p-adic Fourier theory of differentiable functions

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Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ of degree d and $\mathcal{O}_{\mathbf{K}}$ its ring of integers; let $\mathbb{C}_p$ be the completed algebraic closure of $\mathbb{Q}_p$. The *Fourier polynomials* $\mathrm{P}_n : \mathcal{O}_{\mathbf{K}} \to \mathbb{C}_p$ prove that the topological algebra of all locally analytic distributions $\mu : \mathscr{C}^{\mathrm{la}}(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p) \to \mathbb{C}_p$ is, by $\mu \mapsto \sum \mu(\mathrm{P}_n)\mathrm{X}^n$, isomorphic to that of all power series in $\mathbb{C}_p[[\mathrm{X}]]$ that converge on the open unit disc of $\mathbb{C}_p$.

Given a real number $r \geq d$, we determine all power series that correspond under this isomorphism to a distribution $\mu : \mathscr{C}^r(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p) \to \mathbb{C}_p$ that extends to all r-times differentiable functions (as arisen in the p-adic Langlands program): A function $f : \mathcal{O}_{\mathbf{K}} \to \mathbb{C}_p$ is r-times differentiable if and only if $f(x) = \sum a_n \mathrm{P}_n(x)$ with $|a_n| n^{r/d} \to 0$ as $n \to \infty$.

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Introduction

Let $\mathbb{Q}_p$ denote the p-adic numbers and $\mathbb{Z}_p$ its ring of p-adic integers; let $\mathbb{C}_p$ be the completed algebraic closure of $\mathbb{Q}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ its ring of integers. Let $\mathscr{C}^0(\mathbb{Z}_p, \mathbb{C}_p)$ be the $\mathbb{C}_p$-Banach space of continuous functions $f \colon \mathbb{Z}_p \to \mathbb{C}_p$ and $\mathscr{D}^0(\mathbb{Z}_p, \mathbb{C}_p)$ its $\mathbb{C}_p$-linear dual. Every continuous function $f \colon \mathbb{Z}_p \to \mathcal{O}_{\mathbb{C}_p}$ can be uniformly approximated by locally constant functions $f_n \in \mathcal{O}_{\mathbb{C}_p}[\mathbb{Z}_p/p^n\mathbb{Z}_p]$ for $n \in \mathbb{N}$; that is, $f_n \to f$ as $n \to \infty$ for the supremum norm. Dually,

$$\mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}_p}[[\mathbb{Z}_p]]$$

is an isomorphism of topological $\mathcal{O}_{\mathbb{C}_p}$-algebras, where

- the multiplication on the left-hand side is the convolution product, and
- the right-hand side is the completed group algebra $\varprojlim_n \mathcal{O}_{\mathbb{C}_p}[\mathbb{Z}/p^n\mathbb{Z}]$ with the projective-limit topology.

The topological group $\mathbb{Z}_p$ is generated by a single element, say $\gamma = \mathbf{1}$, yielding the *Iwasawa isomorphism* of topological $\mathcal{O}_{\mathbb{C}_p}$ algebras

$$\mathcal{O}_{\mathbb{C}_p}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}_p}[[X]]$$

defined by $\gamma + 1 \mapsto X$. The composed isomorphism

$$\begin{aligned}\mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) &\xrightarrow{\sim} \mathcal{O}_{\mathbb{C}_p}[[X]] \\ \mu &\mapsto \mu\binom{\cdot}{0} + \mu\binom{\cdot}{1}X + \mu\binom{\cdot}{2}X^2 + \cdots\end{aligned}$$

sends a continuous linear map $\mu\colon \mathscr{C}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$ to the power series whose coefficients are the values of μ on the *Mahler polynomials*, given by $\binom{x}{n} := x(x-1)\cdots(x-n+1)/n!$.

The Mahler polynomials $\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \ldots$ are an orthogonal basis of the Banach space $\mathscr{C}^0(\mathbb{Z}_p, \mathbb{C}_p)$; more generally, for ν in $\mathbb{N}$, an orthogonal basis of all ν-times differentiable functions: A function $f\colon \mathbb{Z}_p \to \mathbb{C}_p$ is ν-times differentiable if and only if its coefficients $(a_n)_{n\in\mathbb{N}}$ fulfill $|a_n| n^\nu \to 0$ as $n \to \infty$.

For a real number $r \geq 0$, this differentiability condition on the Mahler polynomial coefficients underlaid in [BB10] the definition of a *$\mathscr{C}^r$-function* $f\colon \mathbb{Z}_p \to \mathbb{C}_p$ for any $r \in \mathbb{R}_{\geq 0}$ by asking its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ to obey $|a_n| n^r \to 0$ as $n \to \infty$.

The notion of r-fold differentiability on $\mathbb{Z}_p$ for a real $r \geq 0$ emerged from the *p-adic Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$ which matches unitary continuous actions of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a, usually infinite-dimensional, p-adic Banach space V with continuous actions of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q})$ of $\mathbb{Q}_p$ on a 2-dimensional p-adic vector space (see [Col14] as a starting point). Let us outline the steps taken to construct this correspondence in the prototypic *crystalline* case (cf. [BB10]):

1. The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on V is *unitary* if the norm of V is invariant under the group action; such a p-adic Banach space V is constructed as quotient space of r-times differentiable functions (*$\mathscr{C}^r$-functions* for short) on two copies of $\mathbb{Z}_p$ for a real number $r \geq 0$.

2. The continuous linear forms on all $\mathscr{C}^r$-functions on $\mathbb{Z}_p$ embed by the *Amice transform* (see [Sch99]) into the ring $\mathscr{A}(B_{<1})$ of all power series that converge on the open unit disc $B_{<1}$ of $\mathbb{C}_p$. This transforms V into a 2-dimensional module D over $\mathscr{A}(B_{<1})$ on which a chosen pair of matrices (φ, Γ) in $\mathrm{GL}_2(\mathbb{Q}_p)$ acts commutatively.

3. This action of (φ, Γ) on D is by Fontaine's Theory of (φ, Γ)-modules equivalent to an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on a 2-dimensional p-adic vector space (see [FO14]).

If $\mathbf{K}$ is a finite extension of $\mathbb{Q}_p$ then a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{K})$ yet has to be formulated. We first introduce r-fold differentiability on the ring of integers $\mathcal{O}_\mathbf{K}$ of $\mathbf{K}$. Recent ([Ber13]) and upcoming work ([Sch13]) indicates that above passage from the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ to that of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is for $\mathrm{GL}_2(\mathbf{K})$ best mimicked via *Lubin-Tate Theory*, used in [ST01] to generalize the Amice transform on $\mathbb{Z}_p$ to the *Fourier transform* on $\mathcal{O}_\mathbf{K}$: it identifies the $\mathbb{C}_p$-linear dual of the $\mathscr{C}^r$-functions $f\colon \mathcal{O}_\mathbf{K} \to \mathbb{C}_p$ with power series that converge on $\mathrm{B}_{<1}$ by

$$\mu \mapsto \mu(\mathrm{P}_0) + \mu(\mathrm{P}_1)\mathrm{X} + +\mu(\mathrm{P}_2)\mathrm{X}^2 + \cdots,$$

sending a continuous linear map $\mu\colon \mathscr{C}^r(\mathcal{O}_\mathbf{K}, \mathbb{C}_p) \to \mathbb{C}_p$ to the power series whose coefficients are the values of μ on certain *Fourier polynomials* $\mathrm{P}_0, \mathrm{P}_1, \ldots$ (implicitly defined in Section 3). We characterize these power series and, dually, all r-times differentiable functions $f\colon \mathcal{O}_\mathbf{K} \to \mathbb{C}_p$ by their *Fourier coefficients*:

Theorem (5.1'). *Let $d = [\mathbf{K} : \mathbb{Q}_p]$ and $r \geq d$. A function $f\colon \mathcal{O}_\mathbf{K} \to \mathbb{C}_p$ is r-times differentiable if and only if $f(x) = \sum_{n\in\mathbb{N}} a_n \mathrm{P}_n(x)$ with $|a_n| n^{r/d} \to 0$ as $n \to \infty$.*

Outline

In Section 1 we define r-fold differentiability on $\mathcal{O}_\mathbf{K}$, as follows: We decompose $r = \nu + \rho$ into an integer part ν in $\mathbb{N}$ and a fractional part ρ in $[0,1[$. Then *ν-fold differentiability* is defined by iterated divided differences and *ρ-fold differentiability* by a strengthened Hölder continuity condition. A function is *r-times differentiable* if its ν-th iterated divided difference is ρ-times differentiable.

In one variable, a function is r-times differentiable if and only if its Taylor polynomial expansion converges (Theorem 1.6).

This equivalence is in Section 2 used to verify the *Cauchy-Riemann conditions* over $\mathbf{K}$: A function f on $\mathcal{O}_\mathbf{K}$ is r-times differentiable as function of one variable in $\mathbf{K}$ if and only if f is r-times differentiable as function of $d = [\mathbf{K} : \mathbb{Q}_p]$ variables in $\mathbb{Q}_p$ and the derivative of f is $\mathbf{K}$-linear.

In Section 3 we review Amice's and Schneider and Teitelbaum's theories that identify locally analytic *distributions* (continuous linear forms on all locally analytic functions) with power series converging on an open unit disc. The Amice transform gives an isomorphism $\mathscr{T}\colon \mathscr{D}^{\mathrm{la}}_{\mathbb{Q}_p}(\mathcal{O}_\mathbf{K}, \mathbb{C}_p) \xrightarrow{\sim} \mathscr{A}(\mathrm{B}^d_{<1})$ between all locally $\mathbb{Q}_p$-analytic distributions on $\mathcal{O}_\mathbf{K} = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ and all power series converging on the open polydisc $\mathrm{B}^d_{<1}$.

The Cauchy-Riemann equations that ensure $\mathbf{K}$-analyticity of such a distribution cut out an analytic variety $\mathrm{R}\colon \widehat{\mathcal{O}}_\mathbf{K} \hookrightarrow \mathrm{B}^d_{<1}$. Schneider and Teitelbaum construct via Lubin-Tate's Theory of formal $\mathcal{O}_{\mathbf{K}}$-modules an analytic isomorphism $\mathrm{F}\colon \mathrm{B}_{<1}/\mathbb{C}_p \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathbf{K}}$. This yields the commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{la}}_{\mathbb{Q}_p}(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p) & \xrightarrow[\text{Amice}]{\sim} & \mathcal{A}(\mathrm{B}^d_{<1}) \\ \downarrow & & \downarrow \mathcal{F}\circ\mathcal{R} \\ \mathcal{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p) & \xrightarrow{\sim} & \mathcal{A}(\mathrm{B}_{<1}) \end{array} \tag{$*$}$$

where $\mathcal{F} \circ \mathcal{R}$ denotes the homomorphism between rings of power series induced from $\mathrm{R} \circ \mathrm{F}$.

In Section 4 we study this diagram for *$\mathcal{C}^r$-distributions*, the continuous linear forms on all $\mathcal{C}^r$-functions. Let us denote by $\mathrm{A}^r_{\mathbb{Q}_p}$ the image under the Amice transform of the continuous dual $\mathcal{D}^r_{\mathbb{Q}_p}(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p)$ of $\mathcal{C}^r(\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p, \mathbb{C}_p)$. It consists of power series subject to a certain boundedness condition. We must compute the pullback $\mathrm{A}^r_{\mathbf{K}} := \mathcal{F} \circ \mathcal{R}(\mathrm{A}^r_{\mathbb{Q}_p})$ of $\mathrm{A}^r_{\mathbb{Q}_p}$ under $\mathrm{R} \circ \mathrm{F}\colon \mathrm{B}_{<1} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathbf{K}} \hookrightarrow \mathrm{B}^d_{<1}$. The restriction from all locally analytic to all $\mathcal{C}^r$-distributions turns the commutative diagram $(*)$ into

$$\begin{array}{ccc} \mathcal{D}^r_{\mathbb{Q}_p}(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p) & \xrightarrow[\text{Amice}]{\sim} & \mathrm{A}^r_{\mathbb{Q}_p} \\ \downarrow & & \downarrow \mathcal{F}\circ\mathcal{R} \\ \mathcal{D}^r_{\mathbf{K}}(\mathcal{O}_{\mathbf{K}}, \mathbb{C}_p) & \xrightarrow{\sim} & \mathrm{A}^r_{\mathbf{K}}. \end{array}$$

To compute $\mathrm{A}^r_{\mathbf{K}}$ we show that the coefficient-wise boundedness condition satisfied by all power series f in $\mathrm{A}^r_{\mathbb{Q}_p}$ is equivalent to a boundedness condition on the values of f on closed subvarieties which exhaust $\widehat{\mathcal{O}}_{\mathbf{K}}$, called *temperedness*. Temperedness translates well under the rigid isomorphism $\mathrm{R} \circ \mathrm{F}$ and allows us to compute $\mathrm{A}^r_{\mathbf{K}} = \mathcal{F} \circ \mathcal{R}(\mathrm{A}^r_{\mathbb{Q}_p})$ (where a technical key point is a uniform bound on the operator norms of these rigid-analytic isomorphisms on all discs of increasing radii below 1 in [BK16]).

In the final Section 5, we conclude by Schikhof duality (cf. [ST02]) the Fourier coefficients of $\mathcal{C}^r$-functions to obey the convergence condition of Theorem 5.1'.

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1 Approaches to fractional differentiability

We define r-fold differentiability for a real number $r \geq 0$. For this, write $r = \nu + \rho \geq 0$ with $\nu \in \mathbb{N}$ and $\rho \in [0,1[$. We first define ν-fold differentiability by iterated divided differences, then ρ-fold differentiability by a strengthened Hölder-continuity condition. Finally a function is r-times differentiable if its ν-th iterated divided difference is ρ-times differentiable.

Iterated linear differentials

The non-Archimedean differentiability condition is more rigorous than the ordinary one to compensate the absence of an analogue of the intermediate value theorem over a non-Archimedean field, due to its total disconnectedness:

Let V be a finite-dimensional $\mathbf{K}$-vector space and X an open subset of V. Let $\mathbf{E}$ be a $\mathbf{K}$-Banach space. The function $f \colon \mathrm{X} \to \mathbf{E}$ is *differentiable* at a in X if there is a linear map $\mathrm{A} \colon \mathrm{V} \to \mathbf{E}$ such that for every $\epsilon > 0$ there is a neighborhood U around a inside X where

$$\|f(x+h) - f(x) - \mathrm{A}h\| \leq \epsilon \|h\| \quad \text{for all } x+h, x \text{ in U.}$$

$\mathscr{C}^\nu$-functions for a natural number ν. The following, equivalent, differentiability condition requires a choice of coordinates on V, but can be iterated, that is, applied again to the obtained differential to define differentiability of higher orders. We fix a basis $e_1, \ldots, e_d$ of V and by this basis identify V with the d-fold direct sum $\mathbf{K} \oplus \cdots \oplus \mathbf{K}$. Let X be an open subset of V.

Definition. The differential $f^{]1[}(x+h, x)$ of f at $x+h, x$ in X with $h \in \mathbf{K}^{*d}$ is the $\mathbf{K}$-linear map $\mathrm{A} \colon \mathrm{V} \to \mathbf{E}$ determined by

$$\mathrm{A} \cdot h_k e_k = f(x + h_1 e_1 + \cdots + h_{k-1} e_{k-1} + h_k e_k) - f(x + h_1 e_1 + \cdots + h_{k-1} e_{k-1})$$

for all $k = 1, \ldots, d$. The function f is a *$\mathscr{C}^1$-function* if $f^{]1[}$ extends to a continuous function $f^{[1]} \colon \mathrm{X} \times \mathrm{X} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$.

Because X is open, $\mathrm{X}^{]1[} = \{(x+h, x) \in \mathrm{X}^2 : h \in \mathbf{K}^{*d}\}$ is dense inside $\mathrm{X}^{[1]}$, and so $f^{[1]}$ is uniquely determined by f.

Starting from this definition, we obtain a notion of ν-fold differentiability for $\nu \geq 0$ as follows: Let $f \in \mathscr{C}^1(\mathrm{X}, \mathbf{E})$ and let us regard the function $f^{[1]}$: Its domain $\mathrm{X} \times \mathrm{X}$ is again included in a finite dimensional $\mathbf{K}$-vector space $\mathrm{V} \times \mathrm{V}$ with an ordered basis, and its codomain $\mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ is again $\mathbf{K}$-Banach space.

So we can define the iterated differential by the differential of $f^{[1]}$. That is, f is *twice differentiable* if $f^{[1]}$ exists and

$$f^{]2[} = (f^{[1]})^{]1[} \colon (\mathrm{X} \times \mathrm{X})^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{V} \times \mathrm{V}, \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E}))$$

extends to a continuous function $f^{[2]}$ on $(\mathrm{X} \times \mathrm{X})^{[1]}$.

Definition. Let ν in $\mathbb{N}$. The function $f \colon \mathrm{X} \to \mathbf{E}$ is a $\mathscr{C}^{\nu+1}$*-function*

- if f is a $\mathscr{C}^{\nu}$-function, and
- if $\mathfrak{X} = \mathrm{X}^{[\nu]}$, $\mathfrak{V} = \mathrm{V}^{[\nu]}$, $\mathfrak{E} = \mathbf{E}^{[\nu]}$ and $\mathfrak{f} = f^{[\nu]}$ then $\mathfrak{f}^{]1[}$ extends to a continuous function $\mathfrak{f}^{[1]} \colon \mathfrak{X} \times \mathfrak{X} \to \mathrm{Hom}_{\mathbf{K}}(\mathfrak{V} \times \mathfrak{V}, \mathfrak{E})$.

Like $f^{[1]}$, also $f^{[\nu]}$ is uniquely determined by f.

$\mathscr{C}^{\rho}$*-functions for* ρ *in* $[0,1[$. Let ρ in $[0,1[$. Roughly, ρ-fold differentiability is stricter Hölder-continuity. Let $\mathfrak{X}$ be a subset of a finite-dimensional $\mathbf{K}$-vector space $\mathfrak{V}$ and let $\mathfrak{E}$ be a non-Archimedean $\mathbf{K}$-Banach space.

Definition 1.1. The function $f \colon \mathfrak{X} \to \mathfrak{E}$ is $\mathscr{C}^{\rho}$ at a in $\mathfrak{V}$ if for every $\varepsilon > 0$, there is a neighborhood U around a inside $\mathfrak{V}$ such that

$$\|f(x) - f(y)\| \leq \varepsilon \cdot \|x - y\|^{\rho} \quad \text{for all } x, y \text{ in } \mathfrak{X} \cap \mathrm{U}.$$

The function $f \colon \mathfrak{X} \to \mathfrak{E}$ is a $\mathscr{C}^{\rho}$*-function* if it is $\mathscr{C}^{\rho}$ at every a in $\mathfrak{X}$.

The above condition on a in $\mathfrak{V}$ is nonvoid only if a is in the closure of $\mathfrak{X}$. This is the case, for example, when $\mathfrak{X}$ is dense inside $\mathfrak{V}$.

The *fractional divided difference* $|f^{]\rho[}|$ of f is defined by

$$|f^{]\rho[}|(x,y) := \|f(x) - f(y)\| / \|x - y\|^{\rho} \quad \text{for all distinct } x, y \text{ in } \mathfrak{X}.$$

The function $f \colon \mathfrak{X} \to \mathfrak{E}$ is a $\mathscr{C}^{\rho}$-function if and only if $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|$ on all of $\mathfrak{X} \times \mathfrak{X}$ that vanishes on the diagonal of $\mathfrak{X} \times \mathfrak{X}$. Because $\mathfrak{X}$ is open, the domain of $|f^{]\rho[}|$ is dense inside $\mathfrak{X} \times \mathfrak{X}$ and f determines $|f^{[\rho]}|$ uniquely. If $\mathfrak{X}$ is compact, then we can endow the $\mathbf{K}$-vector space of $\mathscr{C}^{\rho}$-functions by the natural norm $\|f\|_{\mathscr{C}^{\rho}} := \max\{\|f\|_{\sup}, \| \, |f^{[\rho]}| \, \|_{\sup}\}$.

$\mathscr{C}^r$-functions for $r \geq 0$. Let $r = \nu + \rho \geq 0$ with $\nu \in \mathbb{N}$ and $\rho \in [0,1[$. We define r-fold differentiability of a function f by requiring its ν-th iterated divided difference $f^{]\nu[}$ to be $\mathscr{C}^\rho$ everywhere.

Definition 1.2. Let X be an open subset of V. The function $f\colon \mathrm{X} \to \mathbf{E}$ is a *$\mathscr{C}^r$-function* if f is a $\mathscr{C}^\nu$-function and $f^{[\nu]}$ is a $\mathscr{C}^\rho$-function.

Let X be compact. Because $f^{[n]}$ for $n = 0, \dots, \nu$ and $|\mathrm{F}^{[\rho]}|$ for $\mathrm{F} = f^{[\nu]}$ are uniquely determined by f, the norm $\|\cdot\|_{\mathscr{C}^r}$ on all $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathbf{E}$ given by $\|f\|_{\mathscr{C}^r} := \max\{\|f^{[0]}\|_{\sup}, \dots, \|f^{[\nu-1]}\|_{\sup}, \|f^{[\nu]}\|_{\mathscr{C}^\rho}\}$ is well-defined.

Iterated divided differences in one variable

The preceding definition is well suited for conceptual questions like that about base change in Section 2. For computations, the textbook definition (see [Sch84, Section 26ff.]) is apter.

Schikhof observed that the divided difference $f^{]1[}$ is a symmetric function; as such, it is differentiable if and only if it is partially differentiability in its first coordinate. This reduces, with increasing degree of differentiability ν, the exponential growth in the number of variables of $f^{]\nu[}$ to a linear growth in the number of variables of a divided difference $f^{>\nu<}$, that we define below:

Definition. Let X be a subset of $\mathbf{K}$ and $f\colon \mathrm{X} \to \mathbf{E}$. For $\nu \in \mathbb{N}$ put

$$\mathrm{X}^{<\nu>} = \mathrm{X}^{\{0,\dots,\nu\}} \quad \text{and} \quad \mathrm{X}^{>\nu<} = \{(x_0, \dots, x_\nu) : \text{ if } i \neq j \text{ then } x_i \neq x_j\}.$$

The *ν-th divided difference* $f^{>\nu<}\colon \mathrm{X}^{>\nu<} \to \mathbf{E}$ of a function $f\colon \mathrm{X} \to \mathbf{E}$ is inductively given by $f^{>0<} := f$ and for $n \in \mathbb{N}$ and $(x_0, \dots, x_\nu) \in \mathrm{X}^{>\nu<}$ by

$$f^{>\nu<}(x_0, \dots, x_\nu) := \frac{f^{>\nu-1<}(x_0, x_2, \dots, x_\nu) - f^{>\nu-1<}(x_1, x_2, \dots, x_\nu)}{x_0 - x_1}.$$

The following definition for $\rho = 0$ is given in [Sch84, Section 29], where *integral* differentiability (that is, for ν in $\mathbb{N}$) is defined. That is, a function f is ν times differentiable if $f^{>\nu<}$ extends to a continuous function on $\mathrm{X}^{<\nu>}$.

Definition 1.3. Fix $r = \nu + \rho \in \mathbb{R}_{\geq 0}$. Let X be a subset of $\mathbf{K}$ and $f\colon \mathrm{X} \to \mathbf{E}$.

- The function f is $\mathscr{C}^r$ (or *r-times differentiable*) at a point $a \in \mathrm{X}$ if $f^{>\nu<}\colon \mathrm{X}^{>\nu<} \to \mathbf{E}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a, \dots, a) \in \mathrm{X}^{<\nu>}$.
- The function f is a *$\mathscr{C}^r$-function* (or an *r-times differentiable function*) if f is $\mathscr{C}^r$ at all a in X. Let $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ denote all $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathbf{E}$.

Note that this differentiability condition is, even for higher orders, given point-wise. If a is an accumulation point then the value $D^\nu f(a)$ to which $f^{>\nu<}$ extends at $\vec{a}$, the *derivative of f at a*, is uniquely determined. If $f^{(\nu)}$ is the ν-fold ordinary derivative of f then $\nu!\, D^\nu f = f^{(\nu)}$ ([Sch84, Theorem 29.5]).

Let X contain no isolated point. Then f is a $\mathscr{C}^r$-function if and only if $f^{>\nu<}$ extends to a *unique* $\mathscr{C}^\rho$-function $f^{<\nu>}\colon X^{<\nu>} \to \mathbf{E}$ ([Nag11, Proposition 2.5]).

Every r-times differentiable function is (by [Nag11, Lemma 2.3]) in particular s-times differentiable for every nonnegative $s \leq r$. Thence, if X is compact without isolated points, then we can endow the $\mathbf{K}$-vector space of $\mathscr{C}^r$-functions with the norm

$$\|f\|_{\mathscr{C}^r} := \max\{\|f^{[0]}\|_{\sup}, \dots, \|f^{[\nu-1]}\|_{\sup}, \|f^{[\nu]}\|_{\mathscr{C}^\rho}\}.$$

This norm is equivalent to that of Definition 1.2 by [Nag16, Proposition A.2].

Taylor Polynomials

We give a differentiability condition of only two arguments by Taylor polynomials (whereas that by iterated linear differentials respectively iterated divided differences have an exponential respectively linear growth in the number of variables for increasing degree of differentiability ν).

Definition. Let V be a normed $\mathbf{K}$-vector space. Let $\mathrm{Sym}^n_{\mathbf{K}}(V, \mathbf{E})$ be all continuous *symmetric* $\mathbf{K}$-multilinear maps $M\colon V \times \cdots \times V \to \mathbf{E}$ of n variables. These form a non-Archimedean $\mathbf{K}$-Banach space by the operator norm

$$\|M\| = \sup\{\|M(x)\| : x \in V^n \text{ with } \|x\| \leq 1\}$$

which is the supremum of M on the unit ball of $V \times \cdots \times V$ with respect to the product norm $\|v_1, \dots, v_n\| = \max\{\|v_1\|, \dots, \|v_n\|\}$.

The following definition generalizes that of onefold differentiability at the beginning of Section 1 to a higher differentiability degree $r \geq 0$.

Definition 1.4. Let X be an open subset of V. The function $f\colon X \to \mathbf{E}$ is a $\mathscr{C}^r_{\mathrm{T}}$-function if there are functions $D^n f\colon X \to \mathrm{Sym}^n(V, \mathbf{E})$ for $n = 0, 1, \dots, \nu$ and $R^\nu f\colon X \times X \to \mathbf{E}$ such that

$$f(x+h) = \sum_{n=0,\dots,\nu} D^n f(x)(h, \dots, h) + R^\nu f(x+h, x)$$

and for every a in X and $\varepsilon > 0$, there is a neighborhood U around a inside X such that

$$\|R^\nu f(x+h, x)\| \leq \varepsilon \|h\|^r \quad \text{for all } x+h, x \text{ in U}.$$

The norm. Let $\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ be the $\mathbf{K}$-vector space of all $\mathscr{C}^r_{\mathrm{T}}$ functions $f\colon \mathrm{X} \to \mathbf{E}$. By [Nag16, Corollary 2.5] the functions $\mathrm{D}^0 f, \mathrm{D}^1 f, \ldots, \mathrm{D}^\nu f$ are uniquely determined and differentiable of degree r, $r-1$, ..., ρ. Hence

1. in particular, the functions $\mathrm{D}^0 f, \mathrm{D}^1 f, \ldots, \mathrm{D}^\nu f$ are continuous, and
2. the remainder $\mathrm{R}^\nu f$ of the Taylor polynomial up to degree ν converges as in Definition 1.4 if and only if the function $\Delta^r f$, defined by

$$\Delta^r f(x,y) = \|\mathrm{R}^\nu f(x,y)\| / \|x-y\|^r \quad \text{for all distinct } x, y \text{ in } \mathrm{X},$$

extends to a continuous function $|\Delta^r f|\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}_{\geq 0}$ that vanishes on the diagonal.

Thus if X is a compact open subset of V then there is a well-defined norm $\|\cdot\|_{\mathscr{C}^r_{\mathrm{T}}}$ on $\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ given by $\|f\|_{\mathscr{C}^r_{\mathrm{T}}} := \max\{\|\mathrm{D}^0 f\|_{\sup}, \ldots, \|\mathrm{D}^\nu f\|_{\sup}\} \cup \{\| \, |\Delta^r f| \, \|_{\sup}\}$.

Necessity. Every r-times differentiable function can be locally approximated by its Taylor polynomial expansion up to degree ν:

Proposition 1.5 ([Nag16, Corollary 3.6]). *We have $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \subseteq \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ and if X is a compact open subset of V then the inclusion $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ is a monomorphism of normed vector spaces.*

Sufficiency in one variable. For a function f of one variable (and also of many variables in $\mathbb{Q}_p$ by [Nag16, Section 3]) the convergence condition on the rest term of its Taylor polynomial of degree ν is sufficient for the r-fold differentiability of f.

Theorem 1.6 ([Nag11, Lemma 2.27, Corollary 2.25 and 2.32]). *Let X be an open subset of $\mathbf{E}$ and $f\colon \mathrm{X} \to \mathbf{E}$. Then $f\colon \mathrm{X} \to \mathbf{E}$ is a $\mathscr{C}^r$-function if and only if there are functions $\mathrm{D}^0 f, \mathrm{D}^1 f, \ldots, \mathrm{D}^\nu f\colon \mathrm{X} \to \mathbf{E}$ and $\mathrm{R}^\nu f\colon \mathrm{X} \times \mathrm{X} \to \mathbf{E}$ such that*

$$f(x+y) = \mathrm{D}^0 f(x) + \mathrm{D}^1 f(x) y + \cdots + \mathrm{D}^\nu f(x) y^\nu + \mathrm{R}^\nu f(x+y, x)$$

and for every a in X and $\varepsilon > 0$ exists a neighborhood U around a inside X where

$$|\mathrm{R}^\nu f(x+y, x)| \leq \varepsilon |y|^r \quad \textit{for all } x+y, x \in \mathrm{U}_\varepsilon.$$

Remark. De Ieso defines in [DI13] an r-times differentiable function over the unit ball of a finite extension $\mathbf{F}$ of $\mathbb{Q}_p$ via a Taylor polynomial expansion by the field embeddings of $\mathbf{F}$ into its normal closure. His normed space of $\mathscr{C}^r$-functions equals by [Nag16, Theorem 4.7] that of $\mathscr{C}^r_{\mathbb{Q}_p}$-functions as defined next (in Section 2).

2 Cauchy-Riemann equations

The differentiability condition on a function f depends on the field of definition $\mathbf{K}$ of the vector space that embraces its domain and codomain. The bigger the base field $\mathbf{K}$, the more restrictive the condition on the derivative of f to commute with the scalar multiplication in $\mathbf{K}$, and so the more restrictive the differentiability condition on f. To emphasize this dependency on the base field $\mathbf{K}$, let a *$\mathscr{C}^r_{\mathbf{K}}$-function* for $r \geq 1$ denote an r-times differentiable function whose domain and codomain have field of definition $\mathbf{K}$.

Let $\mathbf{L}$ be a non-Archimedean field and $\mathbf{K}$ a finite extension of $\mathbf{L}$. Let X be an open subset of $\mathbf{K}$, let $\mathbf{E}$ be a Banach space over $\mathbf{K}$ and $f\colon \mathrm{X} \to \mathbf{E}$. We show that if f is a $\mathscr{C}^r_{\mathbf{L}}$-function and additionally all its differentials commute with the scalar multiplication in $\mathbf{K}$ then f is a $\mathscr{C}^r_{\mathbf{K}}$-function (in analogy to $\mathbb{R}$ and its unique algebraic extension $\mathbb{C}$ where these additional conditions on the differentials are called the *Cauchy-Riemann conditions*).

Let V be a $\mathbf{K}$-vector space. We embed $\mathbf{K}$ into the $\mathbf{L}$-vector space $\mathrm{End}_{\mathbf{L}}(\mathrm{V})$ of all $\mathbf{L}$-linear endomorphisms over V by $\lambda \mapsto \lambda\cdot$. An $\mathbf{L}$-multilinear map $\Phi\colon \mathrm{V} \times \cdots \times \mathrm{V} \to \mathbf{E}$ is *$\mathbf{K}$-multilinear* if $\Phi(\ldots,\lambda\cdot,\ldots) = \lambda \cdot \Phi$ for every $\lambda \in \mathbf{K}$. Let $\mathrm{Mult}^n_{\mathbf{L}}(\mathrm{V},\mathbf{E})$ denote all $\mathbf{L}$-multilinear maps of n variables in V that take values in $\mathbf{E}$.

Proposition 2.1. *Let $f\colon \mathrm{X} \to \mathbf{E}$ be a $\mathscr{C}^r_{\mathbf{L}}$-function. If for every x in X the maps $\mathrm{D}^1 f(x),\ldots,\mathrm{D}^\nu f(x)$ are $\mathbf{K}$-multilinear, then f is a $\mathscr{C}^r_{\mathbf{K}}$-function.*

Proof: Let us assume $f \in \mathscr{C}^r_{\mathbf{L}}(\mathrm{X},\mathbf{E})$ and $\mathrm{D}^n f(x) \in \mathrm{Mult}^n_{\mathbf{K}}(\mathbf{K},\mathbf{E})$ for $n = 0,\ldots,\nu$. By Proposition 1.5

$$f(x+h) = \sum_{i=0,\ldots,\nu} \mathrm{D}^i f(x)(h,\ldots,h) + \mathrm{R}^\nu f(x+h,x) \quad \text{for all } x+h, x \in \mathrm{X},$$

such that for every $a \in \mathrm{X}$ and $\varepsilon > 0$ there is a neighborhood U around a where

$$|\mathrm{R}^\nu f(x+h,x)| \leq \varepsilon |h|^r \quad \text{for all } x,y \in \mathrm{U}.$$

Let us write $h \in \mathbf{K}$ as $h = h_1 e_1 + \cdots + h_d e_d$ with $\{e_1 = 1, e_2, \ldots, e_d\}$ a basis of the $\mathbf{L}$-vector space $\mathbf{K}$. For $i = 0,\ldots,\nu$, by $\mathbf{K}$-multilinearity of $\mathrm{D}^i f$,

$$\begin{aligned}\mathrm{D}^i f(x)(h,\ldots,h) &= \sum_{j_1,\ldots,j_i \in \{1,\ldots,d\}} \mathrm{D}^i f(x)(1,\ldots,1) h_{j_1} \cdots h_{j_i} \cdot e_{j_1} \cdots e_{j_i} \\ &= \mathrm{D}^i f(x)(1,\ldots,1)(h_1 e_1 + \cdots + h_d e_d)^i\end{aligned}$$

Putting $\mathrm{D}^i_{\mathbf{K}} f(x) = \mathrm{D}^i f(x)(1,\ldots,1)$, we therefore conclude that there are functions $\mathrm{D}^0_{\mathbf{K}} f,\ldots,\mathrm{D}^\nu_{\mathbf{K}} f \colon \mathrm{X} \to \mathbf{E}$ and $\mathrm{R}^\nu f \colon \mathrm{X}\times\mathrm{X} \to \mathbf{E}$ such that

$$f(x+h) = \sum_{i=0,\ldots,\nu} \mathrm{D}^i_{\mathbf{K}} f(x) h^i + \mathrm{R}^\nu f(x+h,x) \quad \text{for all } x+h, x \in \mathrm{X},$$

and for every $a \in \mathrm{X}$ and $\varepsilon > 0$ exists a neighborhood U around a inside X where

$$|\mathrm{R}^\nu f(x+h,x)| \leq \varepsilon |h|^r \quad \text{for all } x+h, x \in \mathrm{U}.$$

This convergence condition on the remainder of the Taylor expansion of f up to degree ν is by Theorem 1.6 above equivalent to $f \in \mathscr{C}^r_{\mathbf{K}}(\mathrm{X},\mathbf{E})$. □

Lemma 2.2. *Let* $f \colon \mathrm{X} \to \mathbf{L}$ *be a* $\mathscr{C}^r_{\mathbf{L}}$*-function. Given* x *in* X, *if* $\mathrm{D}^1 f(x)$ *is* $\mathbf{K}$*-linear, then* $\mathrm{D}^2 f(x),\ldots,\mathrm{D}^\nu f(x)$ *are* $\mathbf{K}$*-multilinear.*

Proof: Let λ in $\mathbf{K}$. We assume that $\lambda \cdot \mathrm{D}^1 f(x) = \mathrm{D}^1 f(x) \circ \lambda$ where on the right-hand side we regard λ as $\mathbf{L}$-linear endomorphism over $\mathbf{K}$.

Let $n = 2,\ldots,\nu$. Because $\mathrm{D}^n f(x) \in \mathrm{Mult}^n_{\mathbf{L}}(\mathbf{K},\mathbf{E})$ is symmetric, it suffices to show that $\mathrm{D}^n f(x)$ is $\mathbf{K}$-linear in the last argument, that is,

$$\mathrm{D}^n f(x)(\ldots,\lambda\cdot) = \lambda \cdot \mathrm{D}^n f(x).$$

By our assumption $\mathrm{D}^1 f \colon \mathrm{X} \to \mathrm{Hom}_{\mathbf{L}}(\mathbf{K},\mathbf{E})$ and by $\mathbf{K}$-linearity of the differential $\mathrm{D}^1 \colon \mathscr{C}^1(\mathrm{X},\mathbf{E}) \to \mathscr{C}^0(\mathrm{X},\mathrm{Hom}_{\mathbf{L}}(\mathbf{K},\mathbf{E}))$

$$\begin{aligned}\mathrm{D}^n f(x)(\cdot,\ldots,\cdot,\lambda\cdot) &= \mathrm{D}^{n-1}(\mathrm{D}^1 f(\lambda\cdot))(x)\\ &= (\mathrm{D}^{n-1}(\lambda\cdot \mathrm{D}^1 f))(x) = \lambda\cdot \mathrm{D}^{n-1}(\mathrm{D}^1 f)(x) = \lambda \cdot \mathrm{D}^n f(x).\end{aligned}$$ □

Corollary 2.3. *Let* $f \in \mathscr{C}^r_{\mathbf{L}}(\mathrm{X},\mathbf{E})$ *and* $\mathrm{D}^1 f(x) \in \mathrm{Hom}_{\mathbf{K}}(\mathbf{K},\mathbf{E})$ *for all* $x \in \mathrm{X}$. *Then* $f \in \mathscr{C}^r_{\mathbf{K}}(\mathrm{X},\mathbf{E})$.

Proof: By Proposition 2.1 and Lemma 2.2. □

3 The Fourier basis

Let us first explain the Amice transform on $\mathbb{Z}_p$, followed by the Fourier transform as its analogue on a finite extension $\mathcal{O}$ of $\mathbb{Z}_p$.

The Amice transform

Let $\mathscr{C}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p})$ denote all continuous functions $f\colon \mathbb{Z}_p \to \mathcal{O}_{\mathbb{C}_p}$ and let $\mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p})$ be its topological dual of all continuous linear maps $\mu\colon \mathscr{C}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$.

Every continuous function $f\colon \mathbb{Z}_p \to \mathcal{O}_{\mathbb{C}_p}$ is uniformly approximated by locally constant functions f_n in $\mathcal{O}_{\mathbb{C}_p}[\mathbb{Z}/p^n\mathbb{Z}]$; dually, the natural map

$$\mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}_p}[[\mathbb{Z}_p]]$$

is an isomorphism of topological $\mathcal{O}_{\mathbb{C}_p}$-algebras, where

- the left-hand side is equipped with the convolution product and the topology of point-wise convergence, and
- the right-hand side is the completed group algebra $\varprojlim \mathcal{O}_{\mathbb{C}_p}[\mathbb{Z}/p^n\mathbb{Z}]$ with the projective-limit topology.

The topological group $\mathbb{Z}_p$ is generated by a single element, say $\gamma = \mathbf{1}$, yielding the *Iwasawa isomorphism* of topological algebras

$$\mathcal{O}_{\mathbb{C}_p}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}_p}[[X]]$$

defined by $\gamma \mapsto 1 + X$. The composed isomorphism

$$\begin{aligned} \mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) &\xrightarrow{\sim} \mathcal{O}_{\mathbb{C}_p}[[X]] \\ \mu &\mapsto \mu\tbinom{\cdot}{0} + \mu\tbinom{\cdot}{1}X + \mu\tbinom{\cdot}{2}X^2 + \cdots \end{aligned}$$

sends a continuous linear map $\mu\colon \mathscr{C}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$ to the power series whose coefficients are its values $\mu(\tbinom{\cdot}{0}), \mu(\tbinom{\cdot}{1}), \ldots$ on the *Mahler polynomials*, given by $\tbinom{x}{n} := x(x-1)\cdots(x-n+1)/n!$.

We apply this isomorphism to all *locally analytic* functions, that is, functions that are locally given by a convergent power series (of, possibly, many variables): Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ of degree d and $\mathcal{O}$ its ring of integers. Let $\mathscr{C}^{\mathrm{la}}_{\mathbb{Q}_p}(\mathcal{O}, \mathbb{C}_p)$ be the Fréchet space of all $\mathbb{Q}_p$*-locally analytic* functions $f\colon \mathcal{O} \to \mathbb{C}_p$, that is, functions that are locally given by a convergent power series of d variables on an open subset of $\mathcal{O} = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$. The multivariate *Amice transform* ([Ami64, Corollaire 10.3.(a)]) is the isomorphism of topological $\mathbb{C}_p$-algebras

$$\begin{aligned} \mathscr{T}\colon \mathscr{D}^{\mathrm{la}}_{\mathbb{Q}_p}(\mathcal{O}, \mathbb{C}_p) &\xrightarrow{\sim} \mathscr{A}(\mathrm{B}^d_{<1}) \\ \mu &\mapsto \sum \mu\tbinom{\cdot}{\boldsymbol{n}} X^{\boldsymbol{n}} \end{aligned}$$

between all continuous linear maps $\mu\colon \mathscr{C}^{\mathrm{la}}_{\mathbb{Q}_p}(\mathcal{O},\mathbb{C}_p) \to \mathbb{C}_p$ and all power series of d variables over $\mathbb{C}_p$ that converge on the open unit disc $\mathrm{B}^d_{<1}$ of $\mathbb{C}_p^d$; here and henceforth for $\boldsymbol{x} = (x_1,\ldots,x_d)$ and $\boldsymbol{n} = (n_1,\ldots,n_d)$ we denote

$$\binom{\boldsymbol{x}}{\boldsymbol{n}} := \binom{x_1}{n_1}\cdots\binom{x_d}{n_d} \quad \text{and} \quad \boldsymbol{x}^{\boldsymbol{n}} := x^{n_1}\cdots x^{n_d}.$$

To conclude, by evaluation on the Mahler polynomials, the continuous linear forms

- on $\mathscr{C}^0(\mathbb{Z}_p,\mathbb{C}_p)$ correspond to all power series that are bounded, and
- on $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p,\mathbb{C}_p)$ correspond to all power series that converge on $\mathrm{B}_{<1}$.

The Fourier transform

Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ of degree d and $\mathcal{O}$ its ring of integers. The *Fourier polynomials* $\mathrm{P}_0,\mathrm{P}_1,\ldots$ parallel the Mahler polynomials $\binom{\cdot}{0},\binom{\cdot}{1},\ldots$ by the *Fourier* isomorphism of topological algebras $\mathscr{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \xrightarrow{\sim} \mathscr{A}(\mathrm{B}_{<1})$ between all *distributions*, continuous linear maps $\mu\colon \mathscr{C}^{\mathrm{la}}(\mathcal{O},\mathbb{C}_p) \to \mathbb{C}_p$ on all locally analytic functions $f : \mathcal{O} \to \mathbb{C}_p$, and all power series that converge on the open unit disc $\mathrm{B}_{<1}$ of $\mathbb{C}_p$. Let

$$\mathrm{I} = \ker \mathscr{D}^{\mathrm{la}}_{\mathbb{Q}_p}(\mathcal{O},\mathbb{C}_p) \to \mathscr{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p)$$

be the kernel of the continuous linear map that restricts a distribution from all $\mathbb{Q}_p$-locally analytic to all $\mathbf{K}$-locally analytic functions $f : \mathcal{O} \to \mathbb{C}_p$. Let $\mathrm{J} := \mathscr{T}(\mathrm{I})$ be its image under the Amice transform. By the Hahn-Banach Theorem ([PGS10, Theorem 4.2.4]) the induced quotient map of the Amice transform

$$\overline{\mathscr{T}}\colon\ \mathscr{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \xrightarrow{\sim} \mathscr{A}(\mathrm{B}^d_{<1})/\mathrm{J}$$

is an isomorphism. The main result Theorem 3.6 of [ST01] establishes the rigid-analytic *Schneider-Teitelbaum isomorphism* $\mathrm{F}\colon\ \mathrm{B}_{<1} \to \widehat{\mathcal{O}}$ between affinoid algebras

$$\mathscr{F}\colon \mathscr{A}(\mathrm{B}^d_{<1})/\mathrm{J} \xrightarrow{\sim} \mathscr{A}(\mathrm{B}_{<1})$$

(where in op. cit. $(\mathrm{F},\mathscr{F})$ is denoted by (κ,κ^*)). The *Fourier transform* $\mathscr{F}\circ\overline{\mathscr{T}}$ is obtained by composing the Amice transform with the Schneider-Teitelbaum isomorphism, yielding the isomorphism of topological $\mathbb{C}_p$-algebras

$$\mathscr{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \xrightarrow{\sim} \mathscr{A}(\mathrm{B}_{<1})$$
$$\mu \mapsto \sum \mu(\mathrm{P}_n)\mathrm{X}^n$$

given by evaluation on the *Fourier polynomials* P_0, P_1, ... (denoted by $P_0(\Omega\cdot)$, $P_1(\Omega\cdot)$ in [ST01]). To define $P_0, P_1, \ldots$,

1. we will parametrize the set $\widehat{\mathcal{O}}$ of all locally **K**-analytic characters $\kappa\colon \mathcal{O} \to \mathbb{C}_p^*$ by the open unit disc $B_{<1}$, and
2. obtain by restriction from $\mathcal{C}^{\mathrm{la}}(\mathcal{O}, \mathbb{C}_p)$ onto $\widehat{\mathcal{O}}$ (and this parametrization) an injective map

$$\mathcal{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p) \to \{\text{ all } f\colon\ B_{<1} \to \mathbb{C}_p\},$$

   whose image is by the Amice transform shown to consist of all analytic functions on $B_{<1}$.

The character variety

We will describe the Schneider-Teitelbaum isomorphism, point-wise, as rigid-analytic map between the variety $\widehat{\mathcal{O}}$ of all **K**-analytic characters and the open unit disc $B_{<1}$ obtained from Lubin-Tate Theory. Via the Amice transform, it will implicitly define the Fourier polynomials P_0, P_1, P_2 ...:

Let $\widehat{\mathbb{Z}}_p^d$ be the set of all $\mathbb{Q}_p$-analytic characters $\chi\colon \mathbb{Z}_p^d \to \mathbb{C}_p^*$ parametrized by

$$\begin{aligned} B^d_{<1} &\xrightarrow{\sim} \widehat{\mathbb{Z}}_p^d \\ z &\mapsto \chi_z := [\boldsymbol{x} \mapsto (1+z)^{\boldsymbol{x}} := \textstyle\sum \binom{\boldsymbol{x}}{\boldsymbol{n}} z^{\boldsymbol{n}}] \end{aligned}$$

(see [Sch99, Section 2]). We have $\mathcal{T}(\mu)(z) = \mu(\chi_z)$.

Let $\widehat{\mathcal{O}}$ be the set of all **K**-analytic characters $\kappa\colon \mathcal{O} \to \mathbb{C}_p^*$. We will define an analogous rigid-analytic parametrization

$$\begin{aligned} F\colon\ B_{<1} &\xrightarrow{\sim} \widehat{\mathcal{O}} \\ z &\mapsto \kappa_z : \mathcal{O} \to \mathbb{C}_p^* \end{aligned}$$

so that $\mathcal{F} \circ \overline{\mathcal{T}}(\mu)(z) = \mu(\kappa_z)$. This character κ_z will be defined as composition of an orbit map on a formal group with a translated group homomorphism.

Formal $\mathcal{O}$-modules. A *formal group* $\mathcal{G}$ is a *commutative one-dimensional formal group law* over $\mathcal{O}$, that is, a power series $G(X, Y)$ in $\mathcal{O}[[X, Y]]$ such that

- (associativity) $\quad G(X, G(Y, Z)) = G(G(X, Y), Z)$,

- (commutativity) $\quad \mathrm{G}(\mathrm{X},\mathrm{Y}) = \mathrm{G}(\mathrm{Y},\mathrm{X})$, and
- (identity element) $\quad \mathrm{G}(\mathrm{X},\mathrm{Y}) \equiv \mathrm{X}+\mathrm{Y}+$ summands of higher degree.

An *endomorphism* of a formal group G is a power series $g(\mathrm{X})$ in $\mathcal{O}[[\mathrm{X}]]$ such that

$$g(\mathrm{G}(\mathrm{X},\mathrm{Y})) = \mathrm{G}(g(\mathrm{X}),g(\mathrm{Y})).$$

A *formal* $\mathcal{O}$*-module* is a formal group $\mathcal{G}$ together with a ring homomorphism $\mathcal{O} \to \mathrm{End}(\mathcal{G})$.

Let $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ be the multiplicative and additive formal group (over $\mathbb{Z}$) given by the group laws $\mathrm{G}(\mathrm{X},\mathrm{Y}) = \mathrm{XY}+\mathrm{X}+\mathrm{Y}$ and $\mathrm{G}(\mathrm{X},\mathrm{Y}) = \mathrm{X}+\mathrm{Y}$. Let us add a subscript (such as $\mathcal{O}_{\mathbb{C}_p}$, $\mathbb{Q}_p$ or $\mathbb{C}_p$) to indicate the base extension of a formal group (to $\mathcal{O}_{\mathbb{C}_p}$,$\mathbb{Q}_p$ or $\mathbb{C}_p$).

Every formal group $\mathcal{G}$ is by its *logarithm* $\log_{\mathcal{G}}\colon \mathcal{G}_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{G}_{\mathrm{a}}|\mathbb{Q}$ isomorphic to the additive formal group over $\mathbb{Q}$. Let $\exp\colon \mathbb{G}_{\mathrm{a}}|\mathbb{Q} \to \mathbb{G}_m|\mathbb{Q}$ be the exponential map. (See [Lan78, Chapter 8] for the definition of either map and a more thorough discussion of formal groups, in particular formal $\mathcal{O}$-modules.)

Let $\mathrm{T}'(\mathcal{G}) = \mathrm{Hom}(\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}}, \mathbb{G}_{\mathrm{m}}|\mathcal{O}_{\mathbb{C}_p})$ be all formal group homomorphisms between $\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}}$ and $\mathbb{G}_{\mathrm{m}}|\mathcal{O}_{\mathbb{C}_p}$. Every $t'\colon \mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}} \to \mathbb{G}_{\mathrm{m}}|\mathcal{O}_{\mathbb{C}_p}$ in $\mathrm{T}'(\mathcal{G})$ decomposes over $\mathbb{C}_p$ as

$$t'\colon \mathcal{G}_{\mathbb{C}_p} \xrightarrow{\log_{\mathcal{G}}} \mathbb{G}_{\mathrm{a}}|\mathbb{C}_p \xrightarrow{\Omega'\cdot} \mathbb{G}_{\mathrm{a}}|\mathbb{C}_p \xrightarrow{\exp} \mathbb{G}_{\mathrm{m}}|\mathbb{C}_p \tag{3.1}$$

for some Ω' in $\mathcal{O}_{\mathbb{C}_p}$. Consequently $\mathrm{T}'(\mathcal{G})$ is a free $\mathcal{O}_{\mathbb{C}_p}$-module of rank one.

To every uniformizer π in $\mathcal{O}$ corresponds (after base extension to the completion of the maximally unramified extension of $\mathcal{O}$) a (unique) formal $\mathcal{O}$-module $\mathcal{G}_\pi$ ([LT65]). For example, p in $\mathbb{Z}_p$ corresponds to $\mathbb{G}_{\mathrm{m}}$.

Let us henceforth fix a uniformizer π in $\mathcal{O}$, the formal $\mathcal{O}$-module $\mathcal{G}$ that corresponds to π and a generator t'_0 of $\mathrm{T}'(\mathcal{G})$ (and its corresponding scalar Ω in $\mathcal{O}_{\mathbb{C}_p}$).

Orbits under formal group actions. We now recall how $\mathrm{B}_{<1}$ parametrizes all locally $\mathbf{K}$-analytic $\mathbb{C}_p^*$-valued characters on $\mathcal{O}$ ([ST01, Section 3]): The power series over $\mathcal{O}$ that defines a formal group law (such as $\mathcal{G}$ or $\mathbb{G}_{\mathrm{m}}$) converges on $\mathrm{B}_{<1}$ and turns $\mathrm{B}_{<1}$ into a group that we denote by the formal group law (such as $\mathcal{G}$ or $\mathbb{G}_{\mathrm{m}}$). Given a in $\mathcal{O}$, let $[a]$ in $\mathrm{End}(\mathcal{G})$ be the formal $\mathcal{O}$-action of a on $\mathcal{G}$.

We attach to $z \in \mathcal{G} = \mathrm{B}_{<1}$ its orbit map $o_z\colon \mathcal{O} \to \mathcal{G}$ given by $a \mapsto [a]z$. Then

$$\begin{array}{ccccccc} \mathcal{O} & \xrightarrow{o_z} & \mathcal{G} & \xrightarrow{t'_0} & \mathbb{G}_{\mathrm{m}} & \xrightarrow{\cdot+1} & \mathbb{C}_p^* \\ a & \mapsto & [a]z & \mapsto & t'_0([a]z) & \mapsto & t'_0([a]z)+1 \end{array}$$

is a locally $\mathbf{K}$-analytic character $\kappa_z\colon \mathbb{O} \to \mathbb{C}_p^*$ (where $\cdot + 1$ translates between the neutral elements 0 and 1 of $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{C}_p^*$). The obtained map

$$\begin{aligned} F\colon \; B_{<1} &\xrightarrow{\sim} \widehat{\mathbb{O}} \\ z &\mapsto \kappa_z \end{aligned}$$

is a bijection between $B_{<1}$ and the set $\widehat{\mathbb{O}}$ of all $\mathbf{K}$-analytic characters on $\mathbb{O}$ with values in $\mathbb{C}_p$ ([ST01, Proposition 3.1]).

We recall that $\hat{\mathbb{O}}$ is cut out of $\widehat{\mathbb{Z}}_p^d$ by the Cauchy-Riemann equations ([ST01, Lemma 1.1]) and is the rigid-analytic subvariety of $B_{<1}^d$ of vanishing ideal J; the Schneider-Teitelbaum isomorphism is a rigid-analytic group homomorphism between $\mathscr{G}$ and $\hat{\mathbb{O}}$.

The Fourier basis. Expressed in power series, Equation (3.1) says

$$f_{t_0'}(Z) = \exp(\Omega \log_{\mathscr{G}}(Z)) \tag{3.2}$$

where $f_{t_0'}(Z)$ is the formal power series that defines t_0'. Let $P_m(Y)$ in $\mathbf{K}[Y]$ (denoted by $P_n(\Omega\cdot)$ in [ST01]) be the polynomial defined by the formal power series expansion

$$\exp(Y \log_{\mathscr{G}}(Z)) = \sum P_n(Y) Z^n.$$

Let $f(X) = \sum_{n \geq 0} a_n X^n$ in $\mathscr{A}(B_{<1})$ and let $\mu\colon \mathscr{C}^{\mathrm{la}}(\mathbb{O}, \mathbb{C}_p) \to \mathbb{C}_p$ be its image under $\mathscr{F} \circ \overline{\mathscr{T}}$. Then $\mu(P_n) = 1/n!(\mathrm{d}^n f/\mathrm{d}X^n)(0) = a_n$ ([ST01, Lemma 4.6.9]).

4 Differentiability as boundedness over the open unit disc

Let us fix a real number $r \geq 0$. In this Section 4 we characterize the $\mathscr{C}^r$-*distributions*, that is, the continuous linear maps $\mu\colon \mathscr{C}^r(\mathbb{O}, \mathbb{C}_p) \to \mathbb{C}_p$, by a bound on their values on the Fourier polynomials $P_0, P_1, \ldots$

Strategy

Cauchy-Riemann equations. Let $\mathbf{L} = \mathbb{Q}_p$ (and $\mathbf{K}$ as before a finite extension of $\mathbf{L}$). If f is an r-times differentiable function over $\mathbf{L}$, then f is r-times differentiable over $\mathbf{K}$ if and only if f satisfies the Cauchy-Riemann equations (by Corollary 2.3) and likewise if f is a locally analytic function (by [ST01, Lemma 1.1]). We obtain a commutative diagram of restriction maps

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{la}}_{\mathbf{L}}(\mathcal{O},\mathbb{C}_p) & \longrightarrow & \mathcal{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \\ \uparrow & & \uparrow \\ \mathcal{D}^{r}_{\mathbf{L}}(\mathcal{O},\mathbb{C}_p) & \longrightarrow & \mathcal{D}^{r}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p). \end{array} \tag{4.1}$$

where we claim that

(i) the arrows pointing upwards are injections, and

(ii) those pointing rightwards surjections (which induce isometries for the quotient norms).

Ad (i): The set of all locally polynomial (in particular all locally analytic) functions is by [Nag11, Proposition 3.30] dense inside $\mathcal{C}^r(\mathcal{O},\mathbb{C}_p)$; dually, the restriction map

$$\begin{aligned} \mathcal{D}^r(\mathcal{O},\mathbb{C}_p) &\to \mathcal{D}^{\mathrm{la}}(\mathcal{O},\mathbb{C}_p) \\ \mu &\mapsto \mu_{\restriction \mathcal{C}^{\mathrm{la}}(\mathcal{O},\mathbb{C}_p)} \end{aligned}$$

over all continuous linear maps $\mu\colon \mathcal{C}^r(\mathcal{O},\mathbb{C}_p) \to \mathbb{C}_p$ is injective. Because this holds for $\mathcal{C}^r$-functions over an arbitrary non-Archimedean field, such as $\mathbf{L}$ or $\mathbf{K}$, and of an arbitrary number of variables, such as $[\mathbf{K} : \mathbf{L}]$, we obtain injectivity of both, left and right, arrows.

Ad (ii): This follows from the Hahn-Banach Theorem for non-Archimedean locally convex vector spaces of countable type ([PGS10, Theorem 4.2.4]).

The Amice transform. The Amice transform $\mathcal{T}$ turns the commutative diagram (4.1) between distribution spaces into one between spaces of formal power series subject to certain convergence conditions:

$$\begin{array}{ccc} \mathcal{A}(\mathrm{B}^d_{<1}) & \longrightarrow & \mathcal{A}(\mathrm{B}^d_{<1})/\mathrm{J} \\ \uparrow & & \uparrow \\ \mathrm{A}^r_{\mathbf{L}} & \longrightarrow & \mathrm{A}^r_{\mathbf{L}}/\mathrm{J} \cap \mathrm{A}^r_{\mathbf{L}}, \end{array}$$

where $\mathrm{J} = \mathcal{T}(\mathrm{I})$ is the image of the ideal generated by the Cauchy-Riemann equations and $\mathrm{A}^r_{\mathbf{L}} = \mathcal{D}^r_{\mathbf{L}}(\mathcal{O},\mathbb{C}_p)$.

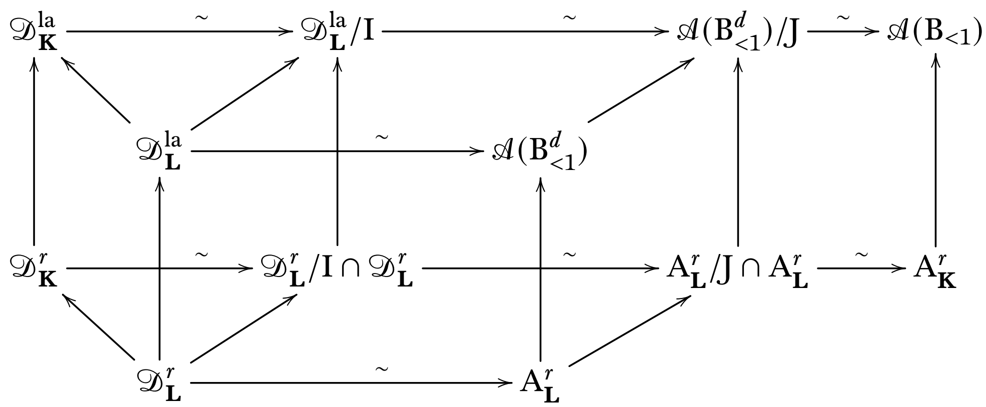
The Schneider-Teitelbaum isomorphism. The main result Theorem 3.6 of [ST01] is a rigid-analytic isomorphism, the *Schneider-Teitelbaum isomorphism*,

$$(\mathrm{F},\mathscr{F})\colon (\mathrm{B}_{<1},\mathscr{A}(\mathrm{B}_{<1})) \xrightarrow{\sim} (\hat{\mathfrak{O}},\mathscr{A}(\mathrm{B}^d_{<1})/\mathrm{J})$$

between the variety $\widehat{\mathfrak{O}}$ of all **K**-analytic characters and the open unit disc $\mathrm{B}_{<1}$. We put

$$\mathrm{A}^r_{\mathbf{K}} := \mathscr{F}(\mathrm{A}^r_{\mathbf{L}}/\mathrm{J} \cap \mathrm{A}^r_{\mathbf{L}}).$$

The $\mathscr{C}^r$-functions under the Fourier transform. The following commutative diagram recollects all homomorphisms that have figured in our above discussion:



Our aim is to describe $\mathscr{D}^r_{\mathbf{K}}(\mathfrak{O},\mathbb{C}_p)$, all continuous linear maps $\mu\colon \mathscr{C}^r_{\mathbf{K}}(\mathfrak{O},\mathbb{C}_p) \to \mathbb{C}_p$, by their values $\mu(\mathrm{P}_0),\mu(\mathrm{P}_1),\ldots$ on the Fourier polynomials. These values are the coefficients of the power series in $\mathrm{A}^r_{\mathbf{K}}$, the image of $\mathscr{D}^r_{\mathbf{K}}(\mathfrak{O},\mathbb{C}_p)$ under the isomorphism given by the bottom rear arrows.

To prove

$$\mathrm{A}^r_{\mathbf{K}} = \left\{ \sum_{n\in\mathbb{N}} a_n \mathrm{X}^n : \{|a_n|/n^{r/d}\} \text{ is bounded} \right\} \tag{4.2}$$

(that is, Corollary 4.8), we take three steps, each carried out in its proper subsection:

1. Because above diagram commutes, $\mathrm{A}^r_{\mathbf{K}}$ is the image of $\mathrm{A}^r_{\mathbf{L}}$ under the epimorphism

   $$\mathscr{A}(\mathrm{B}^d_{<1}) \twoheadrightarrow \mathscr{A}(\mathrm{B}^d_{<1})/\mathrm{J} = \mathscr{A}(\hat{\mathfrak{O}}) \xrightarrow{\sim} \mathscr{A}(\mathrm{B}_{<1}).$$

   Given a power series in $\mathscr{A}(\mathrm{B}^d_{<1})$, show that it is in $\mathrm{A}^r_{\mathbf{L}}$ if and only if it is *tempered*, that is, its values over $\mathrm{B}^d_{<1}$ are bounded in a prescribed manner.

2. Apply the Schneider-Teitelbaum isomorphism $F\colon \hat{\mathcal{O}} \xrightarrow{\sim} B_{<1}$ to this temperedness condition that singles out $A^r_{\mathbf{L}}/J \cap A^r_{\mathbf{L}}$ from $\mathcal{A}(B^d_{<1})/J$.

3. Show that the (temperedness) condition on the values over $B_{<1}$ obtained under the Schneider-Teitelbaum isomorphism is equivalent to the coefficient-wise boundedness condition (4.2).

Differentiability as analytical temperedness

We describe the power series in $A^r_{\mathbf{L}}/J \cap A^r_{\mathbf{L}}$ by a bound on their values on larger and larger closed discs inside the open unit disc. For this we first describe the distributions in $\mathcal{D}^r_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p)$ by their values on locally analytic functions, and afterwards apply the Amice transform.

For a subfield $\mathbf{F}$ of $\mathbb{C}_p$ let $v\colon \mathbf{F}^* \to \mathbb{Q}$ be the additive valuation standardized by $v(p) = 1$, and $|x| = p^{-v(x)}$ for x in $\mathbf{F}^*$. A *ball of radius* $\delta > 0$ of $\mathbf{F}$ is a subset of $\mathbf{F}$ each two of whose elements x and y fulfill $|x - y| \leq \delta$.

Definition. For a field $\mathbf{F}$ in-between $\mathbf{L}$ and $\mathbf{K}$ and n in $\mathbb{N}$, put

$$\mathcal{C}^{n-\mathrm{an}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p) := \{\text{all } f\colon \mathcal{O} \to \mathbb{C}_p \text{ that are } \mathbf{F}\text{-analytic on every ball of radius } p^{-n}\}$$

These *n-analytic functions* form a $\mathbb{C}_p$-Banach space for the natural norm that restricts to the analytic norm on every neighborhood of radius p^{-n}. Let

$$\mathcal{D}^{n-\mathrm{an}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p) := \{\text{all continuous } \mathbb{C}_p\text{-linear } \mu\colon\ \mathcal{C}^{n-\mathrm{an}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p) \to \mathbb{C}_p\}$$

be its continuous dual. These *n-analytic distributions* form a $\mathbb{C}_p$-Banach space for the operator norm.

Given a field $\mathbf{F}$ in-between $\mathbf{L}$ and $\mathbf{K}$ (such as $\mathbf{L}$ or $\mathbf{K}$), a natural number n and an ideal I of $\mathcal{D}^{\mathrm{la}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p)$ (such as that generated by the Cauchy-Riemann equations), we will denote the ideal of $\mathcal{D}^{n-\mathrm{an}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p)$ that is generated by the image of I under $\mathcal{D}^{\mathrm{la}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p) \to \mathcal{D}^{n-\mathrm{an}}_{\mathbf{F}}(\mathcal{O}, \mathbb{C}_p)$ likewise by I.

The Amice transform turns the n-analytic distributions into formal power series that converge on a closed disc of radius $\rho < 1$:

Definition. Let $B^d_{\leq\rho}$ be the closed polydisc of radius $\rho > 0$ of $\mathbb{C}_p^d$. Put

$$\mathcal{A}(B^d_{\leq\rho}) := \{\text{all power series over } \mathbb{C}_p \text{ of } d \text{ variables that converge on } B^d_{\leq\rho}\}.$$

Then $\mathcal{A}(B^d_{\leq\rho})$ is a $\mathbb{C}_p$-Banach algebra for the norm $\|\cdot\|_\rho$ defined by

$$\|\sum_{\boldsymbol{i}\in\mathbb{N}^d} a_{\boldsymbol{i}} X_1^{i_1} \cdots X_d^{i_d}\|_\rho := \sup\{|a_{\boldsymbol{i}}|\rho^{i_1+\cdots+i_d} : \boldsymbol{i} \in \mathbb{N}^d\}.$$

Given d in $\mathbb{N}$, a radius $\delta > 0$ and an ideal J of $\mathscr{A}(\mathrm{B}^d_{<1})$ (such as that generated by the Cauchy-Riemann equations), we will denote the ideal of $\mathscr{A}(\mathrm{B}^d_{\leq\delta})$ that is generated by the image of J under $\mathscr{A}(\mathrm{B}^d_{<1}) \to \mathscr{A}(\mathrm{B}^d_{\leq\delta})$ likewise by J.

For an affinoid algebra A, let A° be its closed unit ball under the *Gauss residue norm*; that is, if $\mathrm{A} = \mathrm{T}/\mathrm{J}$ is the quotient of the Tate algebra T by the ideal J then

$$\mathrm{A}^\circ := \{\text{all } a \in \mathrm{A} \text{ such that for every } \varepsilon > 0 \text{ there is } t \text{ in } a + \mathrm{J} \text{ with } \|t\| \leq 1 + \varepsilon\}$$

Fix once for all a sequence $\lambda_1, \lambda_2, \ldots$ in $\mathbb{C}_p$ such that $\{[v(\lambda_1) + \cdots + v(\lambda_h)]/rh : h \in \mathbb{N}\}$ is bounded. (For example, if r in $\mathbb{Q}$ then, for some λ in $\mathbb{C}_p$ with $v(\lambda) = r$, fix the constant sequence $\lambda, \lambda, \ldots$) Then for a sequence of increasing radii (δ_h) below 1 and d in $\mathbb{N}$, let

$$\cdots \to \mathscr{A}(\mathrm{B}^d_{\leq\delta_{h+1}}) \to \mathscr{A}(\mathrm{B}^d_{\leq\delta_h}) \to \cdots \to \mathscr{A}(\mathrm{B}^d_{\leq\delta_1})$$

be the transition maps defined by $f \mapsto \lambda_h \cdot f_{|\mathrm{B}^d_{\leq\delta_h}}$ for every h in $\mathbb{N}$.

Proposition 4.1 (Temperedness under the Amice Transform). *Let* (ρ_h) *be the sequence of increasing radii below* 1 *defined by* $\rho_0 := p^{-1/(p-1)} < 1$ *and* $\rho_h := \rho^{1/p^h(p-1)}$. *Then the natural map*

$$\mathrm{A}^r_{\mathbf{L}}/\mathrm{J} \cap \mathrm{A}^r_{\mathbf{L}} \xrightarrow{\sim} \varprojlim_{h \in \mathbb{N}} (\mathscr{A}(\mathrm{B}^d_{\leq\rho_h})/\mathrm{J})^\circ \otimes_{\mathscr{O}_{\mathbb{C}_p}} \mathbb{C}_p \tag{4.3}$$

is an isomorphism of $\mathbb{C}_p$*-Banach spaces.*

Proof: The map (4.3) is on the very right of the commutative diagram

$$\begin{array}{ccccc}
\varprojlim (\mathscr{D}^{n-\mathrm{an}}_{\mathbf{K}})^\circ \otimes \mathbb{C}_p & \longrightarrow & \varprojlim (\mathscr{D}^{n-\mathrm{an}}_{\mathbf{L}}/\mathrm{I})^\circ \otimes \mathbb{C}_p & \longrightarrow & \varprojlim (\mathscr{A}(\mathrm{B}^d_{\leq\rho_n})/\mathrm{J})^\circ \otimes \mathbb{C}_p \\
\uparrow & & \uparrow & & \uparrow {\scriptstyle (4.3)} \\
\mathscr{D}^r_{\mathbf{K}} & \longrightarrow & \mathscr{D}^r_{\mathbf{L}}/\mathrm{I} \cap \mathscr{D}^r_{\mathbf{L}} & \longrightarrow & \mathrm{A}^r_{\mathbf{L}}/\mathrm{J} \cap \mathrm{A}^r_{\mathbf{L}}
\end{array}$$

where

- in the *left-hand* rectangle both arrows pointing rightwards are the natural quotient isomorphisms,
- in the *right-hand* rectangle those are given by the Amice transform, and

- all arrows pointing upwards are given by the natural inclusion maps.

To prove that (4.3) is an isomorphism, it suffices by commutativity of the diagram to prove that all other arrows in the commutative diagram given by the right-hand rectangle are isomorphisms:

1. The bottom-right arrow $\mathscr{D}^r_{\mathbf{L}}/\mathrm{I} \cap \mathscr{D}^r_{\mathbf{L}} \to \mathrm{A}^r_{\mathbf{L}}/\mathrm{J} \cap \mathrm{A}^r_{\mathbf{L}}$ is an isomorphism as quotient map of the Amice isomorphism (and because $\mathrm{A}^r_{\mathbf{K}}$ is by definition the image of $\mathscr{D}^r_{\mathbf{L}}$ under the Amice isomorphism).

2. The top-right arrow $\varprojlim \mathscr{D}^{n-\mathrm{an}}_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)/\mathrm{I} \to \varprojlim \mathscr{A}(\mathrm{B}^d_{\leq \rho_n})/\mathrm{J}^\circ \otimes \mathbb{C}_p$ is an isomorphism by [Nag15, Lemma 7.1], which is a suitably formulated version of the (multivariate) Amice Theorem.

3. It rests to show that the middle arrow

$$\mathscr{D}^r_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)/\mathrm{I} \cap \mathscr{D}^r_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p) \to \varprojlim (\mathscr{D}^{n-\mathrm{an}}_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)/\mathrm{I})^\circ \otimes \mathbb{C}_p$$

   is an isomorphism. For this it suffices by commutativity of the diagram to prove that all other arrows in the commutative diagram given by the left-hand rectangle are isomorphisms:

   3.1. The top-left arrow $\varprojlim \mathscr{D}^{n-\mathrm{an}}_{\mathbf{K}}(\mathbb{O}, \mathbb{C}_p) \to \varprojlim \mathscr{D}^{n-\mathrm{an}}_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)/\mathrm{I}$ is an isomorphism because all $\mathscr{D}^{n-\mathrm{an}}_{\mathbf{K}}(\mathbb{O}, \mathbb{C}_p) \to \mathscr{D}^{n-\mathrm{an}}_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)/\mathrm{I}$ are isometric isomorphisms by the ultrametric Hahn-Banach theorem [PGS10, Theorem 4.2.4] for normed spaces of countable type; and

   3.2. the bottom-left arrow $\mathscr{D}^r_{\mathbf{K}}(\mathbb{O}, \mathbb{C}_p) \to \mathscr{D}^r_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)/\mathrm{I} \cap \mathscr{D}^r_{\mathbf{L}}(\mathbb{O}, \mathbb{C}_p)$ is likewise an isomorphism by the ultra-metric Hahn-Banach Theorem.

   3.3. The left arrow $\mathscr{D}^r_{\mathbf{K}}(\mathbb{O}, \mathbb{C}_p) \to \varprojlim \mathscr{D}^{n-\mathrm{an}}_{\mathbf{K}}(\mathbb{O}, \mathbb{C}_p)^\circ \otimes \mathbb{C}_p$ is an isomorphism by [Nag15, Corollary 6.1], which expresses the $\mathscr{C}^r$-norm by the locally analytic ones. □

Temperedness under the Schneider-Teitelbaum isomorphism

We transfer by the Schneider-Teitelbaum isomorphism the isomorphism (4.3) from $\mathrm{A}^r_{\mathbf{L}}/\mathrm{J} \cap \mathrm{A}^r_{\mathbf{L}}$ in Proposition 4.1 to $\mathrm{A}^r_{\mathbf{K}}$. In coordinates, the Schneider-Teitelbaum isomorphism $\mathrm{F} \colon \mathrm{B}_{<1} \to \widehat{\mathbb{O}}$ (inside $(1, \ldots, 1) + \mathrm{B}^d_{<1}$) is given by

$$z \mapsto (1 + f_{e_1 t'_0}(z), \ldots, 1 + f_{e_d t'_0}(z)),$$

where

- $e_1, \dots, e_d$ is a basis of $\mathcal{O}_{\mathbf{K}}$ over $\mathbb{Z}_p$,
- t'_0 is the generator of the $\mathcal{O}$-module $\operatorname{Hom}(\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}}, \mathbb{G}_m | \mathcal{O}_{\mathbb{C}_p})$ fixed in Section 3, and
- $f_{t'}$ for t' in $\operatorname{Hom}(\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}}, \mathbb{G}_m | \mathcal{O}_{\mathbb{C}_p})$ denotes its defining power series (as in (3.2)).

The restrictions F_h of F onto closed discs of certain radii $\zeta_h < 1$, to be specified below, are given by the same power series.

Lemma 4.2. *Fix two sequences* (σ_h) *and* (ς_h) *of increasing radii below* 1 *given by*

- $\sigma_0 := p^{-1/(p-1)-1/e}$ *and* $\sigma_h := \sigma_0^{1/p^h}$,
- $\varsigma_0 := p^{-q/e(q-1)}$ *and* $\varsigma_h := \varsigma_0^{1/p^{dh}}$

and let (F_h) *with*

$$\mathcal{F}_h \colon \mathcal{A}(B^d_{\leq \sigma_h})/J \xrightarrow{\sim} \mathcal{A}(B_{\leq \varsigma_h})$$

be the compatible family of isomorphisms between affinoid algebras given by [ST01, Theorem 3.6]. Then the natural map

$$A^r_{\mathbf{K}} \xrightarrow{\sim} \varprojlim_{h \in \mathbb{N}} \mathcal{F}_h(\mathcal{A}(B^d_{\leq \sigma_h})/J)^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

is an isomorphism of $\mathbb{C}_p$*-Banach spaces.*

Proof: Let (ρ_h) be the sequence of increasing radii below 1 given by $\rho_0 := p^{-1/(p-1)}$ and $\rho_h := \rho^{1/p^h(p-1)}$. By Proposition 4.1 the natural map

$$A^r_{\mathbf{L}}/J \cap A^r_{\mathbf{L}} \xrightarrow{\sim} \varprojlim_{h \in \mathbb{N}} (\mathcal{A}(B^d_{\leq \rho_h})/J)^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

is an isomorphism of $\mathbb{C}_p$-Banach spaces. If δ'_0 and δ''_0 are two positive numbers < 1 then the sequences (δ'_h) and (δ''_h) given by $\delta_h := \delta_0^{1/p^h}$ are cofinal. This conclusion applies in particular to $\delta'_0 := \rho_0$ and $\delta''_0 := \sigma_0$. Thus

$$\varprojlim_{h \in \mathbb{N}} (\mathcal{A}(B^d_{\leq \rho_h})/J)^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p \xrightarrow{\sim} \varprojlim_{h \in \mathbb{N}} (\mathcal{A}(B^d_{\leq \sigma_h})/J)^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

We apply the compatible sequence of Schneider-Teitelbaum isomorphisms $(\mathcal{F}_h)$ to the compatible sequence of maps that induces this isomorphism of projective limits. Because the transition maps are in particular injective, we conclude by left exactness of the projective-limit functor

$$A^r_{\mathbf{K}} = \mathcal{F}(A^r_{\mathbf{L}}/J \cap A^r_{\mathbf{L}}) \xrightarrow{\sim} \varprojlim_{h\in\mathbb{N}} \mathcal{F}_h(\mathcal{A}(B^d_{\leq\sigma_h})/J)^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

□

We fix once for all the sequence of increasing radii (σ_h) below 1 given by $\sigma_0 := p^{-1/(p-1)-1/e}$ and $\sigma_h := \sigma_0^{1/p^h}$. To make the inverse limit $\varprojlim \mathcal{F}_h(\mathcal{A}(B^d_{\leq\sigma_h})/J)^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p$ explicit, we define two $\mathcal{O}_{\mathbb{C}_p}$-*Banach modules* A_h and C_h, that is, closed unit balls of $\mathbb{C}_p$-Banach spaces, such that

$$A_h \hookrightarrow \mathcal{F}_h(\mathcal{A}(B^d_{\leq\sigma_h})/J)^\circ \hookrightarrow C_h. \tag{4.4}$$

Lemma 4.3 (Definition of C_h in (4.4)). *Let $(\sigma_h)_{h\in\mathbb{N}}$ and $(\varsigma_h)_{h\in\mathbb{N}}$ be the sequences of increasing radii defined in Lemma 4.2. Then the isomorphism of affinoid algebras*

$$\mathcal{F}_h\colon \mathcal{A}(B^d_{\leq\sigma_h})/J \xrightarrow{\sim} \mathcal{A}(B_{\leq\varsigma_h})$$

defined in [ST01, Theorem 3.6] restricts to a monomorphism of $\mathcal{O}_{\mathbb{C}_p}$-Banach modules

$$\mathcal{F}_h\colon (\mathcal{A}(B^d_{\leq\sigma_h})/J)^\circ \hookrightarrow \mathcal{A}(B_{\leq\varsigma_h})^\circ.$$

Proof: Let $\bar{f}$ in $\mathcal{A}(B^d_{\leq\sigma_h})/J$. Because

1. by the maximum principle on the closed unit disc, the supremum norm is equal to the Gauss norm,
2. every isomorphism of affinoid algebras is an isometry of Banach spaces for the affinoid supremum norms, and
3. the supremum norm is bounded above by the Gauss residue norm,

we obtain

$$\|\mathcal{F}_h(\bar{f})\|_{\varsigma_h} = \|\mathcal{F}_h(\bar{f})\|_{\varsigma_h,\sup} = \|\bar{f}\|_{\sigma_h,\sup} \leq \|\bar{f}\|_{\sigma_h},$$

and conclude the claimed inclusion of unit balls. □

Lemma 4.4 (Definition of A_h in (4.4)). *Let $(\rho_h)_{h\in\mathbb{N}}$ and $(\varsigma_h)_{h\in\mathbb{N}}$ be the sequences of radii defined in Proposition 4.1 and Lemma 4.2. Put*

$$\mathcal{A}'(B_{\leq\varsigma_h}) := \{\textit{all } f \textit{ in } \mathbb{C}_p[[X]] \textit{ such that } f = \sum a_i X^i \textit{ and } |a_i| i \varsigma_h^i \to 0 \textit{ as } i \to \infty\}$$

together with its natural norm

$$\|\sum_{i\in\mathbb{N}} a_i X^i\|'_{\varsigma_h} := \sup\{|a_i| i \varsigma_h^i : i \in \mathbb{N}\}.$$

Then $\mathscr{A}'(B_{\leq\varsigma_h})$ is a $\mathbb{C}_p$-Banach space and the sequence of inverses of the rigid-analytic isomorphisms $F_0, F_1, \ldots$ given in [ST01, Theorem 3.6] induces a sequence of monomorphisms of $\mathcal{O}_{\mathbb{C}_p}$-Banach modules

$$\mathscr{F}_h^{-1}\colon \mathscr{A}'(B_{\leq\varsigma_h})^\circ \hookrightarrow (\mathscr{A}(B^d_{\leq\rho_h})/J)^\circ$$

with operator norm at most $c_h := c_0 p^{-h}$ (where $c_0 := |\bar{\gamma}(0)| p^{p/(p-1)+1/(e(q-1))}$ and the constant $\bar{\gamma}(0)$ is defined in [BK16, Section 1]).

Proof: Let $\mathscr{T}^h_{\mathbf{L}}\colon \mathscr{D}^{h-\mathrm{an}}_{\mathbf{L}}(\mathbb{Z}_p,\mathbb{C}_p) \hookrightarrow \mathscr{A}(B_{\leq\rho_h})$ for $h \in \mathbb{N}$ be the compatible family of Amice transforms. Because $\mathrm{im}\,\mathscr{T}^h_{\mathbf{L}} \supseteq \mathscr{A}(B_{\leq\rho_{h+1}})$, there is an inverse map

$$\mathscr{A}(B_{\leq\rho_{h+1}}) \hookrightarrow \mathscr{D}^{h-\mathrm{an}}_{\mathbf{L}}(\mathbb{Z}_p,\mathbb{C}_p)$$

of uniform operator norm $C = p^{p/(p-1)}$ independent of h (see [Nag15, Lemma 3.2]); accordingly there are maps $\mathscr{T}^h_{\mathbf{K}} : \mathscr{D}^{h-\mathrm{an}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \hookrightarrow \mathscr{A}(B_{\leq\rho_h})/J$ together with their inverses. By [BK16, Corollary 4.4] the map

$$(\mathscr{F} \circ \mathscr{T}^h_{\mathbf{K}})^{-1}\colon \mathscr{A}'(B_{\leq\varsigma_h}) \hookrightarrow D^{h-\mathrm{an}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \tag{$*$}$$

is a monomorphism of $\mathbb{C}_p$-Banach spaces with operator norm at most $c = c_0 p^{-h}$ where c_0 as defined above (and $\mathscr{F} = \mathscr{F}_i$ for sufficiently small i).

Let us show that the Amice transform

$$\mathscr{T}^h_{\mathbf{K}}\colon D^{h-\mathrm{an}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \hookrightarrow \mathscr{A}(B^d_{\leq\rho_h})/J \tag{$**$}$$

is norm-nonincreasing. First the map

$$\mathscr{T}^h_{\mathbf{L}}\colon D^{h-\mathrm{an}}_{\mathbf{L}}(\mathbb{Z}_p,\mathbb{C}_p) \hookrightarrow \mathscr{A}(B_{\leq\rho_h}).$$

is norm-nonincreasing: On the right-hand side, by the maximum principle, the supremum norm is equal to the Gauss norm. The character $\chi_z\colon x \mapsto x^z = \sum_{n\in\mathbb{N}} z^n \binom{x}{n}$ is h-analytic if z is in $B_{\leq\rho_h}$. Thence the restriction from all h-analytic functions to all h-analytic characters is injective and norm-nonincreasing. By the ultrametric Hahn-Banach Theorem ([PGS10, Theorem 4.2.4]), the surjection $\mathscr{D}^{h-\mathrm{an}}_{\mathbf{L}}(\mathcal{O},\mathbb{C}_p) \twoheadrightarrow \mathscr{D}^{h-\mathrm{an}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p)$ induces an isometry for the quotient-norm on the right-hand side. Let I be its kernel. We conclude that $\mathscr{T}^h_{\mathbf{K}}$ decomposes into the two norm-nonincreasing monomorphisms

$$D^{h-\mathrm{an}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p) \xrightarrow{\sim} D^{h-\mathrm{an}}_{\mathbf{L}}(\mathcal{O},\mathbb{C}_p)/I \hookrightarrow \mathscr{A}(B^d_{\leq\rho_h})/J.$$

Corollary 4.5 (Temperedness under the Fourier Transform). *Let $(\varsigma_h)_{h\in\mathbb{N}}$ be the sequence of radii defined in Lemma 4.2. The natural maps*

$$\varprojlim_h p^{-h}\mathscr{A}'(\mathrm{B}_{\leq\varsigma_h})^\circ \otimes_{\mathscr{O}_{\mathbb{C}_p}} \mathbb{C}_p \hookrightarrow \mathrm{A}^r_{\mathbf{K}} \hookrightarrow \varprojlim_h \mathscr{A}(\mathrm{B}_{\leq\varsigma_h})^\circ \otimes_{\mathscr{O}_{\mathbb{C}_p}} \mathbb{C}_p \tag{4.5}$$

are monomorphisms of $\mathbb{C}_p$-Banach spaces.

Proof: The natural map

$$\mathrm{A}^r_{\mathbf{K}} \xrightarrow{\sim} \varprojlim_{h\in\mathbb{N}} \mathscr{F}_h(\mathscr{A}(\mathrm{B}^d_{\leq\sigma_h})/\mathrm{J})^\circ \otimes_{\mathscr{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

is by Lemma 4.2 an isomorphism of $\mathbb{C}_p$-Banach spaces. We use it to prove that:

- The left-hand map in (4.5) is a monomorphism: By Lemma 4.4, if λ_0 in $\mathbb{C}_p$ satisfies $v(\lambda_0) = v(\bar{\gamma}(0)) + p/(p-1) + 1/e(q-1)$ then the natural map

  $$\lambda_0 \cdot p^{-h}\mathscr{A}'(\mathrm{B}_{\leq\varsigma_h})^\circ \hookrightarrow \mathscr{F}_h(\mathscr{A}(\mathrm{B}^d_{\leq\rho_h})/\mathrm{J})^\circ$$

  is a monomorphism of $\mathscr{O}_{\mathbb{C}_p}$-Banach modules. Thus if h_0 in $\mathbb{N}$ satisfies $\sigma_0 \leq \rho_{h_0}$ then the natural map

  $$\lambda_0 \cdot p^{-(h+h_0)}\mathscr{A}'(\mathrm{B}_{\leq\varsigma_{h+h_0}})^\circ \hookrightarrow \mathscr{F}_h(\mathscr{A}(\mathrm{B}^d_{\leq\rho_{h+h_0}})/\mathrm{J})^\circ \hookrightarrow \mathscr{F}_h(\mathscr{A}(\mathrm{B}^d_{\leq\sigma_h})/\mathrm{J})^\circ.$$

  is a monomorphism of $\mathscr{O}_{\mathbb{C}_p}$-Banach modules; thus, as projective limit of all these maps running over h in $\mathbb{N}$, the left-hand map in (4.5) is by cofinality a monomorphism as well.

- The right-hand map in (4.5) is a monomorphism: By Lemma 4.3 the natural map

  $$\mathscr{F}_h(\mathscr{A}(\mathrm{B}^d_{\leq\sigma_h})/\mathrm{J})^\circ \hookrightarrow \mathscr{A}(\mathrm{B}_{\leq\varsigma_h})^\circ.$$

  is a monomorphism; thus, as projective limit of all these maps running over h in $\mathbb{N}$, the left-hand map in (4.5) is a monomorphism as well. □

Temperedness as coefficient-wise convergence

We show that the temperedness condition of Corollary 4.5 on a power series f in $\mathrm{A}^r_{\mathbf{K}}$ that bounds the values of f on the open unit disc is equivalent to a condition that bounds the coefficients of f.

An *unbounded norm* on a vector space V is a map $\|\cdot\|\colon \mathrm{V} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ that satisfies all axioms of a norm (with the conventions for every $c \geq 0$ that $\infty \geq c$,

$\infty + c = \infty + c = \infty$ and $c \cdot \infty = \infty \cdot c = \infty$ if $c \neq 0$, respectively $0 \cdot \infty = \infty \cdot 0 = 0$). Every unbounded norm restricts to a norm over its set of *bounded elements* given by $\{v \in V : \|v\| < \infty\}$. Two unbounded norms $\|\cdot\|'$ and $\|\cdot\|''$ on V are *equivalent* if they have the same set of bounded elements V^{bd} and equivalent restricted norms on V^{bd}.

Lemma 4.6 (Adaption of [Col03, Lemme V.3.19]). *For* $\rho < 1$ *and* $s \geq 0$, *let* $\|\cdot\|_{\rho,s}$ *be the unbounded norm on* $\mathbb{C}_p[[X]]$ *given by*

$$\|\sum_{k\in\mathbb{N}} a_k X^k\|_{\rho,s} := \sup\{|a_0|\} \cup \{|a_k|\rho^k/k^s : k = 1,2,\ldots\}.$$

Let (ρ_n) *be the sequence of increasing radii below* 1 *given by* $\rho_n = \rho^{1/p^n} < 1$. *Then the unbounded norms*

$$\|f\|' = \sup\{p^{-nr}\|f\|_{\rho_n,s} : n \in \mathbb{N}\}$$

and

$$\|f\|'' = \sup\{|a_0|\} \cup \{|a_k|/k^{r+s} : k = 1,2,\ldots\}$$

on $\mathbb{C}_p[[X]]$ *are equivalent.*

Proof: Let us fix $f = \sum a_k X^k$. We show that there is $C > 1$, independent of f, such that

$$\|f\|'' \leq C \cdot \|f\|' \tag{$*$}$$

Put $v' := \|f\|' = \sup\{p^{-nr}\|f\|_{\rho_n,s} : n \in \mathbb{N}\}$. For every k and n,

$$|a_k|\rho_n^k/k^s \leq v' p^{nr}.$$

This inequality is in particular true for $n = \lfloor \log_p k \rfloor$ where $\log_p \cdot := \log \cdot / \log p$. Because $n \leq \log_p k$

$$|a_k| \leq v' p^{nr} k^s \rho_n^{-k} \leq v' k^{r+s} p^{-k \log_p \rho_n}.$$

By definition of ρ_n and because $\log_p k - 1 \leq n$,

$$-\log_p \rho_n = \log_p(1/\rho)/p^n \leq \log_p(1/\rho)/p^{\log_p k - 1}.$$

Together

$$|a_k| \leq v' k^{r+s} p^{\log_p(1/\rho)(k/p^{\log_p k-1})} \leq v' k^{r+s}(1/\rho)^p.$$

We conclude that if $C := (1/\rho)^p$ then $(*)$ holds.

Conversely we show that there is $C > 1$, independent of f, such that

$$\|f\|' \leq C \cdot \|f\|''. \tag{$**$}$$

Let $u > 0$ and $0 < a < 1$. The function $x \mapsto x^u a^x$ on $\mathbb{R}_{>0}$ has its maximum at $-u/\log a$ with value $e^{-u}(-u/\log a)^u$ (and if $u = 0$ then at 0 with value 1). Hence, if $u = r$ and $a = \rho_n$ then

$$\begin{aligned}\|f\|_{\rho_n,s} &= \sup\{|a_0|\} \cup \{|a_k|\rho_n^k/k^s : k = 1,2,\ldots\}\\ &\leq \sup\left(\{|a_0|\} \cup \{|a_k|/k^{r+s} : k = 1,2,\ldots\}\right)e^{-r}(-r/\log\rho_n)^r\\ &= \|f\|'' p^{nr}\left[r/\left(e\log 1/\rho\right)\right]^r\end{aligned}$$

where the last equality holds by definition of $\|\cdot\|''$ and ρ_n.

We conclude that, putting $C := \left[r/\left(e\log 1/\rho\right)\right]^r$,we have $p^{-nr}\|f\|_{\rho_n,s} \leq C\cdot\|f\|''$ for every n in $\mathbb{N}$. Therefore $(**)$ holds. □

Let henceforth $r \geq d$. For such $r \geq d$, we compute the middle term of the inclusion chain (4.5) in Corollary 4.5:

Lemma 4.7. *Let $\|\cdot\|$ be the unbounded norm on $\mathbb{C}_p[[X]]$ defined by*

$$\|\sum_{k\in\mathbb{N}} a_k X^k\| := \sup\{|a_0|\} \cup \{|a_k|/k^{r/d} : k = 1,2,\ldots\}.$$

If $r \geq d$, then, in the notation of Corollary 4.5,

$$\varprojlim_h p^{-h}\mathscr{A}'(B_{\leq\varsigma_h})^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p = \{f \in \mathbb{C}_p[[X]] : \|f\| < \infty\} = \varprojlim_h \mathscr{A}(B_{\leq\varsigma_h})^\circ \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

Proof: We have to show that the unbounded norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}_p[[X]]$ given by

$$\|f\|' = \sup\{p^{-n(r-1)}\|f\|_{\varsigma_n,1} : n \in \mathbb{N}\} \quad\text{and}\quad \|f\|'' = \sup\{p^{-nr}\|f\|_{\varsigma_n} : n \in \mathbb{N}\}$$

are equivalent to $\|\cdot\|$. Let $r \geq 0$ and $s \in \mathbb{R}$. For $r' := r/d$ and $\varsigma'_h := \varsigma_0^{1/p^h}$,

$$\{p^{-hr}\|\cdot\|_{\varsigma_h,s} : h \in \mathbb{N}\} = \{p^{-hdr'}\|\cdot\|_{\varsigma'_{hd},s} : h \in \mathbb{N}\} = \{p^{-hr'}\|\cdot\|_{\varsigma'_h,s} : h \in d\mathbb{N}\}.$$

Therefore, because $\|\cdot\|_{\varsigma'_h,s}$ is monotone in h, the unbounded norms

$$\sup\{p^{-hr}\|\cdot\|_{\varsigma_h,s} : h \in \mathbb{N}\} \quad\text{and}\quad \sup\{p^{-hr'}\|\cdot\|_{\varsigma'_h,s} : h \in \mathbb{N}\}.$$

are equivalent. We conclude that

- $\|\cdot\|'$ is equivalent to $\|\cdot\|$ by Lemma 4.6 for ς_0, $r/d-1$ and $s=1$, and
- $\|\cdot\|''$ is equivalent to $\|\cdot\|$ by Lemma 4.6 for ς_0, r/d and $s=0$. □

Recall the Fourier transform $\mathcal{F}\circ\overline{\mathcal{T}}$ defined in Section 3, the isomorphism of topological $\mathbb{C}_p$-algebras given by composition of the (quotient map of the) Amice transform $\overline{\mathcal{T}}$ and the Schneider-Teitelbaum isomorphism $\mathcal{F}$, given by

$$\mathcal{F}\circ\overline{\mathcal{T}}\colon\ \mathcal{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p)\xrightarrow{\sim}\mathcal{A}(\mathrm{B}_{<1})$$
$$\mu\mapsto\mu(\mathrm{P}_0)+\mu(\mathrm{P}_1)\mathrm{X}+\mu(\mathrm{P}_2)\mathrm{X}^2+\cdots$$

where $\mathrm{P}_0,\mathrm{P}_1,\ldots$ are the Fourier polynomials (denoted by $\mathrm{P}_0(\Omega\cdot),\mathrm{P}_1(\Omega\cdot),\ldots$ in [ST01]).

Corollary 4.8. *If $r\geq d$, then the map*

$$\begin{aligned}\mathcal{D}^{r}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p)&\xrightarrow{\sim}\ \{\textit{all}\ \textstyle\sum a_n\mathrm{X}^n\ \textit{in}\ \mathbb{C}_p[[\mathrm{X}]]\ \textit{with}\ \{|a_n|/n^{r/d}\}\ \textit{bounded}\,\}\\ \mu&\mapsto\ \mu(\mathrm{P}_0)+\mu(\mathrm{P}_1)\mathrm{X}+\mu(\mathrm{P}_2)\mathrm{X}^2+\cdots\end{aligned}$$

is an isomorphism of $\mathbb{C}_p$-Banach spaces (where the right-hand side is equipped with its natural norm $\|\sum_{n\in\mathbb{N}}a_n\mathrm{X}^n\|:=\max\{|a_0|\}\cup\{|a_n|/n^{r/d}:n=1,2,\ldots\}$).

Proof: The map is by [ST01, Lemma 4.6.9] the restriction of the isomorphism of $\mathbb{C}_p$-Fréchet algebras

$$\mathcal{D}^{\mathrm{la}}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p)\xrightarrow{\sim}\mathcal{A}(\mathrm{B}_{<1})$$

to $\mathcal{D}^{r}_{\mathbf{K}}(\mathcal{O},\mathbb{C}_p)$. Its image is $\mathrm{A}^r_{\mathbf{K}}$ and by Lemma 4.7 the $\mathbb{C}_p$-Banach space $\mathrm{A}^r_{\mathbf{K}}$ is equal to the right-hand side. □

Remark 4.9. This isomorphism holds by [ST01, Corollary 3.7 ff.] over every complete subfield $\mathbf{E}$ of $\mathbb{C}_p$ that includes $\mathbf{K}$ and contains the transcendent number Ω (of Equation (3.1)). If $\mathbf{E}$ does not necessarily contain Ω then, by the descent condition of [ST01, Corollary 3.8], still $\mathcal{D}^{r}_{\mathbf{K}}(\mathcal{O},\mathbf{E})$ is given by

$$\begin{aligned}\mathcal{D}^{r}_{\mathbf{K}}(\mathcal{O},\mathbf{E})\xrightarrow{\sim}\{&\text{all}\ \textstyle\sum a_n\mathrm{X}^n\ \text{in}\ \mathbb{C}_p[[\mathrm{X}]]\ \text{with}\ \{|a_n|/n^{r/d}\}\ \text{bounded}\\ &\text{and}\ \sigma(a_n)=\tau(\sigma)^n\cdot a_n\ \text{for all}\ n\in\mathbb{N}\ \text{and}\ \sigma\in\mathrm{G}_{\mathbf{E}}\}\end{aligned}$$

where $\mathrm{G}_{\mathbf{E}}=\mathrm{Gal}(\bar{\mathbf{E}}/\mathbf{E})$ is the absolute Galois group of $\mathbf{E}$ and $\tau\colon\mathrm{G}_{\mathbf{E}}\to\mathcal{O}^*$ is the character that defines the Galois action on $\mathrm{Hom}(\mathcal{G},\mathbb{G}_m)\cong\mathcal{O}$.

5 Dualizing

For $s \geq 0$ put $c_0^s(\mathbb{N}, \mathbb{C}_p) := \{(a_n)_{n\in\mathbb{N}} \colon |a_n| n^s \to 0\}$. It is a $\mathbb{C}_p$-Banach space for the natural norm $\|(a_n)\| = \max\{|a_n| n^s : n \in \mathbb{N}\}$.

Theorem 5.1. *Let* $\mathrm{P}_0, \mathrm{P}_1, \ldots$ *denote the Fourier polynomials. If* $r \geq d$*, then the map*

$$\begin{aligned} c^{r/d}(\mathbb{N}, \mathbb{C}_p) &\xrightarrow{\sim} \mathscr{C}^r_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p) \\ (a_n) &\mapsto \sum_{n\in\mathbb{N}} a_n \mathrm{P}_n. \end{aligned}$$

is an isomorphism of topological $\mathbf{K}$*-vector spaces.*

Proof: For $s \geq 0$, let $c_b^s(\mathbb{N}, \mathbb{C}_p) = \mathrm{Hom}_{\mathbb{C}_p}(c^s(\mathbb{N}, \mathbb{C}_p), \mathbb{C}_p)$ be the continuous dual of $c_0^s(\mathbb{N}, \mathbb{C}_p)$; it is explicitly given by

$$c_b^s(\mathbb{N}, \mathbb{C}_p) = \{(a_n)_{n\in\mathbb{N}} : \{|a_n|/n^s : n \in \mathbb{N}\} \text{ bounded }\}$$

with its natural supremum norm. As topological $\mathbf{K}$-vector space $\mathrm{A}^r_{\mathbf{K}}$ is identical to $c_b^{r/d}(\mathbb{N}, \mathbb{C}_p)$.

Consider the homomorphism of topological $\mathbf{K}$-vector spaces given by

$$\begin{aligned} \phi\colon\ c_0^{r/d}(\mathbb{N}, \mathbb{C}_p) &\to \mathscr{C}^r_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p) \\ (a_n)_{n\in\mathbb{N}} &\mapsto \sum a_n \mathrm{P}_n. \end{aligned}$$

It is well-defined, that is, $\sum a_n \mathrm{P}_n$ converges in $\mathscr{C}^r_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p)$, because by [Sch02, Lemma 9.9] for every n in $\mathbb{N}$ the norm $\|\cdot\|_{\mathscr{C}^r}$ is equivalent to its double dual norm whose values on $\mathrm{P}_0, \mathrm{P}_1, \ldots$ are given by Corollary 4.8; for this, note that the homomorphism

$$\begin{aligned} \psi : \mathscr{D}^r_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p) &\to c_0^{r/d}(\mathbb{N}, \mathbb{C}_p) \\ \mu &\mapsto \mu \circ \phi. \end{aligned}$$

dual to ϕ sends μ to the sequence $\mu(\mathrm{P}_0), \mu(\mathrm{P}_1), \ldots$ That is, ψ is the isomorphism given in Corollary 4.8. This isomorphism holds for the strong topology of uniform convergence; consequently, for the bounded weak topology.

We conclude by Schikhof duality ([ST02, Theorem 1.2]) that ϕ is an isomorphism of topological $\mathbf{K}$-vector spaces. In particular f in $\mathscr{C}^r_{\mathbf{K}}(\mathcal{O}, \mathbb{C}_p)$ if and only if $f(x) = \sum_{n\in\mathbb{N}} a_n \mathrm{P}_n(x)$ for all $x \in \mathcal{O}_{\mathbf{F}}$ with $|a_n| n^{r/d} \to 0$ as $n \to \infty$. □

Remark (dual to Remark 4.9). This isomorphism holds for every complete subfield $\mathbf{E}$ of $\mathbb{C}_p$ that includes $\mathbf{K}$ and contains Ω. If $\mathbf{E}$ does not necessarily contain Ω then $\mathscr{C}^r_{\mathbf{K}}(\mathcal{O}, \mathbf{E})$ is given by all functions $f : \mathcal{O} \to \mathbb{C}_p$ of the form

$$f(x) = \sum_{n \in \mathbb{N}} a_n \mathrm{P}_n(x) \quad \text{with} \quad |a_n| n^{r/d} \to 0 \text{ as } n \to \infty$$

such that $\sigma(a_n) = \tau(\sigma)^n \cdot a_n$ for all $n \in \mathbb{N}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbf{E}}/\mathbf{E})$.

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