Mod p Principal Series and injective envelopes of $\mathrm{GL}_2(\mathbf{F}_q)$

Enno Nagel

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Contents

0 Prerequisites

Let p be a prime and $q = p^f$ for some $f \in \mathbb{N}$. Fix an embedding $\mathbf{F}_q \hookrightarrow \mathbf{E} := \overline{\mathbf{F}}_p$.

Let $\mathrm{G} = \mathrm{GL}_2(\mathbf{F}_q)$ and $\mathrm{U} = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\} \subset \mathrm{B} = \{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)\} \subset \mathrm{G}$ its unipotent and Borel subgroup. We define the flip matrix $s := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A *weight* will be an irreducible representation of G over $\mathbf{E}$. For $n \in \mathbb{Z}_{\geq 0}$, we denote the reduplicating tuple $\boldsymbol{n} = (n, \ldots, n)$.

1 The principal series of $\mathrm{GL}_2(\mathbf{F}_q)$

Let $\chi \colon \mathrm{B} \to \mathbf{E}^*$ be a character. Since U is the commutator of B, it factors over $\mathrm{B} \twoheadrightarrow \mathrm{T}$. Therefore it is of the form $\chi : \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$. Since $\mathbf{F}_q^* = \mathbb{Z}/ < q-1 >$ as group, we have $\chi = \cdot^r$ for a unique $r \in \{0, \ldots, q-2\}$. Therefore χ can be written

$$\chi : \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto a^r (ad)^m \quad \text{for } r, m \in \{0, \ldots, q-2\},$$

that is, up to a determinant twist the characters $\chi\colon \mathrm{B} \to \mathbf{E}^*$ are determined by the number $r = 0, \ldots, q-2$.

If $\mathrm{H} \subset \mathrm{G}$ are groups and σ an $\mathbf{E}$-representation of H with underlying vector space V, recall that the *induced representation* $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} \sigma$ is defined by

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} \sigma = \oplus_{\bar{g} \in \mathrm{G}/\mathrm{H}} \mathrm{V}^{\bar{g}} = \{f\colon \mathrm{G} \to \mathrm{V} | f(hg) = \sigma(h) \cdot f(g) \text{ for } h \in \mathrm{H}, g \in \mathrm{G}\}$$

with G-action by $f^g := f(\cdot g)$.

Lemma 1.1. *It holds* $\dim \operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi = q + 1$ *for every character* $\chi\colon \mathrm{B} \to \mathbf{E}^*$.

Proof: If $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $[v] \in \mathbb{P}^1(\mathbf{F}_q)$ the spanned line of an element $v \in \mathbf{F}_q^2$, then

$$\mathrm{G}/\mathrm{B} = \mathrm{G}/\mathrm{Fix}_{[e_1]} = \mathrm{G} \cdot [e_1] = \mathbb{P}^1(\mathbf{F}_q).$$

By its classification, the weights of G have maximal dimension q. Therefore $\mathrm{I} := \operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi$ has to have proper G-stable subspaces. Since $p = \operatorname{char} \mathbf{E}$ divides #G, unfortunately I is in general not semisimple. But as $\mathbf{E}$ is a field and G finite, we find E[G] to be Artinian. Hence every finite E[G] module such as I is Artinian and therefore has a decomposition series.

Notice that $\operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \det^m \otimes \chi = \det^m \otimes \operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi$. Hence we are reduced to the case of χ acting in the first coordinate, determined by the power $r \in \{0, \ldots, q-2\}$ it takes elements to.

Therefore fix a character $\chi : \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto a^r$ for $r \in \{0, \ldots, q-2\}$. We had the following recipe to obtain all the (distinct) Jordan Hoelder factors of $\operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi$:
For a subset $\mathrm{J} \subset \{0, \ldots, f-1\} = \mathbb{Z}/ < f >$, *there corresponds a unique f-tuple* λ *in*

$$\prod_{i=0,\ldots,f-1} \{x_i, x_i - 1, p-1-x_i, p-1-x_i-1\} \text{ by } i \in \mathrm{J} \Leftrightarrow \lambda_i \in \{p-1-x_i, p-2-x_i\}.$$

P-adically expanding $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$ with $r_0, r_1, \ldots, r_{f-1} \in \{0, \ldots, p-1\}$, we plugged $\boldsymbol{r} = (r_0, \ldots, r_{f-1})$ into λ to obtain the weight of G uniquely described by the f-parameters $\lambda(\boldsymbol{r})$ and twist it by $\det^e$ for a power $e = e(\lambda(\boldsymbol{r}))$ determined by a formula in $\lambda(\boldsymbol{r})$.

Theorem 1.2 (F. Diamond). *The rule*

$$\{0, \ldots, f-1\} \supset \mathrm{J} \mapsto \lambda(\boldsymbol{r}) \otimes \det^{e(\lambda(\boldsymbol{r}))} = \textit{weight of}\, \mathrm{G}$$

yields all the multiplicity free JH-factors of $\operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi$.

Proof: By a Brauer character computation. □

Remark 1.3. We let $\chi^s := \chi(\cdot^s) = \chi(s \cdot s^{-1})$, the character given by $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto d^r = a^{q-1-r} \cdot (ad)^r$. Then χ^s equals $\chi^* \otimes \det^r$, where $\chi^* \leftrightarrow q-1-r$. By the above recipe, we obtain the same JH-factors of $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi$ and $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi^s$.

We refine this result by exhibiting the semisimple graded pieces of the socle and co-socle filtrations of I and its submodules. Recall the definition of the socle and co-socle filtrations of an E[G] module R.

Definition 1.4. We define the *Socle filtration* by

$$\mathrm{Soc}_0 \mathrm{R} := 0 \text{ and } \mathrm{Soc}_{i+1} \mathrm{R} := \text{the max. semisimple submodule mod. } \mathrm{Soc}_i \mathrm{R},$$

the *Co-Socle filtration* by

$$\mathrm{Cos}_0 \mathrm{R} := \mathrm{R} \text{ and } \mathrm{Cos}_{i+1} \mathrm{R} := \text{the min. semi-simple submodule mod. } \mathrm{Cos}_i \mathrm{R}$$

and the graduation steps of the socle filtration by $\mathrm{Gr}^i \mathrm{R} = \mathrm{Soc}_{i+1} \mathrm{R} / \mathrm{Soc}_i \mathrm{R}$.

Lemma 1.5. *Consider a fixed Artinian module* I. *If two distinct submodules* N *and* M *have the same simple quotients, a JH-factor of* I *has to appear twice.*

Proof: In case $\mathrm{N} \subset \mathrm{M}$ or $\mathrm{M} \subset \mathrm{N}$ this is clear. Otherwise consider

$$0 \longrightarrow \mathrm{N} \xrightarrow{\text{incl.}} \mathrm{N} + \mathrm{M} \xrightarrow{\text{res. map}} \mathrm{N} + \mathrm{M}/\mathrm{N} \longrightarrow 0\ .$$

Then the right-hand side $\mathrm{N} + \mathrm{M}/\mathrm{N} = \mathrm{M}/\mathrm{N} \cap \mathrm{M}$ is nonzero. Therefore a JH-factor has to appear twice in the filtration between N and N + M. □

Because the JH-factors of I are all distinct, the submodules of I are therefore described by their simple quotients, that is, their co-socle or *head*. In particular for each JH-factor τ, there is a unique submodule $\mathrm{U} \subset \mathrm{I}$ who has τ as its sole quotient respectively submodule (otherwise we had two submodules with the same set of simple quotients). This is needed to state the Theorem 1.6 below (stated for $r = 1, \ldots, q-2$ as the case $r = 0$ was treated before).

Theorem 1.6 (M. Bardoe, P.Sin). *It obtains*

(i) We have

$$\mathrm{Gr}^i \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi^s = \bigoplus_{\#\mathrm{J}=i} \tau$$

for $i = 0, \ldots, f$, where the direct sum runs over all τ with corresponding parameter set $\tau \leftrightarrow \lambda \leftrightarrow \mathrm{J}$ satisfying $\#\mathrm{J} = i$ (and the socle and co-socle filtrations are reverse to each other).

(ii) *If* $\tau \in \mathrm{JH}\,\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi^s$, *let* U=*"the unique submodule with sole simple quotient* τ*" inside* I. *Then*

$$\mathrm{Gr}^i\,\mathrm{U} = \bigoplus_{\tilde{\mathrm{J}}\subset\mathrm{J} \text{ with } \#\tilde{\mathrm{J}}=i} \tilde{\tau}$$

for $i = 0, \ldots, \#\mathrm{J}$ *(and the socle and co-socle filtrations are reverse to each other). Here* J *respectively* $\tilde{\mathrm{J}}$ *are the parameter sets corresponding to* τ *respectively* $\tilde{\tau}$.

Remark 1.7. We remark that

(i) For $i = 0, \ldots, f-1$, we have the dualities

$$\mathrm{Gr}^i\,\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi^s = \mathrm{Gr}_{f-i}\,\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi.$$

(ii) If τ is a JH-factor of $\mathrm{I} := \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi$, let U be "the unique submodule with sole simple submodule τ" $\subseteq$ I. Then

$$\mathrm{Gr}^i\,\mathrm{U} = \bigoplus_{\tilde{\mathrm{J}}\supset\mathrm{J} \text{ with } \#\tilde{\mathrm{J}}=i} \tilde{\tau}$$

for $i = \#\mathrm{J}, \ldots, f$ (and the socle and co-socle filtrations are reverse to each other). Here J respectively $\tilde{\mathrm{J}}$ are the parameter sets corresponding to τ respectively $\tilde{\tau}$.

Proof: Ad (i): This follows from the description $\chi^s = \chi^* \otimes \det^r$ with $\chi^* \leftrightarrow q-1-r$.

Ad (ii): Can be seen through the general structure theory leading to the proof of Theorem 1.6. □

In particular, as there are unique $\mathrm{J} \subset \{0, \ldots, f-1\}$ with $\#\mathrm{J} = 0, f$, namely $\mathrm{J} = \emptyset, \{0, \ldots, f-1\}$, we find by the above correspondence $\mathrm{Soc}\,\mathrm{I} := \mathrm{Soc}_1\,\mathrm{I} = \boldsymbol{p-1-r}$ and $\mathrm{Cos}\,\mathrm{I} := \mathrm{R}/\mathrm{Cos}_1\,\mathrm{R} = \boldsymbol{r}$ - up to a determinant twist. In particular I's quotient $\sigma = \boldsymbol{r}$ found by Frobenius reciprocity is the sole simple quotient of $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi$.

Proof of Theorem 1.6: In [BrdSn], M. Bardoe and P. Sin consider the following situation: G acts naturally on $\mathrm{X} = \mathbf{F}_q^2 - \{0\}$. Let $\mathrm{A} = \mathrm{E}[\mathrm{X}]$ be the permutation module. Then $\mathrm{A} = \oplus_{r=0,\ldots,q-2}\mathrm{A}[r]$, where the center $\mathrm{Z} \subset \mathrm{G}$ operates on $\mathrm{A}[r]$ by the character $[r] : \mathrm{F}_q^* \ni z \mapsto z^r$. We notice that $\mathrm{A}[r] = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi$ with $\chi : \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto a^r$. This is seen as follows: The module $\mathrm{A}[r]$ contains the vector $w := \sum_{\lambda\in\mathbf{F}_q^*} \lambda^{-r}\left[\begin{smallmatrix} \lambda \\ 0 \end{smallmatrix}\right]$. Since B acts on this sum by permutation, we find

$$\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)\cdot w = a^r \cdot w.$$

Moreover $\{\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\cdot w\} \cup \{\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right)\cdot w | c \in \mathbf{F}_q\}$ is a set of cardinality $q+1$, which is linearly independent: the sums of vectors run over $\left[\begin{smallmatrix} 0 \\ \lambda \end{smallmatrix}\right]$ in the first and $\left[\begin{smallmatrix} \lambda \\ c \end{smallmatrix}\right]$ in the second set. Since $\dim \mathrm{A} = (q-1)(q+1)$, we have therefore $\dim \mathrm{A}[r] = q+1$, so that $\mathrm{A}[r]$ is spanned by the G-translates of $\mathrm{W} := \mathrm{E}\cdot w$. In other words, $\mathrm{A}[r] = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\chi$, where the underlying vector space of χ is W.

Remark. Identifying $\mathrm{X} = \mathbf{F}_q^2 - \{0\} = \mathbf{F}_{q^2}^*$, the module $\mathrm{A} = \mathrm{E}[\mathrm{X}]$ is the regular representation of $\mathrm{H} := \mathbf{F}_{q^2}^* \subset \mathrm{GL}_2(\mathbf{F}_q) = \mathrm{G}$. Therefore $\mathrm{A} = \oplus_{\chi\colon \mathrm{H} \to \mathbf{E}^*} \chi$, where the sum runs over the $q-1$ distinct characters $\chi : \mathbf{F}_{q^2}^* \ni a \mapsto a^r$ for $r = 0, \ldots, q-2$ (noting $\operatorname{char} \mathrm{E} = p$ not dividing $\#\mathrm{H} = q^2 - 1$ - in general the injective envelopes of the direct summands have to be considered). In particular the JH-factors of $\mathrm{A}[r] \subset \mathrm{A}$ are distinct (as H- and a fortiori G-modules).

We have $\mathbf{E}[\mathbf{F}_q^2] = \mathrm{A} \oplus \mathbf{E}[\begin{smallmatrix}0\\0\end{smallmatrix}]$ as G-modules and the right-hand side equals the polynomial ring $\mathbf{E}[\mathbf{F}_q^2] = \mathbf{E}[\mathrm{X}, \mathrm{Y}]/ < \mathrm{X}^q = \mathrm{X}, \mathrm{Y}^q = \mathrm{Y} >$. Then $\mathrm{A}[r]$ has the monomial basis $\{\mathrm{X}^i \mathrm{Y}^j : i + j \equiv r \bmod q-1 \text{ for } i, j = 0, \ldots, q-1\}$. The degree filtration of $\mathbf{E}[\mathrm{X}, \mathrm{Y}]$ induces the filtration $\mathcal{F}_n[r]$ on $\mathrm{A}[r]$ defined by

$$\mathcal{F}_n[r] =< \{\text{monomials } \mathrm{X}^i \mathrm{Y}^j \text{ with } i + j = r + m(q-1) \text{ for some } m \leq n\} > .$$

Then the Frobenius $\mathrm{Frb} \in \mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ acts on $\mathrm{X} = \mathbf{F}_q^2 - \{0\}$ and hence $\mathrm{A} = \mathbf{E}[\mathrm{X}]$. On the direct summands $\mathrm{A}[r] \subset \mathrm{A}$, it sends $\mathrm{Frb}(\mathrm{A}[r]) \hookrightarrow \mathrm{A}[pr]$: If $z \in \mathrm{Z}$ and $v \in \mathrm{A}[r]$, then $z \cdot \mathrm{Frb}\, v = \mathrm{Frb}\, z \cdot v = \mathrm{Frb}\, z^r \cdot v = z^{pr} v$. *To a tuple* $\boldsymbol{j} \in \{0,1\}^f$ *corresponds a unique subset*

$$\mathrm{J} \subset \{0, \ldots, f-1\} = \mathbb{Z}/ < f > \textit{ by } i \in \mathrm{J} \Leftrightarrow j_{i+1} = 1.$$

We define the submodules $\mathcal{F}[r]_{\boldsymbol{j}}$ by

$$\mathcal{F}[r]_{\boldsymbol{j}} = \mathcal{F}[{}^0 r]_{j_0} \cap \mathrm{Frb}\, \mathcal{F}[{}^1 r]_{j_1} \cap \ldots \cap \mathrm{Frb}^{f-1} \mathcal{F}[{}^{f-1} r]_{j_{f-1}}$$

for the unique ${}^e r = 1, \ldots, q-2$ to assure $\mathrm{Frb}^e \mathcal{F}[{}^e r]_{j_e} \subset \mathrm{A}[r]$, that is, such that $p^e\, {}^e r \equiv r \bmod q-1$. It holds $\mathcal{F}[r]_{\tilde{\boldsymbol{j}}} \subset \mathcal{F}[r]_{\boldsymbol{j}}$ if $\tilde{\boldsymbol{j}} \leq \boldsymbol{j}$.

We fix $r = 0, \ldots, q-2$ and an isomorphism $\mathrm{I} = \mathrm{A}[r]$. By [BrdSn, Corollary 6.1], the module $\mathrm{U}_{\boldsymbol{j}} := \mathcal{F}[r]_{\boldsymbol{j}} \subset \mathrm{I}$ is for each $\boldsymbol{j} \in \{0,1\}^f$ the unique submodule with sole simple quotient $\sigma \leftrightarrow \mathrm{J} \leftrightarrow \boldsymbol{j}$ given by

$$\sigma = \mathrm{U}_{\boldsymbol{j}} / \sum_{\tilde{\boldsymbol{j}} < \boldsymbol{j}} \mathrm{U}_{\tilde{\boldsymbol{j}}}.$$

Moreover, given a submodule $\mathrm{U} \subset \mathrm{I}$, let $\mathrm{JH}\, \mathrm{U} \subset \{0,1\}^d$ be its set of JH-factors, parameterized by $\boldsymbol{j} \leftrightarrow \mathrm{J} \leftrightarrow \sigma$. Then by [BrdSn, Comment before Lemma 3.2], we find $\tilde{\mathrm{U}} \subset \mathrm{U}$ if and only if $\mathrm{JH}\, \mathrm{U} \subset \mathrm{JH}\, \mathrm{U}$.

Ad (i): The filtration $\mathcal{F}_n \subset \mathrm{I}$ given by

$$\mathcal{F}_n = \sum_{\boldsymbol{j} \in \{0,1\}^f \text{ with } |\boldsymbol{j}| = n} \mathrm{U}_{\boldsymbol{j}}$$

equals the socle filtration (and the co-socle filtration in reverse): By the construction of the quotients $\sigma_{\boldsymbol{j}}$ of $\mathrm{U}_{\boldsymbol{j}}$, we must have

$$\mathrm{Gr}^n \mathrm{I} := \mathcal{F}_n / \mathcal{F}_{n-1} = \sum_{|\boldsymbol{j}| = n} (\mathrm{U}_{\boldsymbol{j}} / \sum_{\tilde{\boldsymbol{j}} < \boldsymbol{j}} \mathrm{U}_{\tilde{\boldsymbol{j}}}) = \sum_{|\boldsymbol{j}| = n} \sigma_{\boldsymbol{j}} = \bigoplus_{|\boldsymbol{j}| = n} \sigma_{\boldsymbol{j}};$$

the second equality since the submodules $\tilde{\mathrm{U}} \subset \mathrm{U}_{\boldsymbol{j}}$ are exactly those with $\mathrm{JH}\,\tilde{\mathrm{U}} \subset \mathrm{JH}\,\mathrm{U} \subset \{\tilde{\boldsymbol{j}} \leq \boldsymbol{j}\}$, the last one as the sum runs over nonisomorphic simple modules. We see that each filtration step $\mathcal{F}_n$ contains all possible mod $\mathcal{F}_{n-1}$-simple modules. As it starts at 0, it must coincide with the socle filtration. Analogous for the co-socle filtration.

Ad (ii): If $\mathrm{U} = \mathrm{U}_{\boldsymbol{j}} \subset \mathrm{I}$ is a submodule corresponding to $\boldsymbol{j} \in \{0,1\}^d$ with induced filtration $\mathcal{G}_n = \mathcal{F}_n \cap \mathrm{U}$, then by its definition above

$$\mathcal{G}_n = \sum_{|\tilde{\boldsymbol{j}}|=n} \mathrm{U}_{\tilde{\boldsymbol{j}}} \cap \mathrm{U}_{\boldsymbol{j}} = \sum_{|\tilde{\boldsymbol{j}}|=n} \mathrm{U}_{\tilde{\boldsymbol{j}}} \cap \sum_{|\tilde{\boldsymbol{j}}|\leq \boldsymbol{j}} \mathrm{U}_{\tilde{\boldsymbol{j}}} = \sum_{|\tilde{\boldsymbol{j}}|=n \text{ and } \tilde{\boldsymbol{j}} \leq \boldsymbol{j}} \mathrm{U}_{\tilde{\boldsymbol{j}}}.$$

2 Injective Envelopes

For a fixed weight σ of G, in this section we will determine the JH-factors of its injective envelope $\operatorname{inj}\sigma$. Moreover, we will see that there is a sub representation $\mathrm{V}_\sigma \subset \operatorname{inj}\sigma$ with the same JH-factors which behaves like the induced representation $\operatorname{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi^s$ of the character $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto a^r$ corresponding to $\sigma = (r_0, \ldots, r_{f-1}) \leftrightarrow r \in \{0, \ldots, q-2\}$ (at least if $r_0, \ldots, r_{f-1} < p-1$). It will in particular be multiplicity free.

Up to a twist with $\det^m$ for $m \in \{0, \ldots, q-2\}$, we may assume $\sigma = (r_0, \ldots, r_{f-1})$ with $r_i \in \{0, \ldots, p-1\}$.

Lemma 2.1. *If* $\sigma = (p-1, \ldots, p-1) = \operatorname{Sym}^{q-1} \mathbf{E}^2 =< \{\mathrm{X}^i \mathrm{Y}^{q-1-i} | i = 0, \ldots, q-1\} >$, *then* σ *is injective, that is,* $\operatorname{inj}\sigma = \sigma$.

Proof: 1. We prove that $\sigma_{|\mathrm{B}}$ is injective as a B-representation.

We note that $v := y^{q-1}$ is invariant under the action of the lower unipotent matrices $\mathrm{U}^* = \{\left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right)\}$. The mapping from the regular representation $\mathbf{E}[\mathrm{U}] \to \sigma$ given by $[u] \mapsto u \cdot v$ meets all $x^i y^{q-1-i}$ and is therefore surjective. Hence bijective by both sides having dimension q. We have

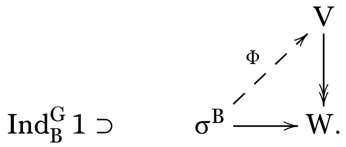
$$\mathrm{E}[\mathrm{U}] = \oplus_{\chi \text{ irred.}} \operatorname{inj}\chi = \oplus_{\chi \text{ char.}} \operatorname{inj}\chi = \operatorname{inj} 1;$$

the first equality by finite representation theory — see [SrFnRp, Excercise 14.1] and more concretely [Psk, Cororllary 4.0.6], the second by commutativity of U and the last one by $\#\mathrm{U} = p^f$ not dividing the order of any $x \in \mathbf{E}^*$. Therefore $\sigma_{|\mathrm{U}} = \operatorname{inj}_{\mathrm{U}} 1$. By [Br, Lemma 6.5] about extensions of injective envelopes for Sylow subgroups (which was the first Proposition in Makis's Talk) for G = B and D = U, we find $\sigma_{|\mathrm{B}} = \operatorname{inj}_{\mathrm{B}} 1$. Hence σ_{B} is in particular injective.
2. We use the injectivity of σ as E[B]-module to prove that σ is injective or equivalently (by self duality, as $\mathbf{E}[\mathrm{G}]$ is finite dimensional) projective as E[G]-module.

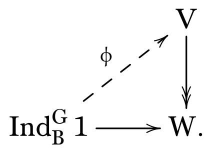
So fix a diagram of $\mathbf{E}[\mathrm{G}]$-module morphisms

$$\begin{array}{ccc} & & \mathrm{V} \\ & \overset{\phi}{\nearrow} & \downarrow \\ \sigma & \longrightarrow & \mathrm{W}. \end{array}$$

We have to construct the dotted lift ϕ such that this diagram commutes. By the first step, it exists as a morphism of $\mathbf{E}[\mathrm{B}]$-modules. Since $\sigma^{\mathrm{B}} = \mathbf{E}x^{q-1}$ and $\sigma = < \{x^{q-1}\} >_{\mathbf{E}[\mathrm{G}]}$, we see that $\operatorname{im}\phi \supsetneq 0$, that is, $\mathrm{Hom}_{\mathrm{B}}(1, \mathrm{V}_{|\mathrm{B}}) \supsetneq 0$. This lets us cook up a nonzero $\Phi \in \mathrm{Hom}_{\mathrm{G}}(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} 1, \mathrm{V})$. By the recipe for Φ, we find that



commutes. Then the G-equivariance yields a commutative diagram



Since $< \sigma^{\mathrm{B}} >_{\mathbf{E}[\mathrm{G}]} = \sigma$ and the inclusion $\sigma \subset \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} 1$ is one of $\mathbf{E}[\mathrm{G}]$-modules by [Br, Lemma 7.4], we find that the restriction to $\sigma = < \sigma^{\mathrm{B}} >_{\mathbf{E}[\mathrm{G}]} \subset \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} 1$ lifts $\sigma \to \mathrm{W}$. □

If $\sigma \leftrightarrow r < q-1$, the socle and co-socle filtrations of $\operatorname{inj}\sigma$ seem yet to complicated. We can only list the JH-factors of $\operatorname{inj}\sigma$, produced as follows: Let $x_0, \ldots, x_{f-1}$ be f variables.

Definition 2.2. If $f = 1$, we let $\mathcal{J} = \{x_0, p-2-x_0 \pm 1\}$. If $f > 1$, let

$$\begin{aligned}\mathcal{J} =\{\lambda \in \prod_{i=0,\ldots,f-1} & \{x_i, x_i \pm 1, p-2-x_i, p-2\pm 1 - x_i\} : \\ & \text{Either } \lambda_i = x_i, x_i \pm 1, \text{ then } x_{i+1} = x_{i+1}, p-2-x_{i+1} \\ & \text{or} \quad \lambda_i = p-2-x_i, p-2\pm 1 - x_i, \\ & \quad\quad \text{then } x_{i+1} = x_{i+1} \pm 1, p-2\pm 1 - x_{i+1}\}\end{aligned}$$

Here $x_f := x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. This can be phrased that the elements are indexed circularly by $\mathbb{Z}/ < f >$.

Concretely $\mathcal{J}$ consists of sequences of the form $(\ldots, p-2-x_{i-1}, p-2-x_i \pm 1, x_{i+1} \pm 1, \ldots)$.

Remark 2.3. At least if $r_0, \ldots, r_{f-1} < p-1$, then we can describe $\mathcal{J}$ by [BrPsk, Definition 4.1] alternatively as follows: Let $\Sigma = \{0, \pm 1\}^f = \{(\varepsilon_0, \ldots, \varepsilon_{f-1})\}$. Then to $\varepsilon \in \Sigma$ corresponds to $\lambda \in \mathcal{J}$ defined by

$$\lambda_i = \begin{cases} \lambda_i + \varepsilon_{i-1}, & \text{if } \varepsilon_i = 0 \\ p-2-\lambda_i + \varepsilon_{i-1}, & \text{if } \varepsilon_i = \pm 1. \end{cases} \qquad \text{(indizes mod } f\text{)}$$

Given such a sequence (which we will attach a JH-factor of our representation $\operatorname{inj}\sigma$ to), there is formula for the appropriate twist by a power of the determinant of this factor.

Definition 2.4. For $\lambda \in \mathcal{J}$ let $e(\lambda)$ be half of

$$\begin{cases} \sum_{i=0,\ldots,f-1} p^i(x_i - \lambda_i(x_i)), & \text{if } \lambda_{f-1}(x_{f-1}) = x_{f-1}, x_{f-1} \pm 1 \\ p^f - 1 + \sum_{i=0,\ldots,f-1} p^i(x_i - \lambda_i(x_i)), & \text{if } \lambda_{f-1}(x_{f-1}) = p-2-x_{f-1} \\ & \text{or } \lambda_{f-1}(x_{f-1}) = p-2-x_{f-1} \pm 1. \end{cases}$$

One can check $e(\lambda) \in \mathbb{Z} \oplus_{i=0,\ldots,f-1} \mathbb{Z}x_i$: The case $p = 2$ is seen directly. If $p > 2$ and not $\lambda_i = x_i$ or $\lambda_i = p - 2 \pm 1 - x_i$ for all $i \in \mathbb{Z}/ < f >$ - in which case $2e(\lambda) \equiv 0 \mod 2$, there must exist a $\lambda_{i+1} = x_{i+1} \pm 1$. Then in-between λ_i is uniquely determined by λ_{i-1} and λ_{i+1}. This can be used to prove $2e(\lambda) \equiv 0 \mod 2$. In other words, we pass on [Br, Exercise 7.8] to our reader.

Theorem 2.5.

(i) (a) *Let* $\sigma = \boldsymbol{r} \neq \boldsymbol{0}, \boldsymbol{p-1}$. *Then*

$$\mathrm{JH}\,\mathrm{inj}\,\sigma = \{\lambda(\boldsymbol{r}) \otimes \det{}^{e(\lambda(\boldsymbol{r}))} | \lambda \in \mathcal{J} \textit{ with } \boldsymbol{0} \leq \boldsymbol{r} \leq \boldsymbol{p-1}\}.$$

(b) *If* $\sigma = \boldsymbol{0} = 1$, *then*

$$\mathrm{JH}\,\mathrm{inj}\,\sigma = \{\lambda(\boldsymbol{0}) \otimes \det{}^{e(\lambda(\boldsymbol{0}))} | \lambda \in \mathcal{J} \textit{ with } \boldsymbol{0} \leq \lambda(\boldsymbol{r}) \leq \boldsymbol{p-1}\} - \{\boldsymbol{p-1}\}.$$

(c) *If* $\sigma = \boldsymbol{p} - 1 = \mathrm{Sym}^{q-1}\,\mathbf{E}^2$, *then* $\mathrm{JH}\,\mathrm{inj}\,\sigma = \mathrm{JH}\,\sigma = \{\sigma\}$.

(ii) There is a unique maximal subrepresentation $V_\sigma \subset \mathrm{inj}\,\sigma$ *with* $\sigma \subset V_\sigma$ *as sole simple submodule such that* $\mathrm{JH}\,V_\sigma = \mathrm{JH}\,\mathrm{inj}\,\sigma$. *If moreover* $\sigma = \boldsymbol{r}$ *with* $r_0, \ldots, r_{f-1} < p-1$, *then* V_σ *is multiplicity free.*

Proof: Ad (ii) - the existence of V_σ. We restrict to the case $\boldsymbol{r} \neq \boldsymbol{0}$ and $r_0, \ldots, r_{f-1} < p-1$. Let the tuple $\boldsymbol{r} = (r_0, \ldots, r_{f-1}) \in \{0, \ldots, p-2\}^f$ correspond to the weight σ. For $r \in \mathbb{N}$ abbreviate $V_r = \mathrm{Sym}^r\,\mathbf{E}^2$. We want to describe the tensor product components $R_{r_0}, R_{r_1}^{\mathrm{Frb}}, \ldots, R_{r_{f-1}}^{\mathrm{Frb}^{f-1}}$ the injective envelope $\mathrm{inj}\,\sigma$ is built of: For $r = p-1$, we have $R_r = V_r$. If $r = 0, \ldots, p-2$, there is an explicitly defined G-subspace $R_r \subset V_{p-1-r} \otimes V_{p-1}$, cf. [Psk, Definition 4.2.10]. Then

$$\mathrm{inj}\,\sigma = \bigotimes_{i=0,\ldots,f-1} R_{r_i}^{\mathrm{Frb}^i}.$$

Put $d' = p-1-r$, $d'' = p-1$ and $d = d'+d'' = 2p-2-r$. Recall that $V_d =< \{X^d - iY^i : i = 0, \ldots, d\} >_{\mathbf{E}}$ and $V_{d'} \otimes V_{d''} =< \{X^{d'-i}Y^i \otimes X^{d''-j}Y^j : i = 0, \ldots, d' \text{ and } j = 0, \ldots, d''\} >_{\mathbf{E}}$. By [Psk, Lemma 4.2.9], we find

$$\begin{array}{ccc} V_d & \hookrightarrow & V_{d'} \otimes V_{d''} \\ \binom{d-i}{i} X^{d-i}Y^i & \mapsto & \sum_{k+l=i} X^{d'-k}Y^k \otimes X^{d''-l}Y^l. \end{array}$$

By [Psk, Definition 4.2.10], this map's image lies in $\mathrm{R}_d \subset \mathrm{V}_{d'} \otimes \mathrm{V}''_d$ and hence induces an inclusion

$$\mathrm{V}_{\boldsymbol{2p-2-r}} := \bigotimes_{i=0,\dots,f-1} \mathrm{V}_{2p-2-r_i}^{\mathrm{Frb}^i} \hookrightarrow \bigotimes_{i=0,\dots,f-1} \mathrm{R}_{r_i}^{\mathrm{Frb}^i} = \operatorname{inj}\sigma .$$

We found our $\mathrm{V}_\sigma := \mathrm{V}_{\boldsymbol{2p-2-r}}$. It will suffice to see that $\mathrm{JH}\,\mathrm{V}_\sigma \supset \mathrm{JH}\operatorname{inj}\sigma$ - meaning the set given above — and that V_σ is multiplicity free.

Ad (i) (a): Firstly, by [BrPsk, Lemma 3.5] we have an exact sequence of $\mathbf{E}[\mathrm{G}]$-modules

$$0 \longrightarrow \mathrm{V}_r \otimes \det^{p-1-r} \longrightarrow \mathrm{V}_{2p-r-2} \longrightarrow \mathrm{V}_{p-r-2} \otimes \mathrm{V}_1^{\mathrm{Frb}} \longrightarrow 0 .$$

By the calculation carried out in [BrPsk, Lemma 3.8(i)] for $r = 0, \dots, p-2$, we find the plausible equality of $\mathbf{E}[\mathrm{G}]$-modules

$$\mathrm{V}_r \otimes \mathrm{V}_1^{\mathrm{Frb}} = \mathrm{V}_{r+1} \oplus \mathrm{V}_{r-1} \otimes \det .$$

Since $\mathrm{V}_{\boldsymbol{p-2-r}} = \mathrm{V}_{2p-2-r_0} \otimes \cdots \otimes \mathrm{V}_{2p-2-r_{f-1}}^{\mathrm{Frb}^{f-1}}$, we see by shifting V_1 that the JH-factors of $\mathrm{V}_{\boldsymbol{2p-2-r}}$ consist of those tensor products $\mathrm{V}_r, \mathrm{V}_{r\pm1}, \mathrm{V}_{p-r-2}, \mathrm{V}_{p-r-2\pm1}$ - up to determinant twists — that are again $\mathbf{E}[\mathrm{G}]$-modules. This is made precise by our above recipe $\mathrm{JH}\,\mathrm{V}_\sigma = \{\lambda(\boldsymbol{r}) \otimes \det^{e(\lambda(\boldsymbol{r}))} | \lambda \in \mathcal{J} \text{ with } \boldsymbol{0} \leq \boldsymbol{r} \leq \boldsymbol{p-1}\}$.

Ad (ii) - V_σ being multiplicity free: That all these JH-factors appear only once is a lot of calculation: One construct a descending tensor product filtration on $\mathrm{V}_\sigma = \mathrm{V}_{\boldsymbol{p-2-r}}$ with graduation steps $\mathrm{Gr}^i\,\mathrm{V}_\sigma = \oplus \mathrm{W}_{\#\mathrm{J}=f-i}\mathrm{W}_\mathrm{J}$ (which turns out to be the co-socle filtration if $r_0, \dots, r_{f-1} < p-1$). Then its summands W_J are tensor products of semi-simple multiplicity free packets L_r, see [BrPsk, Proposition 3.9]. Therefore $\mathrm{Gr}^i\,\mathrm{V}_\sigma$ is multiplicity free and hence V_σ is. □

The socle and co-socle filtrations of the multiplicity free submodule $\sigma \subset \mathrm{V}_\sigma \subset \operatorname{inj}\sigma$ for $\sigma = \boldsymbol{r}$ with $r_0, \dots, r_{f-1} < p-1$ behave similarly to those of the principal series $\operatorname{Ind}_\mathrm{B}^\mathrm{G} \chi \twoheadrightarrow \sigma$ with σ as its sole simple quotient, that is, $\chi = \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto a^r$ with $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$. The following makes this precise.

Definition 2.6. We introduce a notion of size and partial order on Σ as follows.

(i) For $\varepsilon \in \Sigma$ define $|\varepsilon| = |\varepsilon_0| + \cdots + |\varepsilon_{f-1}|$.

(ii) For $\tilde{\varepsilon}, \varepsilon \in \Sigma$ define $\tilde{\varepsilon} \leq \varepsilon$, if $\tilde{\varepsilon}_i = \pm 1$ implies $\tilde{\varepsilon}_i = \varepsilon_i$.

Proposition 2.7. *Let $\sigma = \boldsymbol{r}$ with $r_0, \dots, r_{f-1} < p-1$. If $\sigma = 1$ (that is, $\boldsymbol{r} = \boldsymbol{0}$), forget below the weight $\boldsymbol{p-1}$.*

(i) *We have*

$$\mathrm{Gr}^i\,\mathrm{V}_\sigma = \bigoplus_{|\varepsilon| = i} \tau$$

for $i = 0, \dots, f-1$, where the direct sum runs over all τ whose corresponding parameter tuple $\varepsilon \leftrightarrow \lambda \leftrightarrow \tau$ satisfies $|\varepsilon| = i$ (and the socle and cosocle filtrations are reverse to each other).

(ii) *If $\tau \in \mathrm{JH}\,\mathrm{V}_\sigma$, let* U=*"the unique submodule with sole simple quotient τ" inside* I. *Then*

$$\mathrm{Gr}^i\,\mathrm{U} = \bigoplus_{\tilde{\varepsilon} \leq \varepsilon} \tilde{\tau}$$

for $i = 0, \dots, |\varepsilon|$ (and the socle and co-socle filtrations are reverse to each other). Here $\tilde{\varepsilon}$ respectively ε are the parameter tuples corresponding to $\tilde{\tau}$ respectively τ.

Proof: Ad (i): By [BrPsk, Equation (8)] and [BrPsk, Lemma 3.8], we have

$$\mathrm{Gr}^i\,\mathrm{V}_\sigma = \bigoplus_{|\varepsilon|=i} \tau,$$

where the direct sum runs over all τ whose corresponding parameter tuple $\varepsilon \leftrightarrow \lambda \leftrightarrow \tau$ satisfies $|\varepsilon| = i$.

Ad (ii): In fact, we have by [BrPsk, Theorem 4.7] a convenient description of the submodules of V_σ, just as in the proof of Theorem 1.6: Recall that we can parameterize $\mathrm{JH}\,\mathrm{V}_\sigma$ by $\Sigma = \{0, \pm 1\}^f$. Then each submodule $\mathrm{W} \subset \mathrm{V}_\sigma$, is completely described by its set of JH-factors $\mathrm{JH}\,\mathrm{W} \subset \Sigma$ and we find $\tilde{\mathrm{W}} \subset \mathrm{W} \subset \mathrm{V}_\sigma$ if and only if $\mathrm{JH}\,\tilde{\mathrm{W}} \subset \mathrm{JH}\,\mathrm{W} \subset \Sigma$. Moreover with each $j \in \mathrm{JH}\,\mathrm{W}$, we find $\{\tilde{j} \leq j\} \subset \mathrm{JH}\,\mathrm{W}$. Together (i), this implies the result. □

A dual version of Proposition 2.7(ii) holds for the unique quotients with sole simple submodule a given weight, see [Br, Theorem 7.14(iii)].

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Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, 48149 Münster
e-mail: enno.nagel@uni-muenster.de