

Fourier Transformation in the p -adic
Langlands program

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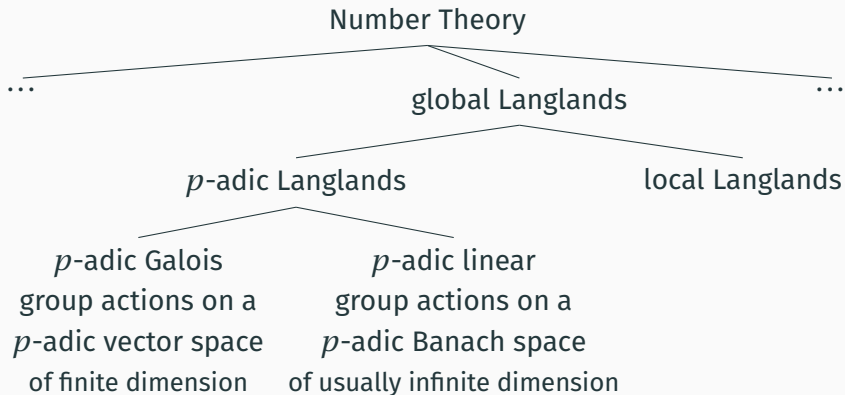
p -ADICS 2015

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- 1 p -adic Langlands program
- 2 From Characteristic 0 to p
- 3 Fourier Transform



p -adic Langlands correspondence

p -adic vector space := vector space over (an extension of) \mathbb{Q}_p

p -adic Banach space := complete normed p -adic vector space

Definition

An action of a group G on a normed space with norm $\|\cdot\|$ is *unitary* if

$$\|g \cdot \|\ = \|\cdot\| \quad \text{for all } g \text{ in } G.$$

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \text{ on} \\ p\text{-adic vector spaces} \\ \text{of dimension } n \end{array} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{l} \text{unitary continuous actions} \\ \text{of } \text{GL}_n(\mathbb{Q}_p) \text{ on} \\ p\text{-adic Banach spaces} \\ \text{of (usually) infinite dimension} \end{array} \right\}$$

Functorial Construction

- ▶ $\mathcal{E} :=$ a ring (in fact field) of p -adic power series in $X^{\pm 1}$
- ▶ *étale* φ, Γ -module over $\mathcal{E} :=$ a module over \mathcal{E} with a *semilinear* action of two commuting matrices φ and Γ

First

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \text{ on} \\ p\text{-adic vector spaces} \\ \text{of dimension } n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{étale } \varphi, \Gamma\text{-modules} \\ \text{over } \mathcal{E} \text{ of dimension } n \end{array} \right\}$$

then

$$\left\{ \begin{array}{l} \text{étale } \varphi, \Gamma\text{-modules} \\ \text{over } \mathcal{E} \text{ of dimension } n \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{unitary continuous actions} \\ \text{of } \text{GL}_n(\mathbb{Q}_p) \text{ on} \\ p\text{-adic Banach spaces} \\ \text{of (usually) infinite dimension} \end{array} \right\}$$

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Cyclotomic Extension

Put

- ▶ $1, \zeta_p, \zeta_{p^2}, \dots :=$ roots of unity of p -power order
- ▶ $\mathbb{Q}_p^{\text{cyc}} := \mathbb{Q}_p(1, \zeta_p, \zeta_{p^2}, \dots)$

Then

$$\overline{\mathbb{Q}_p} \xrightarrow{H} \mathbb{Q}_p^{\text{cyc}} \xrightarrow{\Gamma} \mathbb{Q}_p$$

where

$$\Gamma := \text{Gal}(\mathbb{Q}_p^{\text{cyc}}/\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Z}_p^*$$

$$\sigma \mapsto x \text{ given by } \zeta^\sigma = \zeta^x \text{ for all } \zeta = 1, \zeta_p, \zeta_{p^2} \dots$$

From characteristic 0 to p

Theorem (Field of Norms, [Fontaine '90])

The absolute Galois groups of $\mathbb{F}_p((t))$ and $\mathbb{Q}_p^{\text{cyc}}$ are isomorphic.

Put $\varphi :=$ Frobenius of $\mathbb{F}_p((t))$

Fact (Katz, '80)

Let \mathbf{E} be a field of characteristic p . There is an equivalence of categories

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \text{Gal}(\overline{\mathbf{E}}/\mathbf{E}) \text{ on} \\ \text{vector spaces over } \mathbb{F}_p \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{semilinear injective actions} \\ \text{of } \varphi \text{ on vector spaces over } \mathbf{E} \end{array} \right\}$$

Corollary

Let $\mathcal{E} :=$ complete ring of p -adic power series in $X^{\pm 1}$ that lifts $\mathbb{F}_p((t))$. There is an equivalence of categories

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\text{cyc}}) \text{ on} \\ p\text{-adic vector spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{semilinear injective actions} \\ \text{of } \varphi \text{ on vector spaces over } \mathcal{E} \end{array} \right\}$$

Proof.

By the preceding theorem using

- ▶ $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\text{cyc}}) \cong \text{Gal}(\overline{\mathbb{F}_p((t))}/\mathbb{F}_p((t)))$, and
- ▶ lifting the vector space coefficients from \mathbb{F}_p to \mathbb{Q}_p by applying the functor of Witt vectors and inverting p . □

Theorem (Fontaine)

Let $\mathcal{E} :=$ ring of p -adic power series in $X^{\pm 1}$ lifting $\mathbb{F}_p((t))$ with

- ▶ $\varphi \curvearrowright \mathcal{E}$ by $t^\varphi := (1+t)^p - 1$, and
- ▶ $\Gamma \curvearrowright \mathcal{E}$ by $t^\gamma := (1+t)^\gamma - 1 = \sum \binom{\gamma}{n} t^n$ where $\Gamma \cong \mathbb{Z}_p^*$

There is an equivalence of categories

$$\left\{ \begin{array}{l} \text{continuous actions} \\ \text{of } \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \text{ on} \\ p\text{-adic vector spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{semilinear injective actions} \\ \text{of commutative } \varphi \text{ and } \Gamma \\ \text{on vector spaces over } \mathcal{E} \end{array} \right\}$$

Proof.

By the preceding theorem for $H = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{cyc}})$ using

$$\overline{\mathbb{Q}}_p \xrightarrow{H \curvearrowright} \mathbb{Q}_p^{\text{cyc}} \xrightarrow{\Gamma \curvearrowright} \mathbb{Q}_p.$$

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Let \mathbf{K} be a finite extension of \mathbb{Q}_p with valuation ring $\mathfrak{o}_{\mathbf{K}}$. Denote

- ▶ $\mathcal{C}^0(\mathbb{Z}_p, \mathbf{K}) := \{ \text{all continuous } f: \mathbb{Z}_p \rightarrow \mathbf{K} \}$, and
- ▶ $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{K}) := \{ \text{all continuous linear } v: \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K}) \rightarrow \mathbf{K} \}$.

Theorem

Then $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \mathbf{K} \otimes_{\mathfrak{o}_{\mathbf{K}}} \mathfrak{o}_{\mathbf{K}}[[X]]$ as topological \mathbf{K} -algebras.

Proof.

- ▶ By density of the locally constant functions, that is, $\mathfrak{o}_{\mathbf{K}}[\mathbb{Z}/p\mathbb{Z}] \cup \mathfrak{o}_{\mathbf{K}}[\mathbb{Z}/p^2\mathbb{Z}] \cup \dots$ inside $\mathcal{C}^0(\mathbb{Z}_p, \mathfrak{o}_{\mathbf{K}})$

$$\mathcal{D}^0(\mathbb{Z}_p, \mathfrak{o}_{\mathbf{K}}) \xrightarrow{\sim} \varprojlim \mathfrak{o}_{\mathbf{K}}[\mathbb{Z}/p^n\mathbb{Z}] =: \mathfrak{o}_{\mathbf{K}}[[\mathbb{Z}_p]], \quad \text{and}$$

- ▶ by the *Iwasawa isomorphism* that maps the generator $\mathbf{1}$ of \mathbb{Z}_p to $1 + X$ $\mathfrak{o}_{\mathbf{K}}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathfrak{o}_{\mathbf{K}}[[X]]$.

Mahler Basis

Theorem (Schikhof Duality)

Let $\binom{\cdot}{n} : \mathbb{Z}_p \rightarrow \mathbf{K}$ with $\binom{x}{n} := x(x-1)\cdots(x-n+1)/n!$. Then

$$\{ \text{all zero sequences over } \mathbf{K} \} \xrightarrow{\sim} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$$

$$(a_n) \mapsto \sum a_n \binom{\cdot}{n}$$

Proof.

Because $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \{ \text{all bounded sequences over } \mathbf{K} \}$. \square

Let $r \geq 0$. Denote

- ▶ $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) := \{ \text{all } r\text{-times differentiable } f: \mathbb{Z}_p \rightarrow \mathbf{K} \},$
- ▶ $\mathcal{D}^r(\mathbb{Z}_p, \mathbf{K}) := \{ \text{all continuous linear } \nu: \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) \rightarrow \mathbf{K} \},$

and

$d^r(\mathbb{N}, \mathbf{K}) := \{ \text{all } \sum a_n X^n \text{ in } \mathbf{K}[[X]] \text{ with } \{|a_n|/n^r\} \text{ bounded} \}.$

Theorem

As normed \mathbf{K} -vector spaces $\mathcal{D}^r(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} d^r(\mathbb{N}, \mathbf{K})$ by

$$\int \mapsto \sum_n a_n X^n \quad \text{with } a_n = \int \binom{\cdot}{n}.$$

Back to φ, Γ -modules

Let $n = 2$, that is,

$$V = \mathbf{K} \oplus \mathbf{K}.$$

If $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \curvearrowright V$ is “effective crystalline” then its φ, Γ -module D over \mathcal{E} is

- ▶ base extended from

a φ, Γ -module N over $d^0(\mathbb{N}, \mathbf{K})$, and

- ▶ and, for some $r, s \geq 0$,

N is a φ, Γ -submodule of $d^r(\mathbb{N}, \mathbf{K}) \oplus d^s(\mathbb{N}, \mathbf{K})$.

Matrix Action

Fourier transform \rightsquigarrow the φ, Γ -module N is

- ▶ a module over $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{K})$,
- ▶ a submodule of $\mathcal{D}^r(\mathbb{Z}_p, \mathbf{K}) \oplus \mathcal{D}^s(\mathbb{Z}_p, \mathbf{K})$.

In particular

- ▶ $\mathbb{Z}_p \curvearrowright N$ by δ_x in $\mathcal{D}^0(\mathbb{Z}_p, \mathbf{K})$ for all x in \mathbb{Z}_p ,
- ▶ $\mathbb{Z}_p^* \curvearrowright N$ by $\mathbb{Z}_p^* = \Gamma$ and $p^{\mathbb{N}} \curvearrowright N$ by $p = \varphi^{-1}$

Thus,

$$M := \begin{pmatrix} p^{\mathbb{N}}\mathbb{Z}_p^* & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix},$$

that is, $M^+ = p^{\mathbb{N}}\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ acts on N .

$N \rightsquigarrow \bar{N} =$ submodule of $\mathcal{D}^r(\mathbb{Q}_p, \mathbf{K}) \oplus \mathcal{D}^s(\mathbb{Q}_p, \mathbf{K})$ over $\mathcal{D}^0(\mathbb{Q}_p, \mathbf{K})$

Then there is an action of

$$\begin{pmatrix} \mathbb{Q}_p^* & \mathbb{Q}_p \\ & \mathbb{Q}_p^* \end{pmatrix}$$

on \bar{N} which extends uniquely to one of $GL_2(\mathbb{Q}_p)$ on \bar{N} .

Dualizing gives the sought-for Banach space

$\bar{N} \rightsquigarrow GL_2(\mathbb{Q}_p) \curvearrowright B :=$ subquotient of $\mathcal{E}^r(\mathbb{Q}_p, \mathbf{K}) \oplus \mathcal{E}^s(\mathbb{Q}_p, \mathbf{K})$

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Notes available at

<http://imj-prg.fr/~enno.nagel>.