Fractional differentiability over non-Archimedean fields

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These note accompany my talk about non-Archimedean fractional differentiability in the realms of the p-adic Langlands program held on the 28th of november 2011 at the "Séminaire Automorphe" in Paris.

Contents

0 Prerequisites

Given a non-Archimedeanly valued complete field $\mathbf{F}$ we denote by $\mathbf{o_F}$ its ring of integers. If its valuation $v\colon \mathbf{F} \to \mathbb{R} \cup \{\infty\}$ is discret, we assume $\pi_\mathbf{F}$ be a an element of positive valuation generating $v(\mathbf{F}^*)$. If its residue field $\kappa_\mathbf{F} := \mathbf{o_F}/\pi_\mathbf{F}\mathbf{o_F}$ is of characteristic $p > 0$, we will assume $v(p) = 1$ and define a multiplicative valuation by $|x| = p^{-v(x)}$ for $x \in \mathbf{F}$.

1 Motivation

p-adic Langlands for crystalline Galois representations

Let $\mathbf{F}$ be a p-adic number field, $\mathbf{E}$ a sufficiently large p-adic number field serving as the field of coefficients and $\mathrm{G} = \mathrm{GL}_n(\mathbf{F})$.

Definition 1.1. By a ***p*-adic Galois representation** we mean a continuous representation of the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F})$ of a finite extension $\mathbf{F}$ over a finite dimensional $\mathbf{E}$-vector space.

Definition 1.2. By a **unitary** G**-Banach space representation** we mean an $\mathbf{E}$-Banach space with a continuous action of G such that its topology can be defined by a G-invariant norm.

Conjecture 1.3. *The p-adic Langlands program:*

$$\{n\text{-dimensional } p\text{-adic Galois-rep's}\} \leftrightarrow \{\text{unitary } \mathrm{G}\text{-Banach space rep's}\}$$

The crystalline case

Example 1.4. Let $\mathbf{F} = \mathbb{Q}_p$. A filtered ϕ-module V over $\mathbf{E}$ is an n-dimensional $\mathbf{E}$-vector space with

- an automorphism ϕ,
- and a decreasing, exhaustive and separated filtration $\ldots \supseteq \mathrm{V}_n \supseteq \mathrm{V}_{n+1} \supset \ldots$ indexed over $n \in \mathbb{Z}$.

Definition. The *filtration type* $\boldsymbol{\tau}$ of a filtered ϕ-module V is defined as the sequence of n integers $\tau_1 \leq \ldots \leq \tau_n$ of the filtration's jumps where an entry's multiplicity is given by the dimension of the associated nonzero graded piece.

Breuil and Schneider in [BS07] then associate a *locally algebraic* representation $\pi(\phi) \otimes_\mathbf{E} \rho(\tau)$ to the datum $(\phi, \boldsymbol{\tau})$ as follows:

- The construction of $\pi(\phi)$: We view the semisimple part ϕ^{ss} of the endomorphism ϕ as the semisimple n-dimensional unramified representation $\sigma\colon W(\mathbf{F}) \to \mathrm{Frob}^{\mathbb{Z}} \to GL_n(\mathbf{E})$ of the Weil group of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ given by $\mathrm{Frob} \mapsto \phi^{ss}$. Then by (a modified version of) the classical local Langlands correspondence σ corresponds to a smooth unramified principal series representation $\pi = \pi(\phi)$ of G.

- The construction of $\rho(\boldsymbol{\tau})$: The n-tuple of increasing integers $\boldsymbol{\tau}$ corresponds (almost canonically) to a dominant character $\psi\colon T \to \mathbf{E}^*$ of the subgroup of diagonal matrices $T \subseteq G$. Then ψ in turn uniquely determines the irreducible rational G-representation $\rho(\boldsymbol{\tau})$ of highest weight ψ.

Conjecture 1.5 (See [BS07]). *There is a* G*-invariant norm on* $\pi(\phi) \otimes_{\mathbf{E}} \rho(\boldsymbol{\tau})$ *if and only if the filtered* ϕ*-module* V *is* admissible.

We will make neither the definition of an *admissible* filtered ϕ-module nor the following definition of a *crystalline* p-adic Galois representation precise. Their importance stems from the following theorem.

Theorem 1.6 (Colmez, Fontaine). *We have an equivalence of categories*

$$\begin{array}{ccc} \{\textit{admissible filtered } \phi\textit{-modules}\} & \leftrightarrow & \{\textit{crystalline } p\textit{-adic Galois representations}\} \\ \bigcap_{\textit{full subcategory}} & & \bigcap_{\textit{full subcategory}} \\ \{\textit{filtered } \phi\textit{-modules}\} & & \{p\textit{-adic Galois representations}\}. \end{array}$$

Remark. Up to this point we have only made use of the data given by the filtration type but not the actual filtration itself. Ideally one would like the family of crystalline Galois representations given by the possible admissible filtrations of V to correspond to different completions of $\pi \otimes_{\mathbf{E}} \rho$ by G-invariant norms, but there is no precise conjecture yet.

2 Our object of study

Setup

Let $\mathbf{F}$ be a p-adic number field, $\mathbf{E}$ a sufficiently large p-adic number field serving as the field of coefficients and $G = GL_n(\mathbf{F})$. Let $T \subseteq G$ be the subgroup of diagonal matrices with entries in $\mathbf{F}^*$ and $T_0 \subseteq T$ its maximal compact open subgroup with entries in $\mathbf{o}_{\mathbf{F}}^*$. Let P respectively $\bar{P}$ be the subgroup of higher respectively lower triangular matrices and N respectively $\bar{N}$ its subgroup of higher respectively lower triangular matrices whose diagonal entries are all 1. Then $N_0 \subset N$ denotes the subgroup of matrices with entries in $\mathbf{o}_{\mathbf{F}}$ above the diagonal.

- Let V be a G-representation. A vector v in V is **smooth** respectively **algebraic** respectively **locally algebraic** if its orbit map

$$\begin{aligned} o_v\colon \mathrm{G} &\to \mathrm{V} \\ g &\mapsto g\cdot v \end{aligned}$$

is a locally constant respectively rational respectively locally rational map.

- Let $\chi\colon \mathrm{T} \to \mathbf{E}^*$ be a character which we extend to P by the natural projection $\mathrm{P} \to \mathrm{T}$. Then we define the G-representation

$$\mathrm{I}(\chi) := \mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}}\chi := \{f\colon \mathrm{G} \to \mathbf{E} : f(\bar{p}g) = \chi(\bar{p})\cdot f(g) \text{ for all } \bar{p}\in\bar{\mathrm{P}}, g\in \mathrm{G}\}$$

with the action of G given by right translation.

- We denote by $\mathrm{I}(\chi)^{\mathrm{lc}}$, $\mathrm{I}(\chi)^{\mathrm{alg}}$ and $\mathrm{I}(\chi)^{\mathrm{la}}$ the G-subrepresentation given by all smooth, algebraic and locally algebraic vectors inside $\mathrm{I}(\chi)$ respectively. By the action of G through right-translation these are given by all locally constant respectively rational respectively locally rational functions inside $\mathrm{I}(\chi)$.

Let $\theta\colon \mathrm{T} \to \mathbf{E}^*$ be an **unramified** character, that is, it is trivial on T_0 and hence factorizing over $\mathrm{T}/\mathrm{T}_0 = (\mathbf{F}/\mathbf{o}_\mathbf{F}^*)^n = \mathbb{Z}^n$ and let $\psi\colon \mathrm{T}\to\mathbf{E}^*$ be an **algebraic** character, that is, its given by a rational function in the entries of T and hence of the form

$$\psi : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{a_1}\cdots t_n^{a_n}$$

for some $a_1,\ldots,a_n \in \mathbb{Z}$.

Then θ gives rise to a smooth principal series representation $\pi(\theta) = \mathrm{I}(\theta)^{\mathrm{lc}}$ and if ψ is *dominant* (that is, $a_1 \leq \ldots \leq a_n$ holds) it gives rise to the unique irreducible rational representation $\rho(\psi) = \mathrm{I}(\psi)^{\mathrm{alg}}$ of highest dominant weight ψ. Then $\pi(\theta)\otimes_\mathbf{E}\rho(\psi) = \mathrm{I}(\chi)^{\mathrm{la}}$ with $\chi = \theta\psi$.

In the following we will write shorthand $\mathrm{I}(\chi)$ for $\mathrm{I}(\chi)^{\mathrm{la}}$.

The universal unitary lattice inside $\mathrm{I}(\chi)$

Recall that in the p-adic Langlands program one wants to link continuous Galois representations with unitary G-Banach space representations. One expects the representation $\mathrm{I}(\chi)$ with respect to a unitary completion to correspond to a crystalline Galois-representation. The smooth part is obtained by the local Langlands correspondence whereas the algebraic part corresponds to the filtration type of associated ϕ-module.

Definition. A lattice $\mathbf{L}$ in an $\mathbf{E}$-vector space V is an $\mathbf{o_E}$-submodule such that for any $v \in V$, there is $\lambda \in \mathbf{E}^*$ such that $\lambda v \in \mathbf{L}$.

Remark. Recall that we have a correspondence between non-Archimedean norms and lattices on an vector space V over a non-Archimedean field $\mathbf{E}$ as follows: Given a norm $\|\cdot\|$, we take $\mathbf{L}$ to be its unit ball $\mathbf{L} = \{v \in V : \|v\| \leq 1\}$. Given a lattice $\mathbf{L}$, it has a corresponding norm $\|\cdot\|_{\mathbf{L}}$ on V given by $\|v\|_{\mathbf{L}} := \inf\{|\lambda| : \lambda \in \mathbf{E}^* \text{ with } \lambda v \in \mathbf{L}\}$.

Recall that we call a G-representation on a $\mathbf{E}$-Banach space V **unitary** if the topology of V may be defined by a G-invariant norm. Given a G-representation V there are a nonzero G-invariant seminorm on V if and only if its universal unitary completion as defined below is nonzero.

Definition. Let V be a G-representation. Then the unitary $\mathbf{E}$-Banach space representation $\widehat{V}$ is the **universal unitary completion** of V if any $\mathbf{E}[G]$-linear map $V \to W$ into a unitary $\mathbf{E}$-Banach space representation W factors uniquely over $\widehat{V}$.

Remark. It is given by the completion with respect to the smallest G-stable lattice.

Here and in the following we will not distinguish between the commensurability class of a lattice and the lattice itself. Correspondingly for the equivalence class of a norm and the norm itself.

Let G be compact. Then given any G-representation V we can equip it by a G-invariant norm by choosing any one of them $\|\cdot\|_0$ on V and putting $\|v\| = \sup\{\|g \cdot v\|_0 : g \in G\}$. This should make the following proposition plausible.

Proposition 2.1. *The universal unitary lattice of the locally algebraic* G*-representation* $I(\chi)$ *is given by any lattice finitely generated as an* $\mathbf{o_E}[P]$*-module.*

Proof: Follows from the following fulfilled conditions:

- That G is locally profinite and its Iwasawa decomposition $G = PK$
- That V locally finite dimensional (that is, for each vector exists a compact open subgroup $K \subseteq G$ such that the $\mathbf{E}$-vector subspace generated by Kv is finite dimensional) and finitely generated as an $\mathbf{E}[P]$-module. □

The universal unitary lattice of the $\mathbf{E}[P]$-module $I(\chi)(N)$

We remark that we have a well-defined notion of support inside $\mathfrak{F} := \bar{P}\backslash G$ for functions in $I(\chi)$ as $\bar{P}$ acts on the left through multiplication with invertible scalars on $I(\chi)$. Then we can view N as an open subset in $\mathfrak{F}$ via the image of the open immersion $N \subseteq G \overset{\text{can.}}{\to} \bar{P}\backslash G$. Thus we can define $I(\chi)(N)$ to be the functions inside $I(\chi)$ whose support lies in the open subset $N \subseteq \mathfrak{F}$. It is an $\mathbf{E}[P]$-module.

Proposition 2.2. *We have a* P*-equivariant injection*

$$\mathrm{I}(\theta)(\mathrm{N})\otimes \mathrm{U}_\psi \xrightarrow{\sim} \mathscr{C}^{\mathrm{lc}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})\otimes_{\mathbf{E}} \mathrm{U}_\psi \hookrightarrow \mathscr{C}^{\mathrm{lc}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})\otimes_{\mathbf{E}} \mathscr{C}^{\mathrm{alg}}(\mathrm{N},\mathbf{E}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E}).$$

Proof: Here the injectivity of the first isomorphism is clear and its surjectivity stems from the fact that there is a system of neighborhoods of unity given by compact open subgroups admitting an Iwahori factorization. The injectivity of the last map comes from the fact that by [Bor91, Theorem 21.20(i)] and the Taylor expansion, any polynomial function on N is in characteristic 0 uniquely determined on an open subset in N. The surjectivity holds by compactness of support. □

Definition 2.3. We denote the image of this injection by $\mathscr{C}^{\psi-\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})$. It can be described by

$$\mathscr{C}^{\psi-\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E}) = \{f:\mathrm{N}\to\mathbf{E} \text{ of cpt. supp. : For all } n\in\mathrm{N} \text{ exists open } \mathrm{U}\ni n \text{ in N such that } f_{|\mathrm{U}} = p_{|\mathrm{U}} \text{ for some } p\in \mathrm{Ind}^{\mathrm{G}}_{\bar{\mathrm{P}}}(\psi)^{\mathrm{alg}}\}.$$

Remark. Because **N** is as an algebraic variety a product of copies of $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{K}[\mathrm{X}])$, the function space $\mathscr{C}^{\mathrm{alg}}(\mathrm{N},\mathbf{E})$ consists of locally polynomial functions.

Corollary 2.4. *We conclude that there is an isomorphism of* $\mathbf{E}[\mathrm{P}]$*-modules*

$$\mathrm{I}(\chi)(\mathrm{N}) \xrightarrow{\sim} \mathscr{C}^{\psi-\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E}),$$

Here $\mathscr{C}^{\psi-\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})$ *is endowed with the* P*-action by* $f^n = (\cdot n)$ *for* $n\in\mathrm{N}$ *and* $f^t = \chi(t)\cdot f(\cdot^t)$ *for* $t\in\mathrm{T}$.

Proposition 2.5. *Let* $\mathrm{I}(\psi)^{\mathrm{alg}}$ *be the unique irreducible algebraic* G*-representation of highest weight* ψ *and denote by* $\bar{u}$ *its unique — up to a scalar — vector fixed by* $\bar{\mathrm{N}}$*. Then*

$$\mathscr{C}^{\psi-\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E}) = \mathbf{E}[\mathrm{P}]\cdot \mathbf{1}_{\mathrm{N}_0}\otimes\bar{u}.$$

Corollary 2.6. *Let* T^+ *be the dominant submonoid of* T *given by*

$$\mathrm{T}^+ := \{\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_d\end{pmatrix} : |t_1|\geq\ldots\geq|t_d|\}\subseteq\mathrm{T}.$$

Then we find the universal unitary lattice **L** *of the* $\mathbf{E}[\mathrm{P}]$*-module* $\mathscr{C}^{\psi-\mathrm{la}}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})$ *to be free if and only if* $|\chi(t)|\leq 1$ *for all* $t\in\mathrm{T}^+$.

Proof: Exhibit a nonzero norm $\|\cdot\| := \|\cdot\|_{\mathscr{C}^r}\leq\|\cdot\|_{\mathrm{L}}$ attached to differentiable functions of multiple variables. For this, it suffices for the norm to satisfy the following two conditions:

(i) It is invariant under translation by N.

(ii) There is a constant $\mathrm{C}>0$ such that $\|1_{{}^t\mathrm{N}_0}\otimes\bar{u}\|\leq \mathrm{C}\cdot 1/|\theta\bar{\psi}(t)|$ for all $t\in\mathrm{T}$. □

Gluing the universal unitary lattice from its open cells

Let W be the Weyl group of G with respect to the choice of P and T whose system of representatives is given by all monomial matrices in G. We have a well-defined operation of W on characters through conjugation.

One can show that $\operatorname{im} \delta_P/\delta_P^w \subseteq p^{2\mathbb{Z}}$ and conclude that we have a well-defined unramified character $(\delta_P/\delta_P^w)^{1/2}\colon T \to \mathbf{E}^*$.

Definition. Put $\theta_w := \theta^w(\delta_{\bar{P}}/\delta_{\bar{P}}^w)^{1/2}$. We call a character $\theta\colon \bar{P} \to \mathbf{E}^*$ **regular** if $\theta_w = \theta$ only if $w = 1$.

Let from now on $\theta\colon T \to \mathbf{E}^*$ be regular.

Corollary 2.7. *For $w \in W$ put $\chi_w := \theta_w\psi$. Then there are nonzero morphisms of G-representations $T_{\chi_w}\colon I(\chi_w) \to I(\chi)$ for all $w \in W$.*

Proof: By tensoring the intertwining operators on the smooth part with the identity morphism on the algebraic one. □

Proposition 2.8. *We have*

$$I(\chi) = \sum_{w\in W} T_{\chi_w}(I(\chi)(N)).$$

Corollary 2.9. *The universal unitary lattice of $I(\chi)$ is given by*

$$\mathbf{L} = \sum_{w\in W} \mathbf{o_E}[P] \cdot T_{\chi_w}(1_{N_0} \otimes \bar{u}).$$

Proof: By Proposition 2.8 and Proposition 2.5. □

3 The simplest example

The universal unitary completion of the compact open cell

Let $\mathbf{F} = \mathbb{Q}_p$ and $n = 2$ so that $G = GL_2(\mathbb{Q}_p)$. Let $\psi = \psi_1 \otimes \psi_2\colon T \to \mathbf{E}^*$ be the dominant algebraic character given by the characters $\psi_1 = \cdot^{l+k}$ and $\psi_2 = \cdot^l$ on $\mathbb{Q}_p^*$ with $l + k \geq l \in \mathbb{Z}$ and $\theta = \theta_1 \otimes \theta_2\colon T \to \mathbf{E}^*$ the unramified character. Put $\chi = \theta\psi$. Then identifying $N = \mathbb{Q}_p$, we obtain the following description.

Proposition.

$$I(\chi)(N) = \mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Q}_p, \mathbf{E}) := \{f\colon \mathbb{Q}_p \to \mathbf{E} : f \text{ loc. pol. of degree} \leq k \text{ with cpt. supp.}\}$$

with the P-action given by $f^t = \chi(t)f(d/a \cdot \_)$ for all $t = \begin{pmatrix} a & \\ & d\end{pmatrix} \in T$ and $f^n = f(\cdot + n)$ for $n \in N$.

Proof: We remark that in this case the unique irreducible algebraic representation $\mathrm{I}(\psi)^{\mathrm{alg}}$ of highest weight ψ has a basis given by k-fold products whose factors consist of the coordinate functions in the upper row and the determinant function. Restricting these functions to N leaves the monomial functions $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \mathrm{x} \\ & 1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & \mathrm{x}^k \\ & 1 \end{smallmatrix}\right)$. □

Let $r := v(\chi(\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)))$. The universal unitary lattice inside $\mathrm{I}(\chi)(\mathrm{N})$ is then generated by $1_{\mathbb{Z}_p} x^k$. One can convince oneself that for the existence of a P-invariant norm on $\mathrm{I}(\chi)(\mathrm{N})$ necessarily $r \geq 0$ and $|\chi(\mathrm{Z})| = 1$. We are therefore looking for the greatest norm $\|\cdot\|$ on $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Q}_p, \mathbf{E})$ such that

(i) There is a constant $\mathrm{C} > 0$ such that $\|1_{p^n \mathbb{Z}_p} x^k\| \leq \mathrm{C} \cdot p^{(r-k)n}$ for all $n \in \mathbb{Z}$.

(ii) It is invariant under translation.

Definition. A function $f \colon \mathbb{Z}_p \to \mathbf{E}$ is **continuously differentiable** if

$$f^{]1[}(x,y) := \frac{f(x) - f(y)}{x - y} \quad \text{with } x, y \in \mathbb{Z}_p \text{ distinct}$$

extends to a continuous function $f^{[1]} \colon \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbf{E}$. We denote by $\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$ the $\mathbf{K}$-Banach space of all continuously differentiable functions with respect to the norm given by $\|f\|_{\mathscr{C}^1} := \max\{\|f\|_{\sup}, \|f^{[1]}\|_{\sup}\}$.

Proposition 3.1. *Let $r = 1$ and $k > 0$. The completion of $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p, \mathbf{E})$ with respect to the restriction of $\|\cdot\|$ is given by $\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$. (And if $k = 0$ it is given by $\mathscr{C}^1_0(\mathbb{Z}_p, \mathbf{E}) = \{f \in \mathscr{C}^1(\mathbb{Z}_p, \mathbf{E}) : f' = 0\}$)*

Proof: Let $\mathbf{L}$ be the universal unitary lattice and let $\mathbf{L}_{\mathscr{C}^1} := \mathscr{C}^{\mathrm{la}\leq k} \cap \mathrm{B}_{\leq 1}(0)$ with $\mathrm{B}_{\leq 1}(0) := \{f \in \mathscr{C}^1(\mathbb{Z}_p, \mathbf{E}) : \|f\| \leq 1\}$ the unit ball inside $\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$. We have to check that $\varprojlim \mathbf{L}/\pi^n_{\mathbf{E}}\mathbf{E} \otimes_{\mathbf{o_E}} \mathbf{E} = \varprojlim \mathbf{L}_{\mathscr{C}^1}/\pi^n_{\mathbf{E}}\mathbf{L}_{\mathscr{C}^1} \otimes_{\mathbf{o_E}} \mathbf{E}$.

By definition $\|\cdot\|_{\mathscr{C}^1}$ is translation invariant and it is quickly checked that

$$\|1_{p^n\mathbb{Z}_p} x^k\|_{\mathscr{C}^1} \leq \left|\frac{p^{kn} - 0}{p^n - p^{n-1}}\right| = \mathrm{C} \cdot p^{(1-k)n}$$

with $\mathrm{C} := p^{-1} > 0$ for all $n \in \mathbb{N}$. Therefore $\mathbf{L}_{\mathscr{C}^1} \supseteq \mathbf{L}$.

The van der Put basis

$$\{e^0_n := 1_{n + p^{l(n)}\mathbb{Z}_p} : n \in \mathbb{N}\} \cup \{e^1_n := 1_{p^{l(n)}\mathbb{Z}_p} x(\cdot - n) : n \in \mathbb{N}\}$$

with $l(0) = 0$ and $l(n) = \lfloor \log_p(n) \rfloor$ for $n > 0$ is an orthogonal basis of $\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$ with

$$\|e^0_n\| = p^{l(n)} \quad \text{and} \quad \|e^1_n\| = 1.$$

In particular $\mathrm{B} := \{p^{l(n)} e^0_n : n \in \mathbb{N}\} \cup \{e^1_n : n \in \mathbb{N}\}$ is a basis of the $\mathbf{o_E}$-module $\mathbf{L}_{\mathscr{C}^1} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E})$. Since $\mathrm{B} \subseteq \mathbf{L} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E})$ we find therefore $\mathbf{L}_{\mathscr{C}^1} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E}) =$

$\mathbf{L} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E})$. We note that since $\mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E}) \subseteq \mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$ is dense and the norm $\|\cdot\|_{\mathscr{C}^1}$ on $\mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E})$ coincides with the one induced by $\mathbf{L}_{\mathscr{C}^1}$ and $\mathbf{L}_{\mathscr{C}^1} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E})$, the completion by $\mathbf{L}_{\mathscr{C}^1} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E})$ is given by $\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$. We therefore have inclusions

$$\mathbf{L}_{\mathscr{C}^1} \cap \mathscr{C}^{\mathrm{la}\leq 1}(\mathbb{Z}_p, \mathbf{E}) \subseteq \mathbf{L} \subseteq \mathbf{L}_{\mathscr{C}^1}$$

corresponding to the following morphisms between their completions

$$\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E}) \hookrightarrow \varprojlim_n \mathbf{L}/\pi_{\mathbf{E}}^n \mathbf{L} \otimes_{\mathbf{o_E}} \mathbf{E} \hookrightarrow \mathscr{C}^1(\mathbb{Z}_p, \mathbf{E}).$$

Then it is a general fact that the completion in the middle is also isomorphic to $\mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$, seen as follows: Let ϕ be right-hand side morphism. Because its image is dense it suffices to see that it is closed as well. We have $\|\cdot\|_{\mathscr{C}^1} \leq \|\cdot\|_{\mathbf{L}} \leq \|\cdot\|_{\mathscr{C}^1}$. Therefore it is a topological isomorphism onto its image. Since $\varprojlim_n \mathbf{L}/\pi_{\mathbf{E}}^n \mathbf{L} \otimes_{\mathbf{o_E}} \mathbf{E}$ is complete its image $\operatorname{im}\phi \subseteq \mathscr{C}^1(\mathbb{Z}_p, \mathbf{E})$ is complete as well and hence closed. □

Remark. Because $\mathrm{I}(\chi) = \mathrm{I}(\chi)(\mathrm{N}_0) \oplus \mathrm{I}(\chi)(\mathrm{N}_1 w)$ with $\mathrm{N}_1 = \left(\begin{smallmatrix} 1 & p\mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathrm{I}(\chi)(\mathrm{N}_1 w) = \mathrm{I}(\chi)(\mathrm{N}_0) = \mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p, \mathbf{E})$, we find $\mathrm{I}(\chi)$ to be a sum of two copies of $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p, \mathbf{E})$. Therefore $\mathbf{L}_u \supseteq \mathbf{L}_{\mathscr{C}^r} := \mathbf{L}_u \cap \mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p, \mathbf{E}) \oplus \mathbf{L}_u \cap \mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p, \mathbf{E})$. By completing with respect to these lattices in $\mathrm{I}(\chi)$ we obtain a mapping

$$\mathrm{I}(\chi)^{\mathscr{C}^r} := \mathscr{C}^r(\mathbb{Z}_p, \mathbf{E}) \oplus \mathscr{C}^r(\mathbb{Z}_p, \mathbf{E}) \to \widehat{\mathrm{I}}(\chi).$$

It is surjective as the lattices' surrounding vector spaces coincide. If $r = 0$ it is an isomorphism. Let $r > 0$. Its kernel L is given by the closed $\mathbf{E}[\mathrm{P}]$-module generated by the functions in $\mathrm{I}(\chi)^{\mathscr{C}^r}$ restricting to $1, x, \ldots, x^\nu$ on the dense subset $\mathrm{N} \hookrightarrow \bar{\mathrm{P}}\backslash \mathrm{G}$ with $\nu = \lceil r \rceil - 1 < r$. This is seen as follows: We firstly have to convince ourselves that indeed $\{1, x, \ldots, x^\nu\} \subset \mathrm{I}(\chi)^{\mathscr{C}^r}$. Let $\mathrm{F} \in \mathrm{I}(\chi)^{\mathscr{C}^r}$. Put $f := \mathrm{F}_{|\mathrm{N}} \colon \mathrm{N} \to \mathbf{E}$ and $f^w \colon \mathrm{N}w \to \mathbf{E}$ with $f^w := f(\cdot w)$ its translate. Then identifying their domains by $\mathrm{N} = \mathbb{Q}_p = \mathrm{N}w$ we compute that $f^w_{|\mathbb{Q}_p^*}$ is given in terms of f by

$$f^w(x) = \theta_1/\theta_2(x)(-x)^k f(1/x).$$

This map is locally constant on $\mathbb{Q}_p^*$ and for $f = 1, \ldots, x^\nu$ can be shown to extend to a $\mathscr{C}^r$-function at 0. (Use the criterion by the Taylor polynomial Formula in [Nag11][Chapter I].) Therefore $1, \ldots, x^\nu \in \mathrm{I}(\chi)^{\mathscr{C}^r}$.

Let $i \in \{0, \ldots, \nu\}$. Let $t = \left(\begin{smallmatrix} p^{-n} & \\ & 1 \end{smallmatrix}\right)$. Then $1_{\mathbb{Z}_p} x^{i\,t} = \chi(t) p^{ni} \mathbf{1}_{p^{-n}\mathbb{Z}_p} x^i = \lambda^n \mathbf{1}_{p^{-n}\mathbb{Z}_p} x^i \in \mathbf{L}_u$ with $\lambda := p^i/\alpha$ and, since $i \leq \nu < r$, we have $|\lambda| > 1$. Thus we find $(f_n)_{n\in\mathbb{N}}$ with $f_n := 1_{p^{-n}\mathbb{Z}_p} x^i \in \mathbf{L}_u$, converging to $f := x^i$ in $\mathrm{I}(\chi)^{\mathscr{C}^r}$, to be a zero sequence in $\widehat{\mathrm{I}}(\chi)$ and thus $1, \ldots, x^\nu \in \mathrm{L}$.

On the other hand, we note that by the relation $x^i = 0$ for $i = 0, \ldots, \nu$ in $\widehat{\mathrm{I}}(\chi)$, we obtain $\mathrm{I}(\chi)(\mathrm{N}_0) \ni \mathbf{1}_{\mathbb{Z}_p} x^i = f \in \mathrm{I}(\chi)(\mathrm{N}_1 w)$. Because $\mathrm{I}(\chi)(\mathrm{N}_0)$ is unitary under $\mathrm{T}^+\mathrm{N}_0$

with $T^+ = \{t = \begin{pmatrix} a & \\ & d \end{pmatrix} \in T \colon |a| \leq |d|\}$ whereas $I(\chi)(N_1 w)$ is unitary under $T^- N_1^{*-1}$ with $T^- = \{t = \begin{pmatrix} a & \\ & d \end{pmatrix} \in T : |a| \geq |d|\}$ and $N_1^* = N_1 - \{1\}$, we find $1_{\mathbb{Z}_p}, \ldots, 1_{\mathbb{Z}_p} x^\nu$ to be unitary unter $TN = T^+ T^- N_0 N_1^{*-1}$. As the $\mathbf{E}[P]$-module generated by $1_{\mathbb{Z}_p}, \ldots, 1_{\mathbb{Z}_p} x^\nu$ is dense inside $\widehat{I}(\chi)$, we obtain that $I(\chi)^{\mathscr{C}^r}/L$ is unitary under $P = TN$. Therefore $I(\chi)^{\mathscr{C}^r}/L \to \widehat{I}(\chi)$ is an isomorphism, the inverse mapping given by the universal property of $\widehat{I}(\chi)$.

4 Definition of R-times differentiable functions for $R \in \mathbb{R}_{\geq 0}$

Comparison with the Archimedean situation

We want to show how the classic notion of differentiability over the real numbers compares to the one over non-Archimedeanly valued fields. We will then introduce the classical approach to iterated differentiability over non-Archimedan vector spaces due to Schikhof.

Let $\mathbf{K}$ be a complete valued field, $X \subseteq \mathbf{K}$ an open subset and $f \colon X \to \mathbf{K}$ some function. Recall the function

$$f^{]1[}(x, y) := \frac{f(x) - f(y)}{x - y} \quad \text{for all distinct } x, y \in X.$$

Proposition 4.1. *Let $\mathbf{K} = \mathbb{R}$. Then f' is continuous if and only if $f^{]1[}$ extends to a continuous functions $f^{[1]} \colon X \times X \to \mathbf{K}$.*

Proof: A direct consequence of the mean value theorem. □

In general, a lot of fundamental facts in differential calculus over $\mathbb{R}$ depends on the intermediate value theorem.

The intermediate value theorem implies the

(i) Mean value theorem, which in turn implies
  - (*a*) the Local invertibility of functions with nonvanishing differential
  - (*b*) the fact that a function of many variables is continuously differentiable if and only if it is partially continuously differentiable.

(ii) The fundamental theorem of calculus, which in turn implies
  - (*a*) the Completeness of $\mathscr{C}^1(X, \mathbb{R})$.

To compensate for the lack of the intermediate value theorem over complete non-Archimedeanly valued fields — which in fact renders all of these last consequences with the common Archimedean definition of differentiability false (With the exception of 1b), which I am not sure about for the moment) - we turn Proposition 4.1 into a definition.

Definition. A function $f\colon \mathrm{X} \to \mathbf{K}$ is **continuously differentiable** if

$$f^{[1]}(x,y) := \frac{f(x) - f(y)}{x - y} \quad \text{with } x, y \in \mathrm{X} \text{ distinct}$$

extends to a continuous function $f^{[1]}\colon \mathrm{X} \times \mathrm{X} \to \mathbf{K}$.

Remark. The space of continuously differentiable functions is then as before endowed with a natural norm. In fact by this definition all of the above final consequences are easily shown to remain true.

Problem: Already the first differential quotient $f^{[1]}$ of a one-variable function is a function of two variables. (In contrast to the situation over an Archimedean field.) Therefore in order to iterate this definition with the aim to obtain a notion of higher differentiability for one-variable functions, we must already have established the definition of differentiability for a function of many variables.

Conceptual approach

Let $\mathrm{V} = \mathbf{K}^d$ and $\mathbf{E}$ be a $\mathbf{K}$-Banach space. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the canonical basis of V.

Definition. Let $\mathrm{X} \subseteq \mathrm{V}$ be open. Define for all $x + h, x \in \mathrm{X}$ with $h \in \mathbf{K}^{*d}$ the function

$$f^{]1[} : (x + h, x) \mapsto \mathrm{A} \in \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$$

by

$$\mathrm{A} h_k \cdot \boldsymbol{e}_k = f(x + h_1 \boldsymbol{e}_1 + \cdots + h_{k-1} \cdot \boldsymbol{e}_{k-1} + h_k \cdot \boldsymbol{e}_k) - f(x + h_1 \boldsymbol{e}_1 + \cdots + h_{k-1} \cdot \boldsymbol{e}_{k-1})$$

for $k = 1, \ldots, d$. Then f is a $\mathscr{C}^1$-function if and only if $f^{]1[}$ extends to a continuous function $f^{[1]}\colon \mathrm{X} \times \mathrm{X} \to \mathbf{E}$.

Example. Let $f\colon \mathrm{X}' \times \mathrm{X}'' \to \mathbf{K}$ with $\mathrm{X}', \mathrm{X}'' \subset \mathbf{K}$ open. Then

$$f^{]1[} = (f^{]1,0[}, f^{]0,1[})$$

with

$$f^{]1,0[}(x + h, x) = \frac{f(x' + h', x'') - f(x)}{h'} \quad \text{and} \quad f^{]0,1[}(x + h, x) = \frac{f(x', x'' + h'') - f(x)}{h''}$$

for $x + h, x \in \mathrm{X}' \times \mathrm{X}''$ with $h = (h', h'') \in \mathbf{K}^* \times \mathbf{K}^*$.

Then we can obtain a notion of ν-fold differentiability for $\nu \geq 0$ as follows: Let $f \in \mathscr{C}^1(\mathrm{X}, \mathbf{E})$ and let us regard the function $f^{[1]}$. Its domain $\mathrm{X} \times \mathrm{X}$ is again a $\mathbf{K}$-vector space with canonical choice of basis and its range $\mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ again $\mathbf{K}$-Banach space. We can therefore iterate this definition by applying it to $f^{[1]}$. That is, we define $f \in \mathscr{C}^2(\mathrm{X}, \mathbf{E})$ if and only if $f^{[1}$ exists and

$$f^{]2[} = (f^{[1]})^{]1[} \colon (\mathrm{X}^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E}), \mathbf{E})$$

extends to a continuous function $f^{[2]}$ on $\mathrm{X}^{[2]}$. (Here $\mathrm{X}^{]1[} = \{(x+h, x) \in \mathrm{X}^2 : h \in \mathbf{K}^{*d}\}$.) We aim to give a definition of r-fold differentiability. Write $r = \nu + \rho \geq 0$ with $\nu \in \mathbb{N}$ and $\rho \in [0, 1[$. Then we define ρ-fold differentiability by a strengthened Lipschitz-continuity condition as follows.

Definition 4.2. Let $\mathrm{A} \subseteq \mathrm{X}$ and $f \colon \mathrm{A} \to \mathbf{E}$. Then f is $\mathscr{C}^\rho$ at a point $a \in \mathrm{X}$ if for all $\varepsilon > 0$ there exits $\mathrm{U}_\varepsilon \ni a$ such that

$$\|f(x) - f(y)\| \leq \varepsilon \cdot \|x - y\|^\rho \quad \text{for all } x, y \in \mathrm{U}_\varepsilon \cap \mathrm{A}.$$

Then we can define r-fold differentiability by demanding the ν-th iterated difference quotient $f^{]\nu[}$ to be $\mathscr{C}^\rho$ everywhere.

Schikhof's observation

This definition can also be given point-wise and more concisely by taking into account the symmetry properties of the difference quotients as observed by Schikhof. We will see that for a symmetric function we are hence brought down to checking solely partial differentiability in its first coordinate, reducing an exponential growth of parameters to a linear one.

Remark 4.3. Let $\mathrm{X} \subseteq \mathbf{K}^d$ be open and let $\mathrm{F} \colon \mathrm{X} \times \mathrm{X} \to \mathbf{E}$ be a symmetric function. Then $\mathrm{F} \in \mathscr{C}^1$ if $\mathrm{A} := \mathrm{F}^{]1,0[}(x+h, x; y) \in \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ defined for all $x' + h, x' \in \mathrm{X}, x'' \in \mathrm{Y}$ with $h \in \mathbf{K}^{*d}$ by

$$\mathrm{A} \cdot h_k \boldsymbol{e}_k := f(x' + h_1 \boldsymbol{e}_1 + \cdots + h_{k-1} \boldsymbol{e}_{k-1} + h_k \boldsymbol{e}_k, x'') - f(x' + h_1 \boldsymbol{e}_1 + \cdots + h_{k-1} \boldsymbol{e}_{k-1}, x'')$$

extends to a continuous function $\mathrm{F}^{[1,0]} \colon (\mathrm{X} \times \mathrm{X}) \times \mathrm{X} \to \mathbf{E}$.

The following is in fact the definition of two-fold differentiability as employed in [Sch84].

Example. Let $f \in \mathscr{C}^1(\mathrm{X}, \mathbf{E})$ and put $\mathrm{F} = f^{[1]}$. Then $\mathrm{F} \in \mathscr{C}^1(\mathrm{X} \times \mathrm{X}, \mathbf{E})$ if and only if $f^{]2[}$ defined by

$$f^{]2[}(x, y; z) = \frac{f^{[1]}(x, z) - f^{[1]}(y, z)}{x - y}$$

for all distinct $x, z \in \mathrm{X}$ and $y \in \mathrm{X}$ extends to a continuous function $f^{[2]} \colon (\mathrm{X} \times \mathrm{X}) \times \mathrm{X} \to \mathbf{E}$.

We will now give the explicit definition of an r-times differentiable function for the case of two variables, $f \colon \mathrm{X} \times \mathrm{Y} \to \mathbf{E}$ with $\mathrm{X}, \mathrm{Y} \subseteq \mathbf{K}$ open.

Definition.
- Let $\mathrm{X} \subseteq \mathbf{K}$ open. Then we define $\mathrm{X}^{[k]} = \mathrm{X}^{\{0,\ldots,k\}}$ and $\mathrm{X}^{]k[} := \{\boldsymbol{x} \in \mathrm{X}^{\{0,\ldots,k\}} : x_i, x_j \text{ pairwise distinct }\} \subseteq \mathrm{X}^{[k]}$.

- We define $f^{[i,j]} \colon \mathrm{X}^{]i[} \times \mathrm{Y}^{]j[} \to \mathbf{E}$ recursively by

$$
\begin{aligned}
& f^{]i,j[}(x_0, x_1, x_2, \ldots, x_i; y_0, \ldots, y_j) \\
:= & \frac{f^{]i-1,j[}(x_0, x_2, \ldots, x_i; y_0, \ldots, y_j) - f^{]i,j[}(x_1, x_2, \ldots, x_i; y_0, \ldots, y_j)}{x_0 - x_1} \\
= & \frac{f^{]i,j-1[}(x_0, \ldots, x_i; y_0, y_2, \ldots, y_j) - f^{]i,j[}(x_0, \ldots, x_i; y_1, y_2, \ldots, y_j)}{y_0 - y_1}.
\end{aligned}
$$

- Then we say that f is $\mathscr{C}^r$ at a point $(a,b) \in \mathrm{X} \times \mathrm{Y}$ if $f^{]i,j[}$ is $\mathscr{C}^\rho$ at $(\vec{a}; \vec{b}) = (a, \ldots, a; b, \ldots, b) \in \mathrm{X}^{[i]} \times \mathrm{X}^{[j]}$ for all i, j with $i + j = \nu$.

- We say that f is a $\mathscr{C}^r$-function if f is $\mathscr{C}^r$ everywhere and put $\mathscr{C}^r(\mathrm{X} \times \mathrm{Y}, \mathbf{E}) = \{\mathscr{C}^r\text{-functions } f \colon \mathrm{X} \times \mathrm{Y} \to \mathbf{E}\}$.

5 Investigation of $\mathscr{C}^r$-functions

Fundamental properties

Proposition 5.1. *Let* $\mathrm{X} \subseteq \mathbf{K}^d$ *and* $\mathrm{Y} \subseteq \mathbf{K}^e$ *be open and* $f \in \mathscr{C}^r(\mathrm{X}, \mathrm{Y}), g \in \mathscr{C}^r(\mathrm{Y}, \mathbf{E})$. *Then* $g \circ f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ *if* $r \geq 1$. *(If* $r < 1$, *then the same conclusion holds provided either function is Locally Lipschitz-continuous.)*

Proof: By the conceptual introduction in the previous chapter, this follows from the fact that the composed function's difference quotient is given by the composition of the linear morphisms given by their difference quotients, that is, a matrix product. Since $\mathscr{C}^\rho$-functions are closed under pointwise products, the proposition follows. □

As a Corollary we find that there is a well behaved notion of a $\mathscr{C}^r$-manifold.

Corollary 5.2. *Let* $r \geq 1$. *Let* M *be a topological manifold with a* $\mathscr{C}^r$*-atlas (that is, a covering by charts whose transition maps are given by* $\mathscr{C}^r$*-functions). Then there is a maximal* $\mathscr{C}^r$*-atlas of* M *and a function* $f \colon \mathrm{M} \to \mathbf{E}$ *is a* $\mathscr{C}^r$*-function with respect to the maximal* $\mathscr{C}^r$*-atlas if and only if it is a* $\mathscr{C}^r$*-function with respect to any* $\mathscr{C}^r$*-atlas.*

Proof: Follows from the following two facts:

(i) The definition is pointwise, in particular local.

(ii) The $\mathscr{C}^r$-functions are closed under composition.

□

Lemma 5.3. *We find* $f \in \mathscr{C}^r(\mathrm{X} \times \mathrm{Y}, \mathbf{E})$ *if and only if* $f^{]i,j[} \colon \mathrm{X}^{]i[} \times \mathrm{Y}^{]j[} \to \mathbf{E}$ *extends to a* $\mathscr{C}^\rho$*-function on* $\mathrm{X}^{[i]} \times \mathrm{X}^{[j]}$ *for all* i, j *with* $i + j = \nu$.

Remark 5.4. By induction, using the fact that if $\mathrm{A} \subseteq \mathrm{X}$ is dense and f is a $\mathscr{C}^\rho$-function (that is, $\mathscr{C}^\rho$ everywhere), then it extends to a $\mathscr{C}^\rho$-function $\mathrm{F} \colon \mathrm{X} \to \mathbf{E}$.

Corollary 5.5. *There is a natural topology on the space of* $\mathscr{C}^r$*-functions which renders it to a complete locally convex* $\mathbf{K}$*-algebra.*

Proof: By the supremum norm of the partial difference quotients $f^{[i,j]}$ over compact subsets. □

Remark 5.6. For the following propositions to be meaningful, we note $\mathscr{C}^{\mathrm{la}}(\mathrm{X}, \mathbf{E}) \subseteq \mathscr{C}^\infty(\mathrm{X}, \mathbf{E}) := \cap_{r \geq 0} \mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

Proposition 5.7. *The locally polynomial functions of degree* $\leq \nu$ *are dense inside* $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

Corollary 5.8. *The polynomial functions are dense inside* $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

Definition 5.9. Define $\mathscr{D}^r(\mathrm{X}, \mathbf{E})$ as the dual of $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$. Let $\mathscr{D}(\mathrm{X}, \mathbf{E}) = \varinjlim \mathscr{D}^r(\mathrm{X}, \mathbf{E})$ with the transition maps $\mathscr{D}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{D}^s(\mathrm{X}, \mathbf{E})$ for $r \leq s$ given by restriction (and injective by the previous propositions). It is given by all distributions $\mu \colon \mathscr{C}^\infty(\mathrm{X}, \mathbf{E}) \to \mathbf{K}$ extending onto some $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ for some $r \geq 0$.

Proposition 5.10. *Let* X *be a group with* $\mathscr{C}^\infty$*-multiplication so that* $\mathscr{D}(\mathrm{X}, \mathbf{E})$ *is a* $\mathbf{K}$*-algebra by the convolution product equipped with a natural filtration. Then* $\mathscr{D}(\mathrm{X}, \mathbf{E})$ *is a filtered* $\mathbf{K}$*-algebra.*

Other more explicit descriptions

The Mahler basis. Let $\mathscr{C}(\mathbb{Z}_p, \mathbf{o}_\mathbf{E})^*$ be the continuous dual of the space $\mathscr{C}(\mathbb{Z}_p, \mathbf{o}_\mathbf{E})$ of continuous functions $f \colon \mathbb{Z}_p \to \mathbf{o}_\mathbf{E}$. Then $\mathscr{C}^{\mathrm{lc}}(\mathbb{Z}_p, \mathbf{o}_\mathbf{E}) := \{ f \colon \mathbb{Z}_p \to \mathbf{o}_\mathbf{E} \text{ loc. cst.} \} \subseteq \mathscr{C}(\mathbb{Z}_p, \mathbf{o}_\mathbf{E})$ is dense and corresponding to $\mathscr{C}^{\mathrm{lc}}(\mathbb{Z}_p, \mathbf{o}_\mathbf{E}) = \cup_{n \geq 0} \mathbf{o}_\mathbf{E}[\mathbb{Z}/p^n\mathbb{Z}]$, we have $\mathscr{C}(\mathbb{Z}_p, \mathbf{o}_\mathbf{E})^* = \mathbf{o}_\mathbf{E}[[\mathbb{Z}_p]]$. The *Iwasawa* isomorphism

$$\begin{aligned} \mathbf{o}_\mathbf{E}[[\mathbb{Z}_p]] &\to \mathbf{o}_\mathbf{E}[[\mathrm{X}]] \\ \mathbf{1} &\mapsto 1 + \mathrm{X} \end{aligned}$$

gives an isomorphism of topological rings where the left-hand side is endowed with the topology of pointwise convergence. Define an $\mathbf{o}_\mathbf{E}$-Banach space to be a complete normed torsionfree $\mathbf{o}_\mathbf{E}$-module. Given an $\mathbf{o}_\mathbf{E}$-Banach space V let

$$\mathrm{V}^d := \mathrm{Hom}_{\mathbf{o}_\mathbf{E}-\mathrm{cts.}}(\mathrm{V}, \mathbf{o}_\mathbf{E})$$

be the compact torsionfree $\mathbf{o_E}$-module of $\mathbf{o_E}$-linear continuous functionals with its topology of pointwise convergence. Given a torsionfree compact $\mathbf{o_E}$-module M let

$$\mathrm{M}^d := \mathrm{Hom}_{\mathbf{o_E}}(\mathrm{M}, \mathbf{o_E})$$

be the $\mathbf{o_E}$-Banach space whose norm is given by $\|\cdot\|_{\sup}$. Then these functors are (quasi-)inverses between the categories

$$\{\text{ torsionfree compact } \mathbf{o_E}\text{-modules }\} \leftrightarrow \{\ \mathbf{o_E}\text{-Banach spaces V with } |\mathbf{o_E}| \supseteq \|\mathrm{V}\|\}.$$

Let $\mathscr{C}_0(\mathbb{N}, \mathbf{o_E})$ be the zero sequences with entries in $\mathbf{o_E}$. We conclude

$$\begin{array}{ccccc} \mathscr{C}(\mathbb{Z}_p, \mathbf{o_E}) & & & & \mathscr{C}_0(\mathbb{N}, \mathbf{o_E}) \\ \downarrow \cdot^d & & & & \downarrow \cdot^d \\ \mathscr{C}(\mathbb{Z}_p, \mathbf{o_E})^d & \tilde{\to} & \mathbf{o_E}[[\mathbb{Z}_p]] & \tilde{\to} & \mathbf{o_E}[[\mathrm{X}]] \\ \downarrow \cdot^d & & & & \downarrow \cdot^d \\ \mathscr{C}(\mathbb{Z}_p, \mathbf{o_E}) = \mathscr{C}(\mathbb{Z}_p, \mathbf{o_E})^{dd} & & \leftarrow & & \mathscr{C}_0(\mathbb{N}, \mathbf{o_E}) = \mathscr{C}_0(\mathbb{N}, \mathbf{o_E})^{dd}. \end{array}$$

The bottom isomorphism is then given by $e_n := (\dots, 0, 1, 0 \dots) \mapsto \binom{*}{n}$ with e_n being the sequence whose sole nonzero entry is 1 at the n-th position. We can therefore write $f = \sum_n a_n \binom{*}{n} \in \mathscr{C}(\mathbb{Z}_p, \mathbf{E})$ and call $\{a_n\}$ its *Mahler coefficients*.

The following proposition identifies the condition on the Mahler coefficients of the subset $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K}) \subseteq \mathscr{C}^0(\mathbb{Z}_p^d, \mathbf{K})$ under this isomorphism given by this orthogonal basis.

Proposition 5.11. *We find $f \in \mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ if and only if $|a_{\boldsymbol{n}}||\boldsymbol{n}|^r \to 0$ (with $|\boldsymbol{n}| = n_1 + \cdots + n_d$).*

Proof: Use that their $\mathbf{K}$-linear span given by all polynomial functions is dense by the previous general result. Their orthogonality is quickly checked and their norm $\|\binom{*}{\boldsymbol{n}}\|_{\mathscr{C}^r}$ directly computed. □

Remark. In fact, in [BB10], the authors defined $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{E})$ for $r \in \mathbb{R}_{\geq 0}$ by demanding $|a_n|n^r \to 0$.

The Taylor polynomial.

Proposition 5.12. *Let $\mathrm{X} \subseteq \mathbf{K}$ be open and $f : \mathrm{X} \to \mathbf{K}$. Then $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K})$ if and only if there are functions $\mathrm{D}_0 f, \dots, \mathrm{D}_\nu f : \mathrm{X} \to \mathbf{K}$ such that putting*

$$f(x+y) = \mathrm{D}_0 f(x) + \mathrm{D}_1 f(x) y + \cdots + \mathrm{D}_\nu f(x) y^\nu + \mathrm{R}_\nu f(x+y, x)$$

there is for all $a \in \mathrm{X}$ and $\varepsilon > 0$ some neighborhood $\mathrm{U}_\varepsilon \ni a$ such that

$$|\mathrm{R}_\nu f(x+y, x)| \leq \varepsilon |y|^r \quad \textit{for all } x+y, x \in \mathrm{U}_\varepsilon.$$

Proof: Express $f^{[\nu]}$ through Taylor polynomials. □

Remark. In fact, one can show this definition to be equivalent to the one given in [Col10] over the domain $X = \mathbb{Z}_p$.

Definition 5.13. Let $X \subseteq \mathbf{K}^d$ be an open subset and $k \in \{1, \ldots, d\}$. We will speak of a **$\mathscr{C}_T^{r \cdot e_k}$-function** $f : X \to \mathbf{K}$ if there are *continuous* functions $\mathscr{D}_0 f, \mathscr{D}_{1 \cdot e_k} f, \ldots, \mathscr{D}_{\nu \cdot e_k} f : X \to \mathbf{K}$ such that if one defines $R_{\nu \cdot e_k} f : X^{[e_k]} \to \mathbf{K}$ on $X^{[e_k]} := \{(x; t) \in X \times \mathbf{K} \text{ with } x + t \cdot e_k \in X\}$ by

$$R_{\nu \cdot e_k} f(x; t) := f(x + t \cdot e_k) - \sum_{i=0,\ldots,\nu} \mathscr{D}_{i \cdot e_k} f(x) t^i,$$

then for every point $a \in X$ and any $\varepsilon > 0$, there will exist a neighborhood $U \ni a$ such that

$$|R_{\nu \cdot e_k} f(x; t)| \leq \varepsilon |t|^r \quad \text{for all} \quad x + t \cdot e_k, x \in U.$$

We will denote the set of all $\mathscr{C}_T^{r \cdot e_k}$-functions $f : X \to \mathbf{K}$ by $\mathscr{C}_T^{r \cdot e_k}(X, \mathbf{K})$.

Corollary 5.14. *Let $X \subseteq \mathbb{Q}_p^d$ be open. Then $f \in \mathscr{C}^r(X, \mathbf{K})$ if and only if $f \in \mathscr{C}_T^{r \cdot e_1}(X, \mathbf{K}) \cap \ldots \cap \mathscr{C}_T^{r \cdot e_d}(X, \mathbf{K})$.*

Proof: (i) We have $f \in \mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ if and only if $f = \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \binom{*}{\boldsymbol{n}}$ with $|a_{\boldsymbol{n}}| |\boldsymbol{n}|^r \to 0$.

(ii) We find $|a_{\boldsymbol{n}}| |\boldsymbol{n}|^r \to 0$ if and only if $|a_{\boldsymbol{n}}| n_1^r \to 0, \ldots, |a_{\boldsymbol{n}}| n_d^r \to 0$.

(iii) We have $\mathscr{C}_T^r(X, \mathbf{K}) = \mathscr{C}^r(X, \mathbf{K})$ and $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) = \{f = \sum_n a_n \binom{*}{n} : |a_n| n^r \to 0\}$.

(iv) We have

$$\mathscr{C}_T^{(r,0,\ldots,0)} = \mathscr{C}_T^r \hat{\otimes} [\mathscr{C}^0 \hat{\otimes} \cdots \hat{\otimes} \mathscr{C}^0] = \{f = \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \binom{*}{\boldsymbol{n}} : |a_{\boldsymbol{n}}| n_1^r \to 0\}$$

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