non-Archimedean Taylor polynomials

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Enno Nagel
http://www.math.jussieu.fr/~nagel

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1 Iterated Differentials

- One-variable functions
- Many-variable functions

Let us fix a *non-Archimedean* field $\mathbf{K}$, that is a field with a real-valued valuation which is

- non-Archimedean,
- complete, and
- non-trivial.

One-variable functions

A priori, we can formulate the definition of differentiability over a general topological field $\mathbf{K}$ like we do over $\mathbb{R}$.

Definition

A function $f : \mathrm{X} \to \mathbf{K}$ over an open subset X of $\mathbf{K}$ is *differentiable* at a in X if there is $f'(a)$ in $\mathbf{K}$ such that

$$\frac{f(x_n) - f(a)}{x_n - a} \to f'(a)$$

for every sequence (x_n) in X such that $x_n \to a$.

The completeness and non-discreteness of $\mathbf{K}$ ensure that the *derivative* $f'(a)$ always exists in $\mathbf{K}$ and is unique.

However, as $\mathbb{K}$ is totally disconnected, there is *no counterpart* of

- the Intermediate Value Theorem, or of
- the Mean Value Theorem.

Thus we run into various *pathologies*.

- The differentiable functions with their natural norm form no longer a Banach space,
- there is a function with everywhere invertible derivative that is nowhere locally invertible, and
- the function

$$f : \mathbb{Z}_p \to \mathbb{Q}_p$$
$$\sum_{n \in \mathbb{N}} a_n p^n \mapsto \sum_{n \in \mathbb{N}} a_n p^{2n}$$

  - is injective, but its derivative is zero everywhere, and
  - infinitely often Archimedeanly differentiable, but its Taylor polynomial of degree greater than 1 does not converge.

[Glö07] and [Yan04] used the analogue of f to refute other differentiability notions in positive characteristic.

To exclude these pathologies, we strengthen the differentiability definition.

Definition

A function $f : \mathrm{X} \to \mathbf{K}$ over an open subset X of $\mathbf{K}$ is continuously differentiable if its *divided difference*

$$f^{]1[}(x, y) = \frac{f(x) - f(y)}{x - y}$$

defined for all distinct x and y in X, extends to a continuous function over all of $\mathrm{X} \times \mathrm{X}$.

For a real-valued function over the real numbers, the mean-value theorem shows that the non-Archimedean and Archimedean differentiability condition are equivalent.

Non-Archimedean differentiability is more natural than Archimedean differentiability in the following sense:

If we consider common facts such as

- the local invertibility around a point in which the derivative is invertible
- the existence of the Taylor polynomial, and
- the completeness of the normed space of differentiable functions

then

- with the non-Archimedean differentiability definition, they follow *straight from the definition*, whereas
- with the Archimedean differentiability definition, they are proved by a *detour* either the mean-value theorem or via the fundamental theorem of calculus.

Many-variable functions

Let V and $\mathbf{E}$ be two $\mathbf{K}$-Banach spaces, X an open subset of V.

Definition

The function $f : X \to \mathbf{E}$ is $\mathscr{C}^1$ at a in X if there is a continuous $\mathbf{K}$-linear map $A : V \to \mathbf{E}$ such that for every $\epsilon > 0$, there is a neighborhood U around a inside of X such that for all $x + h, x$ in U holds

$$\|f(x+h) - f(x) - A(h)\| \leq \epsilon \|h\|.$$

This non-Archimedean differentiability is stricter than Archimedean differentiability, because

- In *non-Archimedean* differentiability the offset h and the expansion point x varies, whereas
- in *Archimedean* differentiability h varies but x is fixed.

To iterate non-Archimedean differentiability, we introduce coordinates on V by choosing an ordered basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$ of V.

Definition

The *divided difference* $f^{]1[}(x+h, x)$ of f at $x+h, x$ in X with $h \in \mathbf{K}^{*d}$ is the $\mathbf{K}$-linear map A with k-th column vector

$$\frac{f(h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1} + h_k\boldsymbol{e}_k) - f(h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1})}{h_k}$$

for each $k = 1, \ldots, d$.

The function f is a $\mathscr{C}^1$*-function* if $f^{]1[}$ extends to a continuous function $f^{[1]} \colon \mathrm{X} \times \mathrm{X} \to \mathbf{E}$.

Let us compare domain and codomain of $f^{[1]}$ with that of f.

- The domain $X^{[1]} := X \times X$ of $f^{[1]}$ is included in the finite-dimensional $\mathbf{K}$-vector space $V^{[1]} = V \times V$ with a canonical ordered basis, like the domain X of f, and
- the codomain $\mathbf{E}^{[1]} := \mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$ of $f^{[1]}$ is a $\mathbf{K}$-Banach space, like the codomain $\mathbf{E}$ of f.

We can therefore iterate the non-Archimedean differentiability definition by applying it to $f^{[1]}$ instead of f.

Definition

The function $f : X \to \mathbf{E}$ is a $\mathscr{C}^2$-function if, firstly $f^{[1]}$ exists, and secondly its divided difference

$$f^{]2[} = (f^{[1]})^{]1[} : (X^{[1]})^{]1[} \to (\mathbf{E}^{[1]})^{[1]}$$

extends to a continuous function $f^{[2]}$ over $X^{[2]} := (X^{[1]})^{[1]} = (X \times X) \times (X \times X)$ that has values in $\mathbf{E}^{[2]} := (\mathbf{E}^{[1]})^{[1]}) = \mathrm{Hom}_{\mathbf{K}}(V \times V, \mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E}))$.

This definition is conceptually sound but practically cumbersome, because already for a one-variable function f

- ν-fold non-Archimedean differentiability is checked by the divided difference $f^{[\nu]}$, a function of 2^ν variables, whereas
- ν-fold Archimedean differentiability is checked by the derivative $f^{(\nu)}$, a function of 1 variable.

Let us ease our cumber.

Observation (Schikhof)

The divided difference $f^{[1]}$ over $\mathrm{X} \times \mathrm{X}$ of f is symmetric in both arguments. This symmetry is passed on to its iterations $f^{[\nu]}$.

Definition (Schikhof)

Let X be an open subset of $\mathbf{K}$. For $\nu \in \mathbb{N}$ put

$$X^{[\nu]} = X^{\{0,\dots,\nu\}} \text{ and } X^{]\nu[} := \{(x_0, \dots, x_\nu) \in X^{[\nu]} \mid \text{diff. entries}\}.$$

The ν-*th divided difference* $f^{]\nu[} : X^{]\nu[} \to \mathbf{E}$ of a function $f : X \to \mathbf{E}$ is inductively given by $f^{]0[} = f$ and for $\nu \in \mathbb{N}$ by

$$f^{]\nu[}(x_0, \dots, x_\nu) = \frac{f^{]\nu-1[}(x_0, x_2, \dots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \dots, x_\nu)}{x_0 - x_1}$$

and f is ν-*times differentiable* if $f^{]\nu[}$ extends to a continuous function $f^{[\nu]}$ on all of $X^{[\nu]}$.

This is a function in $\nu + 1$ arguments instead of 2^ν arguments.

Let us look at other descriptions of non-Archimedean differentiability.

2 Mahler Basis

- Construction
- Differentiability

Mahler Basis

Let $\binom{\cdot}{n}\colon \mathbb{Z}_p \to \mathbf{K}$ be the *n-th Mahler polynomial* defined by

$$\binom{x}{n} = x(x-1)\cdots(x-n+1)/n!.$$

Theorem

There is an isomorphism of normed $\mathbf{K}$-vector spaces

$$\{\text{ all zero sequences in } \mathbf{K}\ \} \to \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$$
$$(a_n) \mapsto \sum a_n \binom{\cdot}{n}$$

Given a continuous function $f\colon \mathbb{Z}_p \to \mathbf{K}$, we call (a_n) the *Mahler coefficients* of f.

That is, $f : \mathbb{Z}_p \to \mathbf{K}$ is continuous if and only if there is a zero sequence $(a_n : n \in \mathbb{N})$ in $\mathbf{K}$ such that $f = \sum_{n \in \mathbb{N}} a_n \binom{\cdot}{n}$.

Proof.

Let $\mathbf{o_K} := \{x \in \mathbf{K} : |x| \leq 1\}$ be the closed unit ball of $\mathbf{K}$. Let

$$\mathscr{D}^0(\mathbb{Z}_p, \mathbf{o_K}) = \{ \text{ all continuous linear } \nu\colon \ \mathscr{C}^0(\mathbb{Z}_p, \mathbf{o_K}) \to \mathbf{o_K} \}.$$

- By density of the loc. const. fct's inside of the cts. fct's

$$\mathscr{D}^0(\mathbb{Z}_p, \mathbf{o_K}) \xrightarrow{\sim} \varprojlim \mathbf{o_K}[\mathbb{Z}/p^n\mathbb{Z}] =: \mathbf{o_K}[[\mathbb{Z}_p]],$$

- by the *Iwasawa isomorphism* that maps the generator 1 of $\mathbb{Z}_p$ to $1 + X$ $$\mathbf{o_K}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathbf{o_K}[[X]].$$

We obtain by composition

$$\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \{ \text{ all bounded sequences in } \mathbf{K} \}.$$

The Mahler basis isomorphism is the *dual* isomorphism. □

Theorem

The function $f : \mathbb{Z}_p^d \to \mathbf{K}$ is ν-times differentiable if and only if its Mahler coefficients $(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$ fulfill $|a_{\boldsymbol{n}}||\boldsymbol{n}|^{\nu} \to 0$ as $|n| \to \infty$.

3 Taylor Polynomial

- One variabe
- Many variables over $\mathbb{Q}_p$
- Representation Theory

One variabe

Theorem (Schikhof)

A function $f : X \to \mathbf{K}$ *is* ν*-times differentiable* if *and only if its* Taylor polynomial converges, *that is, there are functions* $D^i f : X \to \mathbf{K}$ *for* $i = 0, \ldots, \nu - 1$ *and a function* $R^\nu f : X \times X \to \mathbf{K}$ *such that*

$$f(x+h) = \sum_{i=0,\ldots,\nu} D^i f(x) h^i + R^\nu f(x+h, x) h^\nu \text{ for all } x+y, x \text{ in } X,$$

and $R^\nu f$ *vanishes continuously on the diagonal.*

The i-th *derivative* $D^i f$ is a uniquely determined $\mathscr{C}^{\nu-i}$-function given by $D^i f(x) = f^{[i]}(x, \ldots, x)$, and satisfies $i!\, D^i f = f^{(i)}$.

The remainder $R^\nu f$ is a function of two arguments.

If X is a subset of $\mathbf{K}$ without isolated points but not necessarily open, then the convergence of Talor polynomials

- implies ν-fold differentiability for $\nu \leq 2$, but
- *fails*, that is, there are counterexamples for $\nu > 2$.

However, it remains true for the following stronger Taylor polynomial convergence.

Definition

The *derived Taylor polynomials of $f : X \to \mathbf{K}$ converge* if there are functions $D^0 f, \ldots, D^\nu f : X \to \mathbf{K}$ such that for every $n = 0, \ldots, \nu$ the Taylor polynomial of $F = D^n f$ converges with $D^0 F = \binom{n}{n} D^n f, D^1 F = \binom{n+1}{n} D^{n+1} f, \ldots, D^{\nu-n} F = \binom{\nu}{n} D^\nu f$.

The divided difference $f^{[\nu]}$ of f is locally a bounded *linear combination*

- of the derivatives $D^0 f, \ldots, D^\nu f$, and
- their Taylor polynomials $R^\nu D^0 f, \ldots, R^0 D^\nu f$

Many variables over $\mathbb{Q}_p$

Theorem

Let X *be an open subset of* $\mathbb{Q}_p^d$*. The function* $f : \mathrm{X} \to \mathbf{K}$ *is* ν*-times differentiable* if *and only if there are*

- *continuous functions* $\mathrm{D}^{\boldsymbol{n}} f : \mathrm{X} \to \mathbf{K}$ *for* $\boldsymbol{n}$ *in* $\mathbb{N}^d$ *with* $n_1 + \cdots + n_d = 0, \ldots, \nu$*, and*
- *a remainder* $\mathrm{R}^\nu f : \mathrm{X} \times \mathrm{X} \to \mathbf{K}$ *such that*

$$f(x+h) = \sum_{n_1+\cdots+n_d=0,\ldots,\nu} \mathrm{D}^{\boldsymbol{n}} f(x) h^{\boldsymbol{n}} + \mathrm{R}^\nu f(x,h)$$

and for every a *in* X *and every* $\varepsilon > 0$*, there is a neighborhood* U *inside of* X *around* a *such that*

$$|\mathrm{R}^\nu f(x,h)| \le \varepsilon |h|^\nu \quad \textit{for all } x+h, x \textit{ in } \mathrm{U}.$$

In fact a stronger variant holds true, and follows from the description in terms of the Mahler basis.

Definition

The function $f : X \to \mathbf{K}$ is ν-times differentiable *along the first coordinate* if there are

- continuous functions $D^{(n,0,\ldots,0)} f : X \to \mathbf{K}$ for $n = 0, \ldots, \nu$, and
- a remainder $R^\nu f : X \times X \to \mathbf{K}$ such that

$$f(x+h) = \sum_{n=0,\ldots,\nu} D^{(n,0,\ldots,0)} f(x) h^n + R^\nu f(x,h) h^\nu$$

and $R^\nu f$ vanishes continuously over the diagonal.

Theorem

The function $f : X \to \mathbf{K}$ is ν-times differentiable if it is differentiable along each coordinate.

Proof.

A function over the unit ball $\mathbb{Z}_p^d$ is ν-times differentiable if and only if its Mahler coefficients $(a_{\boldsymbol{n}})$ satisfy

- $a_{\boldsymbol{n}}(n_1 + \cdots + n_d)^\nu \to 0$ as $n_1 + \cdots + n_d \to \infty$, *if and only if*
- $|a_{\boldsymbol{n}}|n_1^\nu, \ldots, |a_{\boldsymbol{n}}|n_d^\nu \to 0$ as $n_1 + \cdots + n_d \to \infty$, *if and only if*
- the function f converges along each coordinate.

$\square$

Representation Theory

Let $\mathbf{F}$ be a p-adic number field with ring of integers $\mathbf{o}$.

In Representation Theory of p-adic Lie groups such as $\mathrm{GL}_2(\mathbf{F})$, there appear $\mathscr{C}^r$-functions $f : \mathbf{o} \to \mathbf{K}$ for a real number $r \geq 0$.

Let's see how they come about.

The most natural method to construct representations of $G = GL_n(\mathbf{F})$ is by

- taking a representation $\chi = \chi_1 \otimes \cdots \otimes \chi_d$ of smaller copies $M = GL_{n_1}(\mathbf{F}) \times \cdots \times GL_{n_d}(\mathbf{F})$ with $n_1 + \cdots + n_d = n$, and
- building the tensor product $\mathrm{ind}_M^G \chi := \mathbf{K}[G] \otimes_{\mathbf{K}[M]} \chi$.

Example

Let $G = GL_2(\mathbf{F})$ and $M = \mathbf{F}^* \times \mathbf{F}^*$. Let $\chi \colon M \to \mathbf{K}^*$ and note that it extends trivially onto the triangular matrices P. Then

$$\operatorname{ind}_P^G \chi = \{f : G \to \mathbf{K} \mid f(p\cdot) = \chi(p)f \text{ for all } p \text{ in } P\}$$

and G acts by right-translation, that is, $g \cdot f = f(\cdot g)$ for all g in G.

Let

$$i(\chi) = \operatorname{ind}_P^G \chi^{\mathrm{lr}} := \{ \text{ all locally rational functions in } \operatorname{ind}_P^G \chi\}$$

Question

Is there a norm $\|\cdot\|$ on $i(\chi)$ *invariant* under the action by G?

The Breuil-Schneider conjecture predicts for which characters χ there is such a norm. Let $m = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and put

$$r := v_{\mathbf{F}}(\chi(m)) \geq 0$$

The completion of $i(\chi)$ is a quotient of two copies of the following function space.

We assume that $\mathbf{K}$ contains the normal closure of $\mathbf{F}$, and let S denote all embeddings $\mathbf{F} \hookrightarrow \mathbf{K}$. For $\boldsymbol{n} \in \mathbb{N}^S$, we put

- $n = \sum_{s \in S} n_s$ and,
- given $z \in \mathbf{o}$, we put $z^{\boldsymbol{n}} = \prod_{s \in S} s(z)^{n_s}$.

Definition (Colmez & de Ieso, [DI13])

The function $f : \mathbf{o} \to \mathbf{K}$ is a $\mathscr{C}^r_{\mathrm{T}}$-*function* if there are bounded functions $\mathrm{D}^{\boldsymbol{i}} f : \mathbf{o} \to \mathbf{K}$ for $\boldsymbol{n} \in \mathbb{N}^S$ and $\mathrm{R}_\nu f : \mathbf{o} \times \mathbf{o} \to \mathbf{K}$ such that

$$f(x+h) = \sum_{\boldsymbol{n} \in \mathbb{N}^S \text{ with } n \leq r} \mathrm{D}^{\boldsymbol{n}} f(y) h^{\boldsymbol{n}} + \mathrm{R}^\nu f(x+h, x),$$

and

$$\Delta^r f(\delta) := \sup_{x_0 \in \mathbf{o}} \sup_{|h| \leq \delta} \frac{|\mathrm{R}^\nu f(x_0 + h, x_0)|}{\delta^r}$$

is a well-defined function $\Delta^r f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ which converges to 0 as δ does.

Question (de Ieso, [DI13])

How does the differentiability condition by de Ieso compare with our differentiability condition?

Let us decompose

$$r = \nu + \rho$$

into an

- *integral* part $\nu = \lfloor r \rfloor \in \mathbb{N}$, and a
- *fractional* part $\rho = \{r\} \in [0, 1[$.

ρ-fold differentiability for $\rho \in [0, 1[$

Let $\rho \in [0, 1[$. Roughly, ρ-fold differentiability is stricter Hölder-continuity.

Definition

Let V be a finite-dimensional $\mathbf{K}$-vector space, let $\mathbf{E}$ be a $\mathbf{K}$-Banach space and X an open subset of V.

The function $f : X \to \mathbf{E}$ is $\mathscr{C}^\rho$ at a in X if for every $\varepsilon > 0$ there is a neighborhood U around a inside of X such that

$$\|f(x) - f(y)\| \le \varepsilon \cdot \|x - y\|^\rho \quad \text{for all } x, y \in U \cap A.$$

The function f is a $\mathscr{C}^\rho$*-function* if f is $\mathscr{C}^\rho$ at all points $a \in A$.

Recall ν-fold differentiability via iterated divided differences:

Definition

The *divided difference* $f^{]1[}(x+h,x)$ of f at $x+h, x$ in X with $h \in \mathbf{K}^{*d}$ is the $\mathbf{K}$-linear map A with k-th column vector

$$\frac{f(h_1 \boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1} + h_k \boldsymbol{e}_k) - f(h_1 \boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1})}{h_k}$$

for each $k = 1, \ldots, d$. The function f is a $\mathscr{C}^1$-*function* if $f^{]1[}$ extends to a continuous function $f^{[1]} \colon \mathrm{X} \times \mathrm{X} \to \mathbf{E}$.

Definition

The function $f \colon \mathrm{X} \to \mathbf{E}$ is a $\mathscr{C}^2$-function if, firstly $\boldsymbol{f} = f^{[1]}$ over $\mathbf{X} = \mathrm{X} \times \mathrm{X}$ exists, and secondly its differential $f^{]2[} = \boldsymbol{f}^{]1[}$ extends to a continuous function $f^{[2]}$ over $\mathrm{X}^{[2]} := \mathbf{X}^{[1]}$.

r-fold differentiability for $r \in \mathbb{R}_{\geq 0}$ ...

...via iterated divided differences:

Definition

Let V be a finite-dimensional $\mathbf{K}$-vector space, let $\mathbf{E}$ be a $\mathbf{K}$-Banach space and X an open subset of V.

The function $f : \mathrm{X} \to \mathbf{E}$ is *r-times differentiable* if f is a $\mathscr{C}^{\nu}$-function, and $f^{[\nu]}$ is a $\mathscr{C}^{\rho}$ function.

...via the Mahler basis:

Theorem

The function $f : \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \to \mathbf{K}$ is r-times differentiable if and only if *its Mahler coefficients $(a_n)_{n \in \mathbb{N}}$ fulfill $|a_n| n^r \to 0$ as $n \to \infty$.*

Definition (Colmez & de Ieso, [DI13])

The function $f : \mathbf{o} \to \mathbf{K}$ is a $\mathscr{C}^r_{\mathrm{T}}$-*function* if there are bounded functions $\mathrm{D}^{\boldsymbol{i}} f : \mathbf{o} \to \mathbf{K}$ for $\boldsymbol{n} \in \mathbb{N}^S$ and $\mathrm{R}_\nu f : \mathbf{o} \times \mathbf{o} \to \mathbf{K}$ such that

$$f(x+h) = \sum_{\boldsymbol{n} \in \mathbb{N}^S \text{ with } n \leq r} \mathrm{D}^{\boldsymbol{n}} f(y) h^{\boldsymbol{n}} + \mathrm{R}^\nu f(x+h, x),$$

and

$$\Delta^r f(\delta) := \sup_{x_0 \in \mathbf{o}} \sup_{|h| \leq \delta} \frac{|\mathrm{R}^\nu f(x_0 + h, x_0)|}{\delta^r}$$

is a well-defined function $\Delta^r f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ which converges to 0 as δ does.

Theorem ([Nag14a])

The one-variable function $f : \mathbf{o} \to \mathbf{K}$ is a $\mathscr{C}^r_{\mathrm{T}}$-function if and only if *it is r-times differentiable as many-variable function on* $\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$.

1. Iterated Differentials
2. Mahler Basis
3. Taylor Polynomial

The survey on fractional p-adic Calculus [Nag14b] is available at

`http://imj-prg.fr/~enno.nagel`.

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