Fourier Theory and Fractional Differentiability

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Motivation

Let $\mathbb{Q}_p$ be the p-adic numbers with $\mathbb{Z}_p$ its ring of integers, and $\mathbf{E}$ a complete non-Archimedeanly non-triv. valued field. Let

- $\mathcal{C}^0(\mathbb{Z}_p, \mathbf{E}) = \{$ all continuous $f : \mathbb{Z}_p \to \mathbf{E}\}$, and
- $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{E}) = \mathcal{C}^0(\mathbb{Z}_p, \mathbf{E}), \mathbf{E})^*$ continuous $\mathbf{E}$-linear dual.

Every continuous $f : \mathbb{Z}_p \to \mathbf{E}$ is uniformly approximated by locally constant functions $f_n \in \mathbf{E}[\mathbb{Z}_p/p^n\mathbb{Z}_p]$ for $n \in \mathbb{N}$,
$\overset{\text{dualizing}}{\Rightarrow}$ Isomorphism of $\mathbf{E}$-algebras $\mathbf{E}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{D}^0(\mathbb{Z}_p, \mathbf{E})$. By the Iwasawa isomorphism of topological $\mathbf{E}$-algebras

$$\begin{aligned} \mathbf{E}[[\mathbb{Z}_p]] &\xrightarrow{\sim} \mathbf{E}[[X]] \\ \mathbf{1} + 1 &\mapsto X \end{aligned}$$

we infer $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{E}) \xrightarrow{\sim} \mathbf{E}[[X]]$ and by Schikhof duality an orthonormal basis $\{$ zero sequences in $\mathbf{E}\} \xrightarrow{\sim} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{E})$. These are the **Mahler polynomials** $\binom{x}{n} = x(x-1)\cdots(x-n)/n!$.

Letting $\mathcal{C}^{\text{loc.pol.}}(\mathbb{Z}_p, \mathbf{E}) = \{f : \mathbb{Z}_p \to \mathbf{E} : f \text{ locally polynomial } \}$, the authors in [Berger and Breuil(2010)] observed that

$$\widehat{\mathcal{C}^{\text{loc.pol.}}}(\mathbb{Z}_p, \mathbf{E}) = \{f : \mathbb{Z}_p \to \mathbf{E} : f = \sum a_n \binom{*}{n} \text{ with } |a_n| n^r \to 0\}$$

for an $r \in \mathbb{R}_{\geq 0}$ with respect to a certain completion $\hat{\cdot}$ of $\mathcal{C}^{\text{loc.pol.}}(\mathbb{Z}_p, \mathbf{E})$ which arises from representation theory.

Definition

Given continuous $f : \mathbb{Z}_p \to \mathbf{E}$, it was well known that

$$|a_n| n^1 \to 0 \quad \Leftrightarrow \quad f \text{ is strictly differentiable.}$$

Let's thus call such f with $|a_n| n^r \to 0$ **r-times differentiable**.

This raises the following ...

Question

- How to extend the notion of r-fold differentiability to general non-Archimedeanly valued domains such as open subsets in $\mathbf{K}^d$ or differentiable manifolds?
- How to extend the notion of a Mahler basis to a finite extension $\mathbf{o}$ of $\mathbb{Z}_p$?
- Describe the $\mathcal{C}^r$-functions with respect to this basis!

Differentiability - conceptually ...

To compensate for the absence of an analogue of the intermediate value theorem, we need to render our differentiability condition more rigorous to preserve basic properties such as the completeness of our function space in the non-Archimedean case.

Definition 2.1

Let $f : X \to \mathbf{E}$ be a function defined on an open subset $X \subseteq V$. Then f is called *differentiable* or $\mathcal{C}^1$ in the point $a \in X$ if there exists a linear map $D_a f : V \to W$ such that for every $\varepsilon > 0$ there is a neighborhood $U \ni a$ in X with

$$\|f(x+h) - f(x) - D_a f \cdot h\| \leq \varepsilon \|h\| \quad \text{for all } x+h, x \in U.$$

In order to iterate this notion, we have to choose coordinates. Let $V = \mathbf{K}^d$ and $\mathbf{E}$ be a $\mathbf{K}$-Banach space. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the canonical basis of V.

Definition

Let $X \subseteq V$ be open. Define for all $x+h, x \in X$ with $h \in {\mathbf{K}^*}^d$ the function

$$f^{]1[} : (x+h, x) \mapsto A \in \mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$$

by specifying, for each coordinate $k = 1, \ldots, d$, the linear map A through

$$A h_k \cdot \boldsymbol{e}_k = f(h_1 \boldsymbol{e}_1 + \cdots + h_{k-1} \cdot \boldsymbol{e}_{k-1} + h_k \cdot \boldsymbol{e}_k) - f(h_1 \boldsymbol{e}_1 + \cdots + h_{k-1} \cdot \boldsymbol{e}_{k-1})$$

for $k = 1, \ldots, d$. Then f is a $\mathcal{C}^1$-function if and only if $f^{]1[}$ extends to a continuous function $f^{[1]} : X \times X \to \mathbf{E}$.

From this, we obtain a notion of ν-fold differentiability for $\nu \geq 0$ as follows: Let $f \in \mathcal{C}^1(X, \mathbf{E})$ and let us regard the function $f^{[1]}$. Then

- its domain $X \times X$ lies again in a finite dimensional $\mathbf{K}$-vector space $V \times V$ with canonical choice of basis, and
- its range $\mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$ is again $\mathbf{K}$-Banach space.

We can therefore iterate this definition by applying it to $f^{[1]}$.

Definition

We have $f \in \mathcal{C}^2(X, \mathbf{E})$ if $f^{[1]}$ exists and

$$f^{]2[} = (f^{[1]})^{]1[} : (X^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(V \times V, \mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E}))$$

extends to a continuous function $f^{[2]}$ on $X^{[2]}$. (Here $X^{]1[} = \{(x+h, x) \in X^2 : h \in {\mathbf{K}^*}^d\}$.)

Recall our aim to be a definition of r-fold differentiability for a real number $r \in \mathbb{R}_{\geq 0}$. To this end, write $r = \nu + \rho \geq 0$ with $\nu \in \mathbb{N}$ and $\rho \in [0, 1[$. Then we define ρ-fold differentiability by a strengthened Lipschitz-continuity condition as follows.

Definition

Let $A \subseteq X$ and $f : A \to \mathbf{E}$. Then f is $\mathcal{C}^\rho$ at a point $a \in X$ if for all $\varepsilon > 0$ there exits $U_\varepsilon \ni a$ such that

$$\|f(x) - f(y)\| \leq \varepsilon \cdot \|x - y\|^\rho \quad \text{for all } x, y \in U_\varepsilon \cap A.$$

Then $f \in \mathcal{C}^\rho(A, \mathbf{E})$ if f is $\mathcal{C}^\rho$ at every point $a \in A$.

Definition

Let $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu \in \mathbb{N}$ and $\rho \in [0, 1[$. Then $f \in \mathcal{C}^r(X, \mathbf{E})$ if $f \in \mathcal{C}^{\nu-1}(X, \mathbf{E})$ and $f^{]\nu[} : (X^{[\nu-1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(V^{[\nu-1]} \times V^{[\nu-1]}, \mathbf{E}^{[\nu-1]})$ extends to a $\mathcal{C}^\rho$-function $f : X^{[\nu]} \to \mathbf{E}^{[\nu]}$.

… and Differentiability classically

By an observation of Schikhof, the difference quotient $f^{]1[}$ is a symmetric function, and for such we are brought down to checking solely partial differentiability in its first coordinate, reducing an exponential growth of parameters to a linear one.

Definition

Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{E}$ a mapping thereon. For $\nu \in \mathbb{N}$ put

$$X^{<\nu>} = X^{\{0,\dots,\nu\}} \quad \text{and} \quad X^{>\nu<} = \{(x_0, \dots, x_\nu) : x_i \neq x_j \text{ if } i \neq j\}.$$

The **ν-th difference quotient** $f^{>\nu<} : X^{>\nu<} \to \mathbf{K}$ of a function $f : X \to \mathbf{E}$ is inductively given by $f^{>0<} := f$ and for $n \in \mathbb{N}$ and $(x_0, \dots, x_\nu) \in X^{>\nu<}$ by

$$f^{>\nu<}(x_0, \dots, x_\nu) = \frac{f^{>\nu-1<}(x_0, x_2, \dots, x_\nu) - f^{>\nu-1<}(x_1, x_2, \dots, x_\nu)}{x_0 - x_1}$$

Definition

Fix $r = \nu + \rho \in \mathbb{R}_{\geq 0}$. Let $X \subseteq \mathbf{E}$ and $f : X \to \mathbf{E}$ a mapping thereon.

(i) We will say that f is $\mathcal{C}^r$ (or r **times continuously differentiable**) at a point $a \in X$ if $f^{>\nu<} : X^{>\nu<} \to \mathbf{E}$ is $\mathcal{C}^\rho$ at $\vec{a} = (a, \ldots, a) \in X^{<\nu>}$.

(ii) Then f will be a $\mathcal{C}^r$**-function** (or an r**-times continuously differentiable function**) if f is $\mathcal{C}^r$ at all points $a \in X$. The set of all $\mathcal{C}^r$-functions $f : X \to \mathbf{K}$ will be denoted by $\mathcal{C}^r(X, \mathbf{E})$.

It can be shown, see [Nagel(2011), Proposition 2.5], that f is a $\mathcal{C}^r$-function if and only if $f^{<\nu>} : X^{>\nu<} \to \mathbf{E}$ extends to a $\mathcal{C}^\rho$-function $f^{<\nu>} : X^{<\nu>} \to \mathbf{E}$.

Motivation

Let $\mu : [0,1] \to \mathbb{R}_{\geq 0}$ be a measure. Then we can identify it by its **Fourier transform**

$$\hat{\mu} : \mathbb{R} \to \{ \text{ all characters } e^{2\pi\omega\cdot} \} \to \mathbb{C}.$$
$$e^{2\pi\omega\cdot} \mapsto \int e^{2\pi\omega x}\mu(dx)$$

Under this identification, we obtain

$$\text{Convolution product} \quad \leftrightarrow \quad \text{Pointwise multiplication}$$

Let $\mathbf{K}$ be a p-adic number field with $\mathbf{o}$ its ring of integers, and $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ its completed algebraic closure. We will do likewise for

$$\mathcal{D}^{\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p) = \{ \text{ all continuous linear } \int : \mathcal{C}^{\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p) \to \mathbb{C}_p \}.$$

The Fourier Transform over $\mathbb{Z}_p$

Let us define

$\mathcal{C}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p) = \{f : \mathbf{o} \to \mathbb{C}_p : f \text{ loc. def. by a conv. power series}\}$,

and its continuous dual (which can be thought of as integrals)

$$\mathcal{D}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p) = \mathcal{C}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)^*.$$

We let

$$\hat{\mathbf{o}} = \{ \text{ all locally analytic characters } \chi : \mathbf{o} \to \mathbb{C}_p^* \}.$$

They generate the complete topological **K**-vector space $\mathcal{C}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)$.

$$\mathcal{D}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p) \hookrightarrow \{ \text{ all functions } f : \hat{\mathbf{o}} \to \mathbb{C}_p \}$$
$$\mu \mapsto \mu_{|\hat{\mathbf{o}}}$$

Under this identification, we obtain

$$\text{Convolution product} \quad \leftrightarrow \quad \text{Pointwise multiplication}$$

$\mathbb{Q}_p$-analytic Characters of $\mathbb{Z}_p$

Proposition

Let $\mathbf{K} = \mathbb{Q}_p$*. Put* $\mathrm{B}_{<1}(1) = \{z \in \mathbb{C}_p : |z-1| < 1\}$*. As groups*

$$\mathrm{B}_{<1}(1) \xrightarrow{\sim} \hat{\mathbf{o}}$$

$$1 + z \mapsto (1+z)^x := \sum_{n \geq 0} \binom{x}{n} z^n.$$

We can thus view that $\hat{\mathbf{o}} = \mathrm{B}_{<1}$ as an analytic variety over $\mathbb{C}_p$.

Theorem (Amice)

We have an isomorphism of topological $\mathbb{C}_p$*-algebras*

$$\mathcal{D}^{\text{loc.an.}}_{\mathbb{Q}_p}(\mathbb{Z}_p, \mathbb{C}_p) \xrightarrow{\sim} \mathscr{O}(\mathrm{B}_{<1}) = \{\text{analytic fct's converging on } \mathrm{B}_{<1}\}$$

$$\mu \mapsto \mu_{|\hat{o}}.$$

Interlude: Formal Groups and ...

Let G be a (one-dimensional) Lie Group. Then, using a chart $U \to \mathrm{B}_{<1}(0)$ in a neighborhood U of $1 \in G$, the multiplication operation $\mu : G \times G \to G$ gives a power series $F(X, Y)$ subject to certain conditions arising from the group axioms.

Definition

A **formal group** $\mathscr{G}$ over $\mathbf{o}$ is then defined as such a power series in $\mathbf{o}[X, Y]$ and reversely, we can turn $\mathrm{B}_{<1}$ into a group by $xy = \mathbf{F}(x, y)$.

Example

Let $\mathbb{G}_m$ be the **multiplicative group** given by $F(X, Y) = (1 + X)(1 + Y) - 1$. It is the power series for the multiplication in $\mathbb{C}_p^*$ by the chart $x \mapsto x - 1 \in \mathrm{B}_{<1}$ in a neighorhood of 1.

Interlude: ... and Formal $\mathbf{o}$-modules

Definition

A **formal o-module** is then a formal group with an action of $\mathbf{o}$, i.e. a morphism $\mathbf{o} \to \operatorname{End}(\mathscr{G})$, denoted by $a \mapsto [a]$.

By Lubin Tate-theory, there is for any $\mathbf{o}$ a (basically unique) formal $\mathbf{o}$-module $\mathscr{G}$ and $T' = \operatorname{Hom}(\mathscr{G}, \mathbb{G}_m)$ is a free $\mathbf{o}$-module of rank one with generator t_0' say.

K-analytic Characters of o

Let us view $\mathscr{G}$ and $\mathbb{G}_m$ as living on the open unit ball $\mathrm{B}_{<1}$ in $\mathbb{C}_p$. Then we associate to $z \in \mathrm{B}_{<1}$

- its orbit map $o_z : \mathbf{o} \to \mathscr{G}$ given by $a \mapsto [a]z$,
- via $t_0' : \mathscr{G} \to \mathbb{G}_m$ the character $\chi_z : \mathbf{o} \to \mathbb{G}_m$ by $a \mapsto t_0'([a]z)$,
- and finally the character $\chi_z : \mathbf{o} \to \mathbb{C}_p^*$ by $a \mapsto t_0'([a]z) + 1$.

Proposition (Schneider and Teitelbaum)

We have a group homomorphism $\mathrm{B}_{<1} \xrightarrow{\sim} \hat{\mathbf{o}}$ *via* $z \mapsto \chi_z$.

Theorem (Schneider and Teitelbaum)

We have an isomorphism of topological $\mathbb{C}_p$*-algebras*

$$\mathcal{D}_{\mathbf{K}}^{\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p) \xrightarrow{\sim} \mathscr{O}(\mathrm{B}_{<1} / \mathbb{C}_p) = \{\text{analytic fct's conv. on } \mathrm{B}_{<1}\}$$

$$\mu \mapsto \mu_{|\hat{\mathbf{o}}}.$$

The Fourier basis

Thus there is $X \in \mathcal{D}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)$ such that every $\mu \in \mathcal{D}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)$ can be written $\mu = \sum_{n \geq 0} a_n X^n$. The **evaluation map** at the n-th coefficient

$$\mathrm{ev}_n : \mu \mapsto a_n$$

is a continuous $\mathbf{K}$-linear map

$$[\mathrm{ev}_n : \mathcal{D}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p) \to \mathbb{C}_p] \in \mathcal{D}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)^*.$$

By reflexivity, there is $P_n \in \mathcal{C}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)$ s.t. $\mathrm{ev}_n(\mu) = \mu(P_n)$.

Definition (Schneider and Teitelbaum)

The functions P_n constitute the *Fourier basis* of $\mathcal{C}^{\text{loc.an.}}_{\mathbf{K}}(\mathbf{o}, \mathbb{C}_p)$.

They are polynomial functions sharing many properties with the classic Mahler basis of binomial functions $\binom{x}{n}$ and coincide with them in case $\mathbf{K} = \mathbb{Q}_p$.

1 p-adic Differentiability

2 p-adic Fourier Theory

3 The Fourier basis of differentiable functions
- The outset
- Differentiability via Analyticity

The outset

Question 4.1

How can we describe the r-times differentiable functions $f : \mathbf{o} \to \mathbb{C}_p$ via this Fourier bases?

Proposition 4.2

The set $\mathcal{C}^{\text{loc.pol.}\leq r}(\mathbf{o}, \mathbb{C}_p)$ *of all locally* $\mathbf{K}$*-polynomial functions* $f : \mathbf{o} \to \mathbb{C}_p$ *of degree* $d \leq r$ *is dense in* $\mathcal{C}^r(\mathbf{o}, \mathbb{C}_p)$.

We obtain an injection $\mathcal{D}^r(\mathbf{o}, \mathbb{C}_p) \hookrightarrow \mathcal{D}^{\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p)$ of the duals. By the isom. $\mathcal{D}^{\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p) \xrightarrow{\sim} \mathscr{O}(\mathrm{B}_{<1} / \mathbb{C}_p)$, we have a mapping

$$\mathcal{D}^r(\mathbf{o}, \mathbb{C}_p) \hookrightarrow \mathscr{O}(\mathrm{B}_{<1} / \mathbb{C}_p).$$

Question (Dual of Question 4.1)

How can we describe its topological image?

Differentiability via Analyticity

Let $e_{n,e} = 1_{\pi^n \mathbf{o}} x^e$. Put $\rho = 1/|\pi| \geq 1$. One can show

$$\|e_{n,e}\|_{\mathcal{C}^r} = C \cdot \rho^{n(r-e)} = C \cdot \rho^{nr} \cdot \|e_{n,e}\|_{\mathcal{C}^{n-\text{loc.an.}}} \quad \text{with } C = \rho^{r-e}.$$

By translation invariance of these norms, this holds for all $e_{a,n,e} = e_{n,e}(\cdot - a) = 1_{a+\pi^n \mathbf{o}}(x-a)^e$ as well. The set $\{e_{a,n,e}\}$ of all these functions contains an orthogonal basis of $\mathcal{C}^{n-\text{loc.pol.}}(\mathbf{o}, \mathbb{C}_p)$, the van der Put-base, and $\mathcal{C}^{n-\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p)$.

Corollary (Above observation and Proposition 4.2)

The operator norm $\|\cdot\|_{\mathcal{D}^r}$ on $\mathcal{D}^r(\mathbf{o}, \mathbb{C}_p) \hookrightarrow \mathcal{D}^{\text{loc.an.}}(\mathbf{o}, \mathbb{C}_p)$ is equivalent to the one given by

$$\sup\{\rho^{-nr} \|\cdot\|_{\mathcal{D}^{n-\text{loc.an.}}} : n \in \mathbb{N}\}.$$

Proposition

Let $0 < \sigma_0 < 1$ *be suitable and* $0 < \sigma_n = \sigma_0^{-p^{dn}} \nearrow 1$. *We identify* $\mathrm{B}_{<1} \xrightarrow{\sim} \hat{\mathbf{o}}$ *and put* $\hat{\mathbf{o}}(\sigma_n) = \mathrm{B}_{\leq \sigma_n}$. *Then* $\hat{\mathbf{o}}(\sigma_n) \subseteq \{f \in \mathcal{C}^{n-\mathrm{loc.an.}}(\mathbf{o}, \mathbb{C}_p) : \|f\| \leq 1\}$. *Then there is* $C > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every* $\mu \in \mathcal{D}^{\mathrm{loc.an.}}(\mathbf{o}, \mathbb{C}_p)$ *holds*

$$\|\mu_{|\hat{\mathbf{o}}(\rho_n)}\|_{\sup} \leq \|\mu\|_{\mathcal{D}^{n-\mathrm{loc.an.}}} \leq C\|\mu_{|\hat{\mathbf{o}}(\rho_{n+n_0})}\|_{\sup}.$$

Corollary

The natural norm on $\mathcal{D}^r(\mathbf{o}, \mathbb{Z}_p)$ *translates* $\mathcal{D}^r(\mathbf{o}, \mathbb{Z}_p) \hookrightarrow \mathscr{O}(\mathrm{B}_{<1} / \mathbb{C}_p)$ *to the one given by*

$$\|f\|_r = \sup\{\rho^{-nr} \|\cdot\|_{\mathrm{B}_{\leq \sigma_n}} : n \in \mathbb{N}\}.$$

Corollary (Elementary Calculus)

Let $f(X) = \sum a_n X^n$. *Then* $\|f(X)\|_r = \sup\{|a_n|/n^{rd} : n \in \mathbb{N}\}$.

Corollary (Schikhof Duality)

Let $f : \mathbf{o} \to \mathbb{C}_p$. *Then* f *is* r*-times differentiable if and only if* $f(x) = \sum_{n\in\mathbb{N}} a_n P_n(x)$ *with* $|a_n| n^{rd} \to 0$.

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