \dots, \chi_d \colon \mathbf{F}^* \to \mathbf{E}^*$ characters.

We obtain as their tensor product simply the character $\chi \colon T \to \mathbf{E}^*$. Since $\mathbf{F}^*$ is abelian, and $T = \bar{P}^{ab}$, the maximal abelian quotient, χ extends uniquely by the natural projection $\bar{P} \to T$ to the character $\chi \colon \bar{P} \to \mathbf{E}^*$.

Example 2.1 (Induced representation of a Torus)

The (abstract) induced representation $\mathrm{ind}_{\bar{P}}^G \chi$ is explicitly given by

$$\mathrm{ind}_{\bar{P}}^G \chi = \{f \colon G \to \mathbf{E} : f(\bar{p}g) = \chi(\bar{p}) \cdot f(g) \text{ for } \bar{p} \in \bar{P},\ g \in G\},$$

where G acts by right translation denoted by $f^g := f(\cdot g)$.

2.2. Smooth, algebraic and locally algebraic

Our matrix group G has two addition structures:

- Its totally disconnected topology stemming from its coefficient field F, and
- its structure as an algebraic variety.

This gives rise to corresponding types of representations of G, and the locally algebraic one makes use of both of these:

Definition

Let V be a G-representation. A vector v in V is **smooth/algebraic/locally algebraic** if its orbit map

$$o_v \colon G \to V$$
$$g \mapsto g \cdot v$$

is a **locally constant/rational/locally rational** map.

We note that o_v is **locally rational** if there is an open subgroup G_0 of G and a finite-dimensional vector space V_0 of V such that $o_v \colon G_0 \to V_0$ is given by (the restriction of the rational points of) a rational function on G. (That is, the G-action is locally given by rational functions in the coordinates of G.)

Definition

Let $i(\chi) = \mathrm{ind}_{\bar{P}}^G \chi$ be the abstract induction of Example Example 2.1. We denote by $i(\chi)^{\mathrm{lc}}$ / $i(\chi)^{\mathrm{alg}}$ / $i(\chi)^{\mathrm{la}}$ the G-subrepresentation given by all *smooth* / *algebraic* / *locally algebraic* vectors in $i(\chi)$.

Since G operates by translation on the functions $f \colon G \to \mathbf{E}$ in $i(\chi)$, we obtain that, inside $i(\chi)$ the subrepresentation

- $i(\chi)^{\mathrm{lc}}$ consists of all locally constant functions,
- $i(\chi)^{\mathrm{alg}}$ of all *rational* functions, that is, polynomial functions in the coordinates $\{X_{ij} : i, j = 1, \dots, n\}$ and determinant $\det(X_{11}, \dots, X_{nn})$ of G, and
- $i(\chi)^{\mathrm{la}}$ of all functions on G which restrict locally to a rational function.

2.3. Our locally algebraic representation of interest

- A character $\theta \colon T \to \mathbf{E}^*$ is **unramified** if it is trivial on $T_0 \subseteq T$ (and thus factors through the quotient $T/T_0 = \mathbb{Z}^n$).
- A character $\psi \colon T \to \mathbf{F}^*$ is **dominant** if, letting $t_1, \dots, t_n$ denote the entries of $t \in T$, it is given by $(t_1, \dots, t_n) \mapsto t_1^{i_1} \cdots t_n^{i_n}$ with integers $i_1 \leq \ldots \leq i_n$.

Example (The representation of Conjecture 1.1)

Let $\chi = \theta\psi$ be the product of a nonramified and a dominant character. The induced locally algebraic representation V of Conjecture 1.1 is given by

$$i(\chi) = \mathrm{ind}_{\bar{P}}^G \chi^{\mathrm{la}}.$$

2.4. A simple P-subrepresentation of $i(\chi)$

We have for a function in $i(\chi)$ a well-defined notion of its support in $\mathcal{F} := \bar{P}\backslash G$. Consider the inclusion of the **open cell** $N \hookrightarrow \mathcal{F}$ (with dense image), and put

$$i(\chi)(N) := \{f \colon G \to \mathbf{E} \text{ in } i(\chi) \text{ with support in } N\}.$$

Proposition

By restriction $f \mapsto f_{|N}$ we obtain an injection of P-representations

$$i(\chi)(N) \hookrightarrow \mathscr{C}^{lp}_{cpt}(N, \mathbf{E}) := \{f \colon N \to \mathbf{E} : \text{ loc. pol. of compact support }\},$$

where the group P acts on $\mathscr{C}^{lp}_{cpt}(N, \mathbf{E})$ by

$$f^p = \chi(p) f(\cdot^t n) \quad \text{for } p = tn \in P \text{ with } t \in T,\ n \in N.$$

We denote its image by $\mathscr{C}^{\psi-lp}(N, \mathbf{E})$, so $i(\chi)(N) \xrightarrow{\sim} \mathscr{C}^{\psi-lp}(N, \mathbf{E})$.

3. The unitary norm of our locally algebraic rep.

Recall that Conjecture 1.1 looks out for a G-invariant norm $\|\cdot\|$ on $V = i(\chi)$. Such a seminorm can equivalently be described by its unit ball $\mathfrak{L} = \{v \in V : \|v\| \leq 1\}$, which is a *lattice*.

Definition

Let $\mathbf{o} = \mathbf{o_E}$ be the ring of integers of the p-adic number field $\mathbf{E}$. A **lattice** in an $\mathbf{E}$-vector space V is an $\mathbf{o}$-module $\mathfrak{L} \subseteq V$ such that $\mathbf{K} \cdot \mathfrak{L} = \{\lambda \cdot v : \lambda \in \mathbf{E}, v \in \mathfrak{L}\} = V$.

(Conversely, a lattice $\mathfrak{L} \subseteq V$ induces the semi-norm $\|v\| := \inf\{|\lambda| \in |\mathbf{E}^*| : v \in \lambda\mathfrak{L}\}$ on V.)
Among all the unitary semi-norms on V, there is one such with smallest unit ball. It is given by the following lattice $\mathfrak{L}_u$ which gives rise to $\hat{V}$ by completion. (Recall that we are identifying equivalent norms or, equivalently, lattices with finite index in each other.)

Observation (Valid for a general algebraic group G)

We observe V to be a finitely generated $\mathbf{E}[G]$-module. Then the Banach space $\hat{V}$ is the completion with respect to any semi-norm $\|\cdot\|_{\mathfrak{L}_u}$ whose unit ball is given by the lattice $\mathfrak{L}_u$ which is finitely generated as $\mathbf{o_E}[G]$-module. We call $\mathfrak{L}_u$ the **universal unitary lattice**.

In what follows, we want to show that $\mathfrak{L}_u$ is actually a free $\mathbf{o}$-module, so that it comes from a unitary norm $\|\cdot\|$ on V. We only arrive at the partial answer Theorem 3.1 though.

3.1. Restriction to P

Let K be a compact group, V a continuous K-representation and $\|\cdot\|_0$ some norm on V. Then the norm $\|\cdot\|$ on V given by $\|v\| = \max\{\|v^g\|_0 : g \in G\}$ is K-invariant. This makes plausible that the shape of $\mathfrak{L}_u$ only depends on the noncompact part of G.

Proposition 3.1

The lattice $\mathfrak{L}_u \subseteq V$ is given by any lattice finitely generated as an $\mathbf{o_E}[P]$-module.

This follows essentially from the *Iwasawa decomposition* $G = KP$ with $K = \mathrm{GL}_n(\mathbf{o_F})$ a maximal compact subgroup of G. (See [Nag14, Corollary 3.3])

3.2. Gluing the open cells

Proposition (Intertwining operators (valid for *regular* θ))

There are finitely many unramified characters $\{\theta_w : w \in W = N_G(T)/T\}$ and isomorphisms

$$T_w \colon i(\chi_w) \to i(\chi) \quad \text{with } \chi_w = \theta_w\chi \quad \text{for all } w \in W.$$

The intertwining operators T_w allow us to glue $i(\chi)$ from the open cell N. More precisely:

Proposition 3.2 (Gluing via the Intertwining operators)

The universal unitary lattice $\mathfrak{L}_u \subseteq i(\chi)$ is of the form

$$\mathfrak{L}_u = \sum_{w \in W} \mathfrak{L}_w$$

where $\mathfrak{L}_w = T_w(\mathfrak{L})$ with $\mathfrak{L}$ the universal unitary lattice of the P-representation $i(\chi_w)(N)$.

⇒ Let us study the universal unitary lattice $\mathfrak{L}_u$ of the P-representation $i(\chi)(N)$.

3.3. The norm of differentiable functions

We want to show the lattice $\mathfrak{L}_u \subseteq i(\chi)(N)$ to be free. It can be shown to be generated by a single element as $\mathbf{o_E}[P]$-module. This explicit description allows us to establish:

Proposition 3.3

*Let T^+ be the **dominant submonoid** of T given by $T^+ = \{t \in T : {}^t N_0 \subseteq N_0\}$. Then the lattice $\mathfrak{L} \subseteq i(\chi)(N)$ is free if and only if $|\chi(t)| \leq 1$ for all $t \in T^+$.*

Together with Proposition 3.2 we obtain the following corollary, our main result.

Theorem 3.1 (Partial answer to the freeness of the universal unitary lattice $\mathfrak{L}_u$ of $i(\chi)$)

Suppose that for all $w \in W$, we have $|\chi_w(t)| \leq 1$ for all $t \in T^+$. Then the universal unitary lattice $\mathfrak{L}_u$ of the G-representation $i(\chi)$ is of the form

$$\mathfrak{L}_u = \sum_{w \in W} \mathfrak{L}_w \quad \text{with free } \mathbf{o_E}\text{-modules } \mathfrak{L}_w.$$

We will sketch how to prove the sufficiency of the conditions given in Proposition 3.3. To this end, we revert back to the standpoint of norms instead of lattices. More precisely, we use the following observation to prove freeness of $\mathfrak{L}_u \subseteq i(\chi)(N)$. It rests on the observation that in the non-Archimedean case, any Banach space possesses an orthonormal base.

Observation

A lattice $\mathfrak{L} \subseteq V$ is *free* if and only if there is a norm $\|\cdot\|$ on V such that $\|v\| \leq \|v\|_{\mathfrak{L}}$.

Assuming that the necessary conditions in Proposition 3.3 hold, we establish such a norm $\|\cdot\|$ for $\mathfrak{L}_u \subseteq i(\chi)(N) \xrightarrow{\sim} \mathscr{C}^{\psi-lp}(N, \mathbf{E})$ in the exemplary case $G = \mathrm{GL}_2(\mathbf{F})$ (when $\mathbb{N} = \mathbf{F}$).

Observation

One checks the lattice $\mathfrak{L}_u \subseteq \mathscr{C}^{\psi-lp}(\mathbf{F}, \mathbf{E})$ to only depend on the absolute value $|\chi| \colon T \to \mathbb{R}_{>0}$ which is seen to factor over $T \to T/T_0 Z = \mathbb{Z}$. Let $t_\alpha \in T$ map to $1 \in \mathbb{Z}$. Thence $\mathfrak{L}_u$ is completely determined by

$$r = v(\chi(t_\alpha)) \in \mathbb{R}_{\geq 0}$$

with $v \colon \mathbf{E} \to \mathbb{R} \cup \{\infty\}$ the normalized (that is, $v(p) = 1$) valuation of $\mathbf{F}$.

Given this parameter $r \in \mathbb{R}_{\geq 0}$, we will now construct the sought for norm $\|\cdot\| \leq \|\cdot\|_{\mathfrak{L}_u}$. It will be the one attached to *r-times differentiable functions*. We will make this explicit for $r = 1$.

Definition

Let $f \in \mathscr{C}^{lp}(\mathbf{F}, \mathbf{E})$. We define a function $f^{]1[}$ by

$$f^{]1[}(x, y) = \frac{f(x) - f(y)}{x - y} \quad \text{for all distinct } x, y \in \mathbf{F}.$$

The norm $\|\cdot\|_{\mathscr{C}^1}$ is then defined by $\mathscr{C}^{lp}(\mathbf{F}, \mathbf{E})$ by $\|f\|_{\mathscr{C}^1} = \|f^{]1[}\|_{\sup}$.

Theorem

The norm $\|\cdot\| = \|\cdot\|_{\mathscr{C}^1}$ satisfies $\|\cdot\| \leq \|\cdot\|_{\mathfrak{L}_u}$ on $\mathscr{C}^{\psi-lp}(N, \mathbf{E}) \subseteq \mathscr{C}^{lp}(N, \mathbf{E})$.

For general split reductive group $G = \mathbf{GF}$) and $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$, see [Nag14, Section 3].

Literature

- C. Breuil and P. Schneider, *First steps towards p-adic Langlands functoriality*, J. Reine Angew. Math. **610** (2007), 149–180. MR 2359853. DOI 10.1515/CRELLE.2007.070.
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