Contemporary Mathematics

From Crystalline to Unitary Representations

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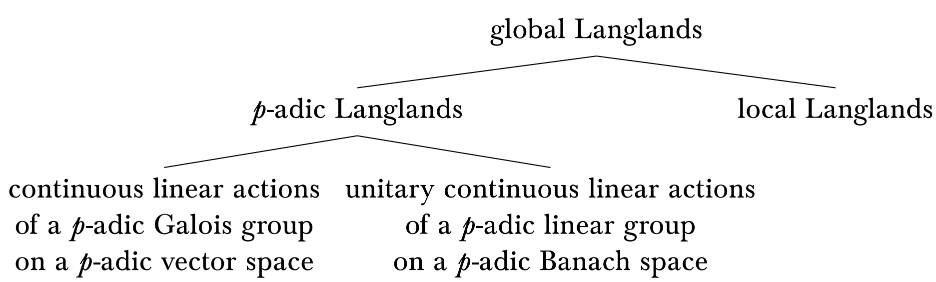
Abstract. We review for the unacquainted a key construction in the p-adic Langlands program: The functor from the category of 2-dimensional crystalline representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of $\mathbb{Q}_p$ to that of unitary actions of the general linear group $\mathrm{GL}_2(\mathbb{Q}_p)$ on a quotient Banach-space of fractionally differentiable functions.

Introduction

Let p be a prime. Whereas the *global* Langlands correspondence links continuous linear actions of the absolute Galois group of $\mathbb{Q}$ on finite-dimensional vector spaces with actions of a general linear group on, usually infinite-dimensional, function spaces, this survey treats specifically the *p-adic* Langlands correspondence that links continuous actions of the absolute Galois group of the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ on p-adic vector spaces of dimension n with unitary continuous linear actions of the general linear group $\mathrm{GL}_n(\mathbb{Q}_p)$ on, usually infinite-dimensional, p-adic Banach spaces. An important distinction is here the topology of the coefficient field of these vector spaces: If it is again a p-adic number field then one speaks of the *p-adic* Langlands correspondence, else (for example $\mathbb{C}$ or $\overline{\mathbb{Q}}_l$ for $l \neq p$) of the *local* Langlands correspondence (as only in the latter, local, case the actions of the absolute Galois group of $\mathbb{Q}_p$ reduce to those of finite image). The p-adic Langlands correspondence hence branches off as follows:



To be more precise: Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$. Let n in $\mathbb{N}$.

- A *p-adic Galois representation* is a continuous linear action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of $\mathbb{Q}_p$ on an n-dimensional vector space over $\mathbf{K}$.
- A *p-adic Banch-space representation* is a continuous linear action of $\mathrm{GL}_n(\mathbb{Q}_p)$ on a Banach space over $\mathbf{K}$ (usually of infinite dimension).

A p-adic Banach space representation is *unitary* if the norm of every vector is invariant under the action of all of $\mathrm{GL}_n(\mathbb{Q}_p)$. Among all p-adic Galois representations, there are the *geometric* ones, those that are subquotients of a (p-adic étale) cohomology group on a (smooth proper) variety. Among all geometric ones, there are the *crystalline* ones, those that are determined by two other cohomology groups, the *de Rham* and *crystalline* one, on which there is no Galois action but a *filtration* and an automorphism, the *Frobenius*. The equivalent data of the de Rham and crystalline cohomology groups is more explicit than that of a Galois representation and used to parametrize all crystalline (p-adic Galois) representations. This article surveys the construction of the functor

$$\left\{\begin{matrix}\text{crystalline representations}\\ \text{of dimension } 2\end{matrix}\right\} \to \left\{\begin{matrix}\text{unitary } p\text{-adic Banach space}\\ \text{representations of } \mathrm{GL}_2(\mathbb{Q}_p)\end{matrix}\right\}$$

as given in [**BB10**]. For a similar construction in the case of reducible, *trianguline*, p-adic Galois representations, see [**Col10b**].

This functor passes through several categories equivalent to that of crystalline representations before a unitary p-adic Banach space representation is obtained:

- As touched upon above, every crystalline representation is determined by a *filtered* ϕ*-module*, a filtration and an automorphism ϕ. The filtered ϕ-modules that are attached to crystalline representations are those that are *admissible*, a condition that bounds the valuation of the eigenvalues of ϕ by the *filtration jumps*, the indices in $\mathbb{Z}$ where the filtration changes. That is, by [**CF00**, Theorem 1] an equivalence of categories:

  $$\left\{\begin{matrix}\text{crystalline representations}\\ \text{of dimension } n\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{admissible} \text{ filtered } \phi\text{-modules}\\ \text{of dimension } n\end{matrix}\right\}$$

  This is reviewed in Part 1.

At this point, already a general map

$$\left\{\begin{matrix}\textit{admissible} \text{ filtered } \phi\text{-modules}\\ \text{of dimension } n\end{matrix}\right\} \to \left\{\begin{matrix}\text{unitary } p\text{-adic Banach space}\\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p)\end{matrix}\right\}$$

can be defined, as follows: Let V be a filtered ϕ-module. Then

$$\mathrm{V} \mapsto \phi^{\mathrm{ss}}, \kappa = \begin{pmatrix}\theta_1 & & \\ & \ddots & \\ & & \theta_n\end{pmatrix}, (k_1, \dots, k_n)$$

where $\theta_1, \ldots, \theta_d$ are the eigenvalues, or semisimplification, of ϕ (which, after taking a finite extension, we may assume to be in $\mathbf{K}$) and $k_1, \ldots, k_d$ are the filtration jumps (with multiplicities, that is, each filtration jump occurs dimension of the graduation step many times).

To ϕ^{ss} and κ, we attach characters θ and ψ on the diagonal matrices T of $\mathrm{GL}_n(\mathbb{Q}_p)$ by

$$\begin{array}{ccc} \mathrm{T} \xrightarrow{\theta} \mathbf{K}^* & & \mathrm{T} \xrightarrow{\psi} \mathbf{K}^* \\ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \theta_1^{v(t_1)} \cdots \theta_d^{v(t_d)} & \text{and} & \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1} \cdots t_n^{k_n} \end{array}$$

Let $\mathrm{G} = \mathrm{GL}_n(\mathbb{Q}_p)$. Let $\overline{\mathrm{B}}$ be the subgroup of all lower triangular matrices of G and $\overline{\mathrm{N}}$ the subgroup of $\overline{\mathrm{B}}$ of all matrices whose diagonal entries are all 1. Because $\overline{\mathrm{B}} = \overline{\mathrm{N}}\mathrm{T}$ and $\overline{\mathrm{N}}$ is the commutator of $\overline{\mathrm{B}}$, the character $\chi = \theta\psi$ extends uniquely from T to $\overline{\mathrm{B}}$. Let

$$\mathrm{ind}_{\overline{\mathrm{B}}}^{\mathrm{G}} := \mathbf{K}[\mathrm{G}] \otimes_{\mathbf{K}[\overline{\mathrm{B}}]} \mathbf{K}$$

where $\mathbf{K}$ is the $\mathbf{K}[\overline{\mathrm{B}}]$-module given by χ. Explicitly,

$$\mathrm{ind}_{\overline{\mathrm{B}}}^{\mathrm{G}} \chi := \left\{ f : \mathrm{G} \to \mathbf{K} : f(\cdot b) = \theta\psi(b) f \text{ for all } b \in \mathrm{B} \right\},$$

where G acts by right translation, and

$$\mathrm{ind}_{\overline{\mathrm{B}}}^{\mathrm{G}} \chi^{\mathrm{lr}} := \left\{ f \in \mathrm{ind}_{\overline{\mathrm{B}}}^{\mathrm{G}} \chi : f \text{ is locally a rational function } \right\},$$

the *locally algebraic* vectors of $\mathrm{ind}_{\overline{\mathrm{B}}}^{\mathrm{G}} \chi$. Let $\mathcal{O}_{\mathbf{K}}$ be the ring of integers of $\mathbf{K}$. Then $i(\chi) := \mathrm{ind}_{\overline{\mathrm{B}}}^{\mathrm{G}} \chi^{\mathrm{lr}}$ is an $\mathbf{K}[\mathrm{G}]$-module of finitely many generators, and the $\mathcal{O}_{\mathbf{K}}[\mathrm{G}]$-module $\mathfrak{L}$ generated by these is a *lattice* of $i(\chi)$, an $\mathcal{O}_{\mathbf{K}}$-submodule that generates the including $\mathbf{K}$-vector space. The lattice $\mathfrak{L}$ is (stable under G if and only if it is) the unit ball of a norm which is *unitary*, that is, invariant under the action of G. The completion

$$\widehat{i(\chi)}$$

of $i(\chi)$ for this norm is the *universal unitary completion*, the unitary Banach $\mathbf{K}[\mathrm{G}]$-module that surjects onto every other unitary completion of $i(\chi)$.

However, the assignment

$$\mathrm{V} \mapsto \widehat{i(\chi)}$$

only keeps the jumps, but forgets the subspaces of the filtration of V! Speculatively, these subspaces correspond to quotients of $\widehat{i(\chi)}$. At the moment, [**BS07**] just cautiously conjectures that $\widehat{i(\chi)}$ is nonzero.

This map is quickly set up, however it tells us little about $\widehat{i(\chi)}$: For example, whether it is nonzero (that is, whether the lattice $\mathfrak{L}$ is the whole vector space $i(\chi)$). For this, we shall show for $n = 2$ that $\widehat{i(\chi)}$ is obtained from V by a functor. This will not only prove that $\widehat{i(\chi)}$ is nonzero, but also irreducible (if V is): The universal unitary completion is irreducible as topological $\mathbf{K}[G]$-module (if V is), and is a quotient of a space of (fractionally) differentiable functions by the closure of a cyclic $\mathbf{K}[G]$-module ([**BB10**, Théorème 4.3.1]). (Indeed, the irreducibility of $\widehat{i(\chi)}$ corresponds to the existence, up to isomorphism, of a single admissible filtration on V.)

To define this functor, we take a detour through the theory of φ, Γ-modules:

- For general n-dimensional p-adic Galois representations, we review in Part 2 another equivalence,

$$\left\{ \begin{array}{c} p\text{-adic Galois representations} \\ \text{of dimension } n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{étale } \varphi, \Gamma\text{-modules} \\ \text{of dimension } n \end{array} \right\}$$

  that to the category of *étale* φ, Γ*-modules* of continuous actions of $\mathbb{Z}_p^\bullet := p^{\mathbb{N}}\mathbb{Z}_p^* = \varphi^{\mathbb{N}}\Gamma$ on an n-dimensional free module over a coefficient field $\mathscr{E}$ of convergent power series in $T^{\pm 1}$ in characteristic 0 whose p-adic unit ball lifts the function field $\mathbb{F}_p((t))$ of characteristic p. Two observations underlie this equivalence:
  - For a field E of characteristic p, with separable closure $\bar{E}$ and absolute Galois group $\mathrm{Gal}(\bar{E}/E)$, the equivalence of categories

$$\left\{ \begin{array}{c} \text{continuous actions of } \mathrm{Gal}(\bar{E}/E) \\ \text{on an } \mathbb{F}_p\text{-vector space} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{étale } \varphi\text{-modules} \\ \text{on an E-vector space} \end{array} \right\}$$
$$V \mapsto (V \otimes_{\mathbb{F}_p} \bar{E})^{\mathrm{Gal}(\bar{E}/E)}$$

  - Let $\mu_{p^{-\infty}}$ be all roots of unity of p-power order. Put

$$\begin{aligned} \mathbb{Q}_{p^{-\infty}} &:= \mathbb{Q}_p(\mu_{p^{-\infty}}) &&\text{and} & \overline{\mathbb{Q}}_p &= \text{ algebraic closure of } \mathbb{Q}_{p^{-\infty}}, \\ E &:= \mathbb{F}_p((t)) &&\text{and} & \bar{E} &= \text{ separable closure of E.} \end{aligned}$$

    Then there is an isomorphism of topological groups (the *field of norms*)

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) \xrightarrow{\sim} \mathrm{Gal}(\bar{E}/E).$$

  To carry the equivalence of categories from operations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}})$ on $\mathbb{F}_p$-vector spaces to those of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\mathbb{Q}_p$-vector spaces, we use
  - that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) =: \Gamma \xrightarrow{\sim} \mathbb{Z}_p^*$, and
  - that the field $\mathbb{F}_p((t))$ of characteristic p lifts to one of characteristic 0 (denoted $\mathscr{E}$).

  The group $\mathbb{Z}_p$ embeds into $\mathcal{O}_{\mathscr{E}}$ and acts via scalar multiplication on our φ, Γ-module D. By choosing a section ψ of φ the action of $p = \varphi$ becomes invertible on

$$\psi^{-\infty}D := \varprojlim_{\psi} D$$

(where all transition maps are given by ψ) and the actions of $\mathbb{Z}_p$ and $\mathbb{Z}_p^\bullet$ on D induce an action of the *mirabolic subgroup*

$$M := \begin{pmatrix} 1 & \mathbb{Q}_p \\ & \mathbb{Q}_p^* \end{pmatrix} = \left\langle \begin{pmatrix} 1 & \mathbb{Z}_p \\ & p^{\mathbb{Z}}\mathbb{Z}_p^* \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \mathbb{Z}_p \\ & \mathbb{Z}_p^\bullet \end{pmatrix}, \right\rangle$$

on $\psi^{-\infty}$D.

- To extend this action on $\psi^{-\infty}$D from M to $\mathrm{GL}_2(\mathbb{Q}_p)$, we regard certain *bounded* (p-adically and T-adically mod p^n for all n) submodules of the module D over $\mathscr{E}$ (= ring of p-adic power series in $\mathrm{T}^{\pm 1}$): In Part 3, we give the image of the above equivalence of categories on the subcategory of all crystalline Galois representations, that of all *Wach modules*, étale φ, Γ-modules that are of *finite height*, that is, already defined over the ring $\mathscr{E}^+$ of all (p-adically) bounded power series in T. p-adic Hodge theory then shows how the induced equivalence of categories

$$\{\text{admissible filtered } \phi\text{-modules}\} \leftrightarrow \{\text{Wach modules}\}$$

passes directly (that is, without passing through the category of étale φ, Γ-modules) from the explicit data of a filtered ϕ-module V to that of a Wach module $\mathscr{N}$.

- For dimension 2, we show in Part 4 how to obtain from the action of M on $\psi^{-\infty}$D an action of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ on a bounded submodule $\mathbb{D}$ of $\psi^{-\infty}$D. To define $\mathbb{D}$, we observe that by compactness of the Galois group, there is a p-adic lattice T in the p-adic Galois representation, and consequently a p-adic lattice $\mathscr{N}(\mathrm{T})$ in $\mathscr{N}$. We put $\mathbb{D} := \psi^{-\infty}\mathscr{N}(\mathrm{T}) \otimes_{\mathscr{O}_{\mathbf{K}}} \mathbf{K}$

  To define the action of G on $\mathbb{D}$, we identify the module over a power series ring $\mathbb{D}$ with the dual of a Banach space representation of G. This identification is given by evaluation on the *Mahler polynomials*:

  Let $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ be the normed (by the supremum norm) vector space of all continuous functions $f : \mathbb{Z}_p \to \mathbf{K}$ and let $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K})$ be its dual of all continuous linear maps $\mu : \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}$. Every continuous function $f : \mathbb{Z}_p \to \mathbf{K}$ is uniformly approximated by locally constant functions f_n in $\mathbf{K}[\mathbb{Z}/p^n\mathbb{Z}]$; dually, the natural map

$$\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \mathbf{K} \otimes \mathscr{O}_{\mathbf{K}}[[\mathbb{Z}_p]]$$

is an isomorphism of topological $\mathbf{K}$-algebras, where
  - the left-hand side is equipped with the convolution product, and
  - the right-hand side is the completed group ring $\mathbf{K} \otimes \varprojlim \mathscr{O}_{\mathbf{K}}[\mathbb{Z}/p^n\mathbb{Z}]$.

The topological group $\mathbb{Z}_p$ is generated by a single element, say $\gamma = \mathbf{1}$, yielding the *Iwasawa isomorphism* of topological algebras

$$\mathscr{O}_{\mathbf{K}}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathscr{O}_{\mathbf{K}}[[t]]$$

defined by $\gamma + 1 \mapsto t$. The composed isomorphism

$$\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \mathbf{K} \otimes \mathscr{O}_{\mathbf{K}}[[t]]$$
$$\mu \mapsto \mu\tbinom{\cdot}{0} + \mu\tbinom{\cdot}{1} t + \mu\tbinom{\cdot}{2} t^2 + \cdots$$

sends a continuous linear map $\mu\colon \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}$ to the power series whose coefficients are its values $\mu(\tbinom{\cdot}{0}), \mu(\tbinom{\cdot}{1}), \ldots$ on the basis of *Mahler polynomials*, given by $\tbinom{x}{n} := x(x-1)\cdots(x-n)/n!$.

After a choice of basis, $\mathscr{N}$ is a submodule over $\mathscr{E}^+$ $(=\mathbf{K} \otimes \mathscr{O}_{\mathbf{K}}[[t]])$ of rank 2 inside two copies $\mathscr{R}^+ \oplus \mathscr{R}^+$ of the ring of all power series that converge on the open unit ball of $\mathbb{C}_p$. Evaluation on the Mahler polynomials embeds $\mathbb{D}$ into the duals of two Banach spaces of (fractionally) differentiable functions of compact support (whose degrees of differentiability r and s are given by the valuation of the eigenvalues of φ)

$$\mathbb{D} \hookrightarrow \mathscr{D}^r_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^s_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}).$$

- Part 5 describes the exact image of $\mathbb{D}$: Let $\overline{\mathrm{B}}$ be the lower triangular matrices in G and $\chi\colon \mathbf{B} \to \mathbf{K}^*$ a character. Let

$$\operatorname{ind}^{\mathrm{lr}} \chi := \{f : \mathrm{G} \to \mathbf{K} \text{ locally rational and } f(\cdot b) = \theta\psi(b) f \text{ for all } b \in \mathrm{B}\}$$

be the *locally algebraic* (or rational) *induction* of χ from $\mathbf{B}$ to G, that is, given by functions that are locally rational on G.

The group G acts on $\operatorname{ind}^{\mathrm{lr}} \chi$ by translation from the right. Let

$$\widehat{\operatorname{ind}^{\mathrm{lr}}} \chi := \text{ \emph{universal unitary completion} of } \operatorname{ind}^{\mathrm{lr}} \chi,$$

the unique *unitary* completion (that is, $\|g\cdot\| = \|\cdot\|$ for all g in G) of $\operatorname{ind}^{\mathrm{lr}} \chi$ that maps onto every other unitary completion of $\operatorname{ind}^{\mathrm{lr}} \chi$. Then

$$\mathbb{D} \xrightarrow{\sim} \widehat{\operatorname{ind}^{\mathrm{lr}} \Psi\theta}^{\,*}$$

the continuous dual of the universal unitary completion of $\operatorname{ind}^{\mathrm{lr}} \Psi\theta$, where, referring us to the filtered φ-module V we started with,
  - the unramified character $\theta\colon \mathrm{B} \to \mathbf{K}^*$ is determined by the eigenvalues of φ,
  - the algebraic character $\Psi\colon \mathrm{B} \to \mathbf{K}^*$ is determined by the filtration jumps of V.

  As corollaries, we obtain that $\widehat{\operatorname{ind}^{\mathrm{lr}} \Psi\theta}$ is nonzero and, if D is irreducible to begin with, then the action of M on $\widehat{\operatorname{ind}^{\mathrm{lr}} \Psi\theta}$ is topologically irreducible.

To trace out, the functor

$$\left\{\begin{matrix}\text{crystalline representations}\\ \text{of dimension 2}\end{matrix}\right\} \to \left\{\begin{matrix}\text{unitary } p\text{-adic Banach space}\\ \text{representations of } \mathrm{GL}_2(\mathbb{Q}_p)\end{matrix}\right\}$$

takes the following route, each arrow being worked out in its proper section:

$$
\begin{array}{ccc}
\{\, p\text{-adic Galois representations} \,\} & \leftrightarrow & \{\, \text{étale } \varphi, \Gamma\text{-modules} \,\} \\
\bigcup & & \\
\{\, \text{crystalline representations} \,\} & & \bigcup \\
\updownarrow & & \\
\{\, \text{admissible filtered } \phi\text{-modules} \,\} & \leftrightarrow & \{\, \text{Wach modules} \,\} \\
 & & \downarrow \\
 & & \left\{ \begin{array}{c} \text{unitary } p\text{-adic Banach space} \\ \text{representations of } \mathrm{GL}_2(\mathbb{Q}_p) \end{array} \right\}.
\end{array}
$$

Part 1. p-adic Hodge Theory

The main tool to construct a (p-adic) Galois representation V comes from geometry, that is, $V = H^{\bullet}_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is a certain sheaf cohomology group of a proper smooth algebraic variety X over $\mathbb{Q}$, the *p-adic étale* cohomology. However, as a p-adic Galois representation, it is hard to compute.

Let V be a p-adic Galois representation. In the following, we define

- a Fréchet algebra $\mathbb{B}_{\text{dR}}$ over $\mathbb{Q}_p$, a field, with a filtration and Galois action, by which we can endow V with a filtration by which we can conjecturally detect whether it is a subquotient of some $H^{\bullet}_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ (up to twist by a power of the cyclotomic character). (If $V = H^{\bullet}_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ then this is the filtration given by another, the *de Rham*, cohomology group.)
- a subalgebra $\mathbb{B}_{\max}$ of $\mathbb{B}_{\text{dR}}$ over $\mathbb{Q}_p$ with a continuous automorphism φ, by which we can endow V with an automorphism ϕ and reconstruct the Galois action on V by the filtration and automorphism ϕ on V. (If $V = H^{\bullet}_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and the proper smooth variety X is of good reduction, that is, it is the base extension of a proper smooth variety over $\mathbb{Z}_p$, then ϕ is the automorphism given by another, *crystalline*, cohomology group.)

All of this, the algebras and the equivalence of vector spaces with a continuous action of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and vector spaces with a filtration and an automorphism φ, will in the following be defined abstractly, without reference to any cohomology theories.

1. Big Rings

Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$. The field $\mathbb{B}^+_{\text{dR}}$ is the canonical complete local field of characteristic 0 and residue field $\mathbb{C}_p$. To construct it, we need the following notions:

- The *characteristic* of a ring A is the nonnegative integer that generates the kernel of the canonical morphism of rings $\mathbb{Z} \to A$. For a ring A of characteristic p, its *Frobenius* is the ring endomorphism $\cdot^p$. A ring A of characteristic p is *perfect* if its Frobenius is an automorphism.

Let A be a topological ring of characteristic p. Then

$$\widetilde{\mathrm{E}}^+(\mathrm{A}) := \varprojlim_{\cdot p} \mathrm{A} \quad \text{whose countably many transition maps are all } \cdot^p$$

is the *universal* topological ring R whose Frobenius is injective and that has a morphism of topological rings $\mathrm{R} \to \mathrm{A}$ (by the universal property of the projective limit).

Let $\overline{\mathbb{Z}}_p$ be the ring of integers of $\overline{\mathbb{Q}}_p$. Then we may in particular apply this to $\mathrm{A} = \overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$: It is a discrete topological ring of characteristic p but not perfect (as its Frobenius is not injective). Let

$$\widetilde{\mathrm{E}}^+ := \widetilde{\mathrm{E}}^+(\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p)$$

be the universal perfect topological ring that maps onto $\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$. It is complete and Hausdorff and its topology is given by the valuation

$$v_{\widetilde{\mathrm{E}}^+}(x) := v_{\overline{\mathbb{Z}}_p}(\lim_{n\to\infty} \widehat{x}_n^{p^n})$$

where $\widehat{x}_n$ is a lift of x_n from $\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$ to $\overline{\mathbb{Z}}_p$ and $v_{\overline{\mathbb{Z}}_p}$ is the p-adic completion of $\overline{\mathbb{Z}}_p$ such that $v_{\overline{\mathbb{Z}}_p}(p) = 1$.

- A *strict p-ring* is a ring
  - in which p is not a zero divisor,
  - that is complete for the p-adic topology, and
  - whose mod-p reduction is a perfect topological ring.

  Let A be a strict p-ring and $a = \mathrm{A}/p\mathrm{A}$ its mod-p reduction. As a set,

$$(*) \qquad \mathrm{A} = \left\{ \sum_{n\geq 0} \widehat{x}_n p^n : x_0, x_1, \ldots \text{ in } a \right\}$$

  for a section $\widehat{\cdot}\colon a \to \mathrm{A}$. Because a is perfect, there is a unique multiplicative section, the *Teichmüller* section

$$[\cdot]\colon x \mapsto \lim \widehat{x^{p^{-n}}}^{p^n}.$$

  For example $\mathbb{Z}_p$ is a strict p-ring and the image of its Teichmüller lift is given by all $p-1$-th roots of unity and 0.

THEOREM. *There is an equivalence of categories*

$$\mathrm{W}\colon \left\{ \begin{matrix} \textit{perfect topological rings} \\ \textit{of characteristic } p \end{matrix} \right\} \to \left\{ \textit{strict } p\textit{-rings} \right\}$$

PROOF: Given a perfect topological ring a of characteristic p, its countable product $\mathrm{A} := a^{\mathbb{N}}$ is a topological space. There is a continuous addition $x + y = z$ and multiplication $x \cdot y = z$ on A where $z_0, z_1, \ldots$ are given by polynomials in $x_0, x_1, \ldots; y_0, y_1, \ldots$ and its roots of order a power of p.

Confer [FO14, Section 0.2.3] for this classic construction (or [CD14] for a recent alternative). □

In particular W lifts every morphism $h\colon a \to b$ between perfect topological rings of characteristic p to a morphism $\mathrm{H}\colon \mathrm{A} \to \mathrm{B}$ between strict p-rings of mod-p reductions a and b respectively.

The topological ring $\widetilde{\mathrm{E}}^+$ being perfect, let

$$\mathcal{O}_{\widetilde{\mathcal{E}}^+} := \mathrm{W}(\widetilde{\mathrm{E}}^+).$$

be the strict p-ring of mod-p reduction $\widetilde{\mathrm{E}}^+$. For example, the Frobenius automorphism $\cdot^p$ on $\widetilde{\mathrm{E}}^+$ lifts to a *Frobenius* automorphism on $\mathcal{O}_{\widetilde{\mathcal{E}}^+}$.

This lifting holds for a general p-adically complete ring B with mod-p reduction b:

Proposition 1.1. *Let* A *be a strict p-ring and let* B *be a p-adically complete ring and a and b the respective mod-p reductions. For every morphism $\phi\colon a \to b$ there is a unique morphism $\Phi\colon \mathrm{A} \to \mathrm{B}$ that lifts ϕ, that is,*

$$\begin{array}{ccc} \mathrm{A} & \xrightarrow{\Phi} & \mathrm{B} \\ \downarrow & & \downarrow \\ a & \xrightarrow{\phi} & b \end{array}$$

commutes.

Proof: Let us fix lifts $\hat{\cdot}$ on a and on b. Recall the Teichmüller lift $[\cdot]\colon a \to \mathrm{A}$ given by

$$[x] = \lim \widehat{x^{p^{-n}}}^{p^n}$$

On $\mathrm{im}[\cdot]$, we must put

$$\Phi\colon \lim \widehat{x^{p^{-n}}}^{p^n} \mapsto \lim \widehat{\phi(x^{p^{-n}})}^{p^n}$$

and extend Φ linearly and continuously by $(*)$ to all of A. Then ϕ is a ring morphism: Because $[\cdot]$ is multiplicative, Φ is multiplicative, and Φ is checked to be additive. □

In particular let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$. Because $\mathcal{O}_{\mathbb{C}_p}$ is p-adically complete with mod-p reduction $\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$ by Proposition 1.1 the morphism of topological rings $\widetilde{\mathrm{E}}^+ \to \overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$ lifts to a morphism of topological rings

$$\mathcal{O}_{\widetilde{\mathcal{E}}^+} \to \mathcal{O}_{\mathbb{C}_p}$$

which

- is surjective because the Frobenius on $\mathcal{O}_{\mathbb{C}_p}$ is surjective, and
- its kernel is
  - generated by any $w(r)$ in $\mathrm{W}(\widetilde{\mathrm{E}}^+)$ whose canonical image $\boldsymbol{r} = (r_0, r_1, \ldots)$ in $\widetilde{\mathrm{E}}^+ = \varprojlim_{\cdot p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ satisfies $r_0 = 0$ and $r_1 \neq 0$, for example $\vartheta := [\bar{p}] - p$ where $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots)$ in $\widetilde{\mathrm{E}}^+$ such that $\bar{p}_0 = p$;

– but not stable under the Frobenius of $\mathcal{O}_{\widetilde{\mathscr{E}}^+}$.

Put

$$\widetilde{\mathscr{E}}^+ := Q(\mathcal{O}_{\widetilde{\mathscr{E}}^+}).$$

The quotient field functor yields an epimorphism of topological rings

$$\theta\colon \widetilde{\mathscr{E}}^+ \to \mathbb{C}_p.$$

Let

$$\mathbb{B}_{\mathrm{dR}}^+ := \varprojlim_n \widetilde{\mathscr{E}}^+/\ker\theta^n \quad \text{and} \quad \mathbb{B}_{\mathrm{dR}} := Q(\mathbb{B}_{\mathrm{dR}}^+).$$

Then $\mathbb{B}_{\mathrm{dR}}$

- has a Galois action (functorially obtained by that of $\mathcal{O}_{\mathbb{C}_p}$) which stabilizes $\ker\theta$,
- has an exhaustive and separated decreasing filtration indexed over $\mathbb{Z}$: Let $\epsilon = (1, \epsilon_1, \ldots)$ in $\widetilde{\mathrm{E}}^+$ be a p^∞-th of unity (for example, $\epsilon_1 \neq 1$). Let $[\epsilon]$ be its Teichmüller-lift to $\mathcal{O}_{\widetilde{\mathscr{E}}^+}$ and $\pi := [\epsilon] - 1$. Put $t = \log[\epsilon]$, that is
$$t := \pi - \pi^2/2 + \pi^3/3 - \cdots .$$
Then t generates $\ker\theta$ and we obtain a filtration on $\mathbb{B}_{\mathrm{dR}}$ by the fractional ideals
$$\mathbb{B}_{\mathrm{dR}}^i := \ker\theta^i$$
whose graded ring is
$$\mathbb{B}_{\mathrm{HT}} := \mathbb{C}_p[t, 1/t] = \cdots \oplus \mathbb{C}_p(-1) \oplus \mathbb{C}_p \oplus \mathbb{C}_p(1) \oplus \cdots .$$
On each, say i-th, graduation step, $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts by the i-th power of the cyclotomic character.

However, because the Frobenius on $\widetilde{\mathscr{E}}^+$ does not stabilize $\ker\theta$, it does not extend (by uniform continuity) from $\widetilde{\mathscr{E}}^+$ onto all of $\mathbb{B}_{\mathrm{dR}}^+$. Let

$$\widetilde{\mathrm{E}} := Q(\widetilde{\mathrm{E}}^+) \quad \text{and} \quad \widetilde{\mathscr{E}} = Q(W(\widetilde{\mathrm{E}}))$$

be the quotient field of $\widetilde{\mathrm{E}}^+$ and $\widetilde{\mathscr{E}}$ the quotient field of the ring of Witt vectors of $\widetilde{\mathrm{E}}$. Then $\mathbb{B}_{\mathrm{dR}}^+$ includes $[\widetilde{\mathrm{E}}]$ because θ is nonzero on every Teichmüller-lift of $\widetilde{\mathrm{E}}^+$. If a series $x = \sum_n [x_n]p^n$ (= the sequence of finite partial sums $x_{\mathrm{N}} := \sum_{n=0,\ldots,\mathrm{N}}[x_n]p^n$) in $\widetilde{\mathscr{E}}$ converges in $\mathbb{B}_{\mathrm{dR}}^+$ then by continuity necessarily $\theta(x)$ (= the sequence $\theta(x_{\mathrm{N}})$) converges in $\mathbb{C}_p$; this condition is also sufficient. Let us therefore define a valuation on $\widetilde{\mathscr{E}}^+$ given by

$$v_0\left(\sum_n [x_n]p^n\right) := \min\{v_{\widetilde{\mathrm{E}}}(x_n) + n : n \in \mathbb{N}\}$$

and the topological subrings of $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathbb{B}_{\mathrm{dR}}$ given by

$$\mathbb{B}_{\max}^+ := \text{ completion of } \widetilde{\mathscr{E}}^+ \text{ for } v_0 \quad \text{and} \quad \mathbb{B}_{\max} := \mathbb{B}_{\max}^+[1/t].$$

Then

- $\mathbb{B}^+_{\max}$ is a topological subring of $\mathbb{B}^+_{\mathrm{dR}}$, and
- because t generates $\ker\theta$, the ring $\mathbb{B}_{\max}$ is a subring of $\mathrm{Q}(\mathbb{B}_{\mathrm{dR}}) = \mathbb{B}^+_{\mathrm{dR}}[1/t]$.

(Finally, if $\mathbb{B}_{\mathrm{cris}}$ as defined in, say, [**FO14**, Section 6.1.1] then $\varphi(\mathbb{B}_{\max}) \subseteq \mathbb{B}_{\mathrm{cris}} \subseteq \mathbb{B}_{\max}$.)

The continuous injective Frobenius on $\widetilde{\mathscr{E}}$ stabilizes $\mathbb{B}_{\max}$. The filtration on $\mathbb{B}_{\mathrm{dR}}$ restricts to a filtration on $\mathbb{B}_{\max}$, which is again separated, exhaustive, indexed over $\mathbb{Z}$ and stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with graded ring $\mathbb{B}_{\mathrm{HT}}$. However, because the Frobenius does not stabilize $\ker\theta$, it does not stabilize the induced filtration on $\mathbb{B}_{\max}$.

We conclude that both rings have

- a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (given functoriality by that on $\mathscr{O}_{\mathbb{C}_p}$), and
- an exhaustive separated filtration indexed over $\mathbb{Z}$, which is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and whose graded ring is $\mathbb{B}_{\mathrm{HT}}$,

and $\mathbb{B}_{\max}$ has

- an injective endomorphism of continuous rings φ (given functorially by the Frobenius $\varphi = \cdot^p$ on $\mathscr{O}_{\mathbb{C}_p}$) which however does not stabilize the filtration.

2. Classes of Geometric Galois Representations

Let $\mathscr{G}_{\mathbb{Q}_p}$ denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of $\mathbb{Q}_p$, and let B be a topological $\mathbb{Q}_p$-algebra on which $\mathscr{G}_{\mathbb{Q}_p}$ acts continuously. Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ and $v_{\mathbf{K}}$ its valuation standardized by $v_{\mathbf{K}}(p) = 1$.

Definition. Let V be a finite-dimensional $\mathbf{K}$-vector space on which $\mathscr{G}_{\mathbb{Q}_p}$ acts continuously and $\mathbf{K}$-linearly. Then V is *admissible* for B if the B-semilinear continuous action of $\mathscr{G}_{\mathbb{Q}_p}$ on $\mathrm{V} \otimes_{\mathbb{Q}_p} \mathrm{B}$ is trivial, that is, there is a basis of the B-module $\mathrm{V} \otimes_{\mathbb{Q}_p} \mathrm{B}$ such that every vector is fixed by $\mathscr{G}_{\mathbb{Q}_p}$. Then V

- is *de Rham* if admissible for $\mathbb{B}_{\mathrm{dR}}$, and
- is *crystalline* if admissible for $\mathbb{B}_{\max}$.

Because $\mathbb{B}_{\max}$ is included in $\mathbb{B}_{\mathrm{dR}}$, if V is crystalline then it is de Rham.

Put

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{V}) := (\mathrm{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{dR}})^{\mathscr{G}_{\mathbb{Q}_p}} \quad \text{and} \quad \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) := (\mathrm{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\max})^{\mathscr{G}_{\mathbb{Q}_p}}.$$

Because

- the filtration respects the Galois action and the graded ring of $\mathbb{B}_{\mathrm{dR}}$ is $\mathbb{C}_p[t, 1/t]$, and
- we have $\mathbb{C}_p^{\mathscr{G}_{\mathbb{Q}_p}} = \mathbb{Q}_p$ ([**FO14**, Corollary 3.57]),

it follows

$$\mathbb{B}_{\mathrm{dR}}^{\mathscr{G}_{\mathbb{Q}_p}} = \mathbb{Q}_p \quad \text{and} \quad \mathbb{B}_{\max}^{\mathscr{G}_{\mathbb{Q}_p}} = \mathbb{Q}_p.$$

Thus, if V is de Rham respectively crystalline then $D_{dR}(V)$ respectively $D_{cris}(V)$ is an **K**-vector space of the dimension $\dim V$.

The **K**-vector space $D_{dR}(V)$ has

- an exhausting and separated filtration indexed over $\mathbb{Z}$,

and $D_{cris}(V)$ has

- an exhausting and separated filtration indexed over $\mathbb{Z}$, and
- an automorphism φ.

If V is crystalline, then it is in particular de Rham. Because $\mathbb{B}_{dR}$ and $\mathbb{B}_{max}$ have isomorphic graded rings, the injection $D_{cris}(V) \hookrightarrow D_{dR}(V)$ is an isomorphism of filtered **K**-vector spaces. We conclude that if V is crystalline then $D_{cris}(V) = D_{dR}(V)$ as filtered **K**-vector spaces.

2.1. Crystalline Galois Representations. The filtration $\mathrm{Fil}^\bullet \mathbb{B}_{dR}$ on $\mathbb{B}_{dR}$ induces by $\mathrm{Fil}^i \mathbb{B}_{max} := \mathrm{Fil}^i \mathbb{B}_{dR} \cap \mathbb{B}_{max}$ a filtration on $\mathbb{B}_{max}$. Let φ denote the Frobenius on $\mathbb{B}_{max}$.

THEOREM 2.1 ([**FO14**, THEOREM 6.26.(1)]). *We have* $\mathrm{Fil}^0 \mathbb{B}_{max}^{\varphi=1} = \mathbb{Q}_p$.

Thence if V is crystalline then it can be recovered from the data of a filtration and action of the Frobenius on $D_{cris}(V)$, as follows: The isomorphism of $\mathbb{B}_{max}$-modules

$$\mathbb{B}_{max} \otimes_{\mathbb{Q}_p} D_{cris}(V) \xrightarrow{\sim} \mathbb{B}_{max} \otimes_{\mathbb{Q}_p} V$$

respects the filtration and the actions of $\mathscr{G}_{\mathbb{Q}_p}$ and the Frobenius φ ([**BC09**, Proposition 9.1.9]). Thus first

$$\mathrm{Fil}^0(\mathbb{B}_{max} \otimes_{\mathbb{Q}_p} D_{cris}(V)) = \mathrm{Fil}^0(\mathbb{B}_{max} \otimes_{\mathbb{Q}_p} V) = \mathrm{Fil}^0 \mathbb{B}_{max} \otimes_{\mathbb{Q}_p} V$$

and then

$$(\mathrm{Fil}^0 \mathbb{B}_{max} \otimes_{\mathbb{Q}_p} V)^{\varphi=1} = V.$$

It follows that the functor D_{cris} that sends a crystalline representation to a $\mathbb{Q}_p$-vector space with filtration and automorphism φ is fully faithful ([**BC09**, Proposition 9.1.11]). Let us make this notion precise:

DEFINITION. A *filtered* φ*-module* over **K** is a **K**-vector space V with

- an **K**-linear automorphism $\varphi \curvearrowright V$, and
- an exhausting and separable filtration $\ldots \supseteq V^{-1} \supseteq V^0 \supseteq V^1 \supseteq \ldots$ on $V := V \otimes_{\mathbb{Q}_p} \mathbf{K}$ indexed by $\mathbb{Z}$.

Admissible filtered φ*-modules.* A *filtration jump index* is an integer i such that $V^i \supsetneq V^{i+1}$, and, oft-used in the literature, a *Hodge-Tate weight* is the inverse $-i$ of a filtration jump index i.

The filtered φ-modules that are attached to crystalline Galois representations are (cf. [**CF00**]) singled out by an *admissibility* condition that bounds above the finitely many filtration-jump indices by the absolute values of the eigenvalues of φ.

Definition. Put

$$t_{\mathrm{H}}(\mathrm{V}) := \sum_{n\in\mathbb{Z}} n \dim \mathrm{V}^n/\mathrm{V}^{n+1}, \quad \text{and} \quad t_{\mathrm{N}}(\mathrm{V}) := v_{\mathbf{K}}(\det(\varphi))$$

A filtered φ-module is *admissible* if

$$t_{\mathrm{H}}(\mathrm{V}) = t_{\mathrm{N}}(\mathrm{V})$$

and, for every vector subspace W of V stable under φ,

$$t_{\mathrm{H}}(\mathrm{W}) \leq t_{\mathrm{N}}(\mathrm{W}).$$

Theorem 2.2 ([**CF00**, Théorème 1]). *The functor* $\mathrm{V} \mapsto \mathrm{D}_{\mathrm{cris}}(\mathrm{V})$ *is an equivalence of categories*

$$\left\{\begin{array}{c}\text{crystalline actions of } \mathscr{G}_{\mathbb{Q}_p} \\ \text{on a } d\text{-dimensional} \\ \mathbf{K}\text{-vector space}\end{array}\right\} \stackrel{\sim}{\leftrightarrow} \left\{\begin{array}{c}\text{admissible filtered } \varphi\text{-modules} \\ \text{on a } d\text{-dimensional } \mathbf{K}\text{-vector space } \mathrm{D}\end{array}\right\}$$

Though originally stated only for $\mathbf{K} = \mathbb{Q}_p$, the above statement carries over to the above equivalence between **K**-linear actions and filtered φ-modules over **K** (for example [**BC09**, Exercise 8.4.3]).

Let $\mathrm{V}, \varphi, (\mathrm{V}^n)$ be a filtered φ-module over **K**. In dimension 1 and 2, admissibility leaves little choice for φ and the filtration:

- If V is of dimension 1 then φ is given by multiplication with a scalar $\lambda\pi^n$ where λ is a unit in $\mathscr{O}_{\mathbf{K}}$ and π generates the maximal ideal of $\mathscr{O}_{\mathbf{K}}$. Then V is admissible if and only if the filtration jumps at $-n$, that is, $\mathrm{V} = \ldots = \mathrm{V}^{-n} \supsetneq \mathrm{V}^{-n+1} = \ldots = 0$. The corresponding Galois representation is $\chi^n\lambda$
  - where $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^*$ is the cyclotomic character, and
  - where $\lambda\colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to \mathscr{O}_{\mathbf{K}}^*$ is the unramified character that sends the Frobenius $\cdot^p$ to λ.
- If V is of dimension 2, let us assume
  - that φ is semisimple and the filtration jumps are distinct, and
  - that
    - ▹ the lowest filtration jump index is 0, that is, the filtration jumps first at 0 (which can be ensured by twisting, that is, taking the tensor product with a power of the cyclotomic character),
    - ▹ we have $v(\beta) \leq v(\alpha)$ (after possibly swapping α and β)
    - ▹ all eigenvalues of φ are contained in **K** (after enlarging the coefficient field).

  Let α and β denote the eigenvalues of φ, and $(0, k-1)$ the filtration jump indices. Then by admissibility
  - $0 \leq v(\beta)$, and
  - $v(\alpha) + v(\beta) = k - 1$

  and there is a basis of eigenvectors $\{v, w\}$ of φ such that

– if $0 < v(\beta)$, then $V^1 = \ldots = V^{k-2} = \mathbf{K}(v + w)$, and
– if $0 = v(\beta)$ then
  ▹ either $V^1 = \ldots = V^{k-2} = \mathbf{K}(v+w)$ (and $\mathbf{K}v$ is an admissible filtered φ-submodule),
  ▹ or $V^1 = \ldots = V^{k-2} = \mathbf{K}v$. (That is, V is the direct sum of two one-dimensional admissible filtered φ-modules.)

We conclude that, in dimension 2, an irreducible semisimple admissible filtered φ-module is, up to twist by a crystalline character, determined by

- an eigenvalue in $\mathbf{K}$, and
- a filtration jump index in $\mathbb{Z}$.

Part 2. φ, Γ-modules

We define an equivalence between

- continuous actions of the big absolute Galois group of $\mathbb{Q}_p$ on finite modules over the small ring $\mathbb{Z}_p$ (or $\mathbb{Q}_p$), and
- continuous actions of the small monoid $\mathbb{Z}_p - \{0\} = p^{\mathbb{N}}\mathbb{Z}_p^*$ on finite modules over a big ring of convergent Laurent series over $\mathbb{Z}_p$ (or $\mathbb{Q}_p$).

This equivalence takes three steps:

1. An action of the absolute Galois group of a field of positive characteristic E on a finite-dimensional vector space V over $\mathbb{F}_p$ is determined by the action of the Frobenius of E on V and E,
2. the absolute Galois group of the function field $\mathbb{F}_p((X))$ is isomorphic to the absolute Galois group of the cyclotomic extension of $\mathbb{Q}_p$ obtained by adjoining all roots of unity of p-power order, and
3. the action of the absolute Galois group of $\mathbb{Q}_p$ on a finite module V over $\mathbb{Z}_p$ (or $\mathbb{Q}_p$) is determined by the action of a "Frobenius" φ and $\mathbb{Z}_p^*$ (= the Galois group of $\mathbb{Q}_{p^{-\infty}}$ over $\mathbb{Q}_p$) on a finite module over a ring of Laurent series over $\mathbb{Z}_p$ that p-adically lift $\mathbb{F}_p((X))$ (or the quotient field of $\mathcal{O}_{\mathcal{E}}$).

3. Galois Representations of fields of positive characteristic

Let E be a topological field of positive characteristic p and let $\bar{E}$ be the separable closure of E. Let $\mathcal{G}_E = \mathrm{Gal}(\bar{E}/E)$ be the absolute Galois group of E. The group $\mathcal{G}_E$ is a profinite topological group. The finite field $\mathbb{F}_p$ is a discrete topological field and every finite-dimensional topological vector space over $\mathbb{F}_p$ is a discrete topological vector space.

Let U be a vector space over $\mathbb{F}_p$ and $\mathcal{G}_E \curvearrowright U$ a continuous linear action of $\mathcal{G}_E$ on U. Let $\varphi = \cdot^p$ on E be the Frobenius of E and let φ on U be the automorphism of U given by the action of φ in $\mathcal{G}_E$ on U. Define a diagonal action of $\mathcal{G}_E$ on $U \otimes_{\mathbb{F}_p} \bar{E}$ by

$$\sigma(u \otimes e) := \sigma(u) \otimes \sigma(e)$$

and define

$$D(U) := (U \otimes_{\mathbb{F}_p} \bar{E})^{\mathscr{G}_E}$$

as the invariants under the diagonal action of $\mathscr{G}_E$.

Definition. Let R be a topological ring and $\varphi \curvearrowright R$ an endomorphism of R. A map $\varphi\colon M \to N$ between modules over R

- is *semilinear for* φ if
  - it is *additive*, that is, $\varphi(m+n) = \varphi(m) + \varphi(n)$ for all m, n in M, and
  - it fulfills $\varphi(rm) = \varphi(r)\varphi(m)$ for all r in R and m in M;
- it is *étale for* φ if it is semilinear for φ and

  $$\varphi^*\colon M \otimes_\varphi R \to N$$

  is an isomorphism

An *étale* φ*-module* is a finite module M over R and a map $\varphi\colon M \to M$ that is étale for φ.

Then D(U) is an étale φ-module over $\bar{E}^{\mathscr{G}_E} = E$: the map $\varphi\colon D(U) \to D(U)$ given by the diagonal action of the Frobenius φ in $\mathscr{G}_E$ is semilinear and étale for $\varphi\colon E \to E$.

Theorem 3.1. *The functor*

$$\left\{\begin{array}{c}\text{continuous actions of } \mathrm{Gal}(\bar{E}/E) \text{ on} \\ \text{an } \mathbb{F}_p\text{-vector space of dimension } d\end{array}\right\} \to \left\{\begin{array}{c}\text{étale } \varphi\text{-modules on} \\ \text{an } E\text{-vector space of dimension } d\end{array}\right\}$$
$$U \mapsto D(U)$$

is an equivalence of categories.

Proof: By Hilbert 90 ([**FO14**, Proposition 2.7]), there is a basis of the $\bar{E}$-vector space $U \otimes_{\mathbb{F}_p} \bar{E}$ such that $\mathscr{G}_E$ fixes each of these basis vectors. In particular,

$$D(U) = (U \otimes_{\mathbb{F}_p} \bar{E})^{\mathscr{G}_E}$$

has dimension d. The functor

$$D \mapsto (D \otimes_E \bar{E})^{\varphi=1},$$

where the right-hand side are all elements that are invariant under the diagonal action of φ, is (quasi-)inverse to D. See [**FO14**, Theorem 2.21] for a detailed proof. □

4. Identifying Galois Groups in Characteristic p and 0

Let $\mu_{p^{-\infty}}$ be all roots of unity of p-power order. Put

$$\mathbb{Q}_{p^{-\infty}} := \mathbb{Q}_p(\mu_{p^{-\infty}}) \quad \text{and} \quad \overline{\mathbb{Q}}_p = \text{ algebraic closure of } \mathbb{Q}_{p^{-\infty}},$$

and

$$E := \mathbb{F}_p((X)) \quad \text{and} \quad \bar{E} = \text{ separable closure of } E.$$

Theorem 4.1 (Field of Norms). *There is an isomorphism of topological groups*

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) \xrightarrow{\sim} \mathrm{Gal}(\bar{\mathrm{E}}/\mathrm{E}).$$

Proof: Let $\overline{\mathbb{Z}}_p$ be the ring of integers of $\overline{\mathbb{Q}}_p$. Recall the topological ring

$$\widetilde{\mathrm{E}}^+ = \varprojlim_n \overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$$

where every transition map is $\cdot^p$ and the topology is the projective limit topology (for the discrete quotient topology on $\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p$). Its quotient field

$$\widetilde{\mathrm{E}} := \mathrm{Q}(\widetilde{\mathrm{E}}^+)$$

is an algebraically closed field of characteristic p ([**FO14**, Proposition 4.8]).

Let $\epsilon := (1, \epsilon_1, \epsilon_2, \ldots)$ in $\widetilde{\mathrm{E}}^+$ be a root of unity of order p^∞ (for example, ϵ_1 is not 1) and put $\mathrm{X} := \epsilon - 1$. Then the topological fields

$$\mathrm{E} := \mathbb{F}_p((\mathrm{X})) \quad \textit{and} \quad \bar{\mathrm{E}} := \text{ the separable closure of } \mathrm{E}$$

are included in $\widetilde{\mathrm{E}}$, and $\bar{\mathrm{E}}$ is dense inside $\widetilde{\mathrm{E}}$ ([**FO14**, Theorem 4.17]). Thus we obtain the isomorphism given by restriction

$$\mathrm{Aut}^{\mathrm{cts}}_{\mathrm{E}}(\widetilde{\mathrm{E}}) \xrightarrow{\sim} \mathrm{Gal}(\bar{\mathrm{E}}/\mathrm{E})$$

where the left-hand side are all continuous automorphisms of the topological E-algebra $\widetilde{\mathrm{E}}$. To conclude, the natural morphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) \xrightarrow{\sim} \mathrm{Aut}^{\mathrm{cts}}_{\mathrm{E}}(\widetilde{\mathrm{E}})$$

is an isomorphism:

- it is injective, because the values of an automorphism $\sigma \curvearrowright \overline{\mathbb{Q}}_p$ on $\mu_{p^{-\infty}}$ and an element x_0 in $\overline{\mathbb{Q}}_p$ determine the values of σ on all roots of x_0 of p-power order,
- it is surjective, because if an automorphism $\sigma \curvearrowright \widetilde{\mathrm{E}}$ fixes E, then it fixes in particular ϵ, therefore $\mu_{p^{-\infty}}$ and thus $\mathbb{Q}_{p^{-\infty}}$.

□

The Frobenius φ on $\widetilde{\mathrm{E}}$ restricts via the monomorphism $\mathrm{E} \hookrightarrow \widetilde{\mathrm{E}}$ to a Frobenius on E given by $\varphi(\mathrm{X}) = (1+\mathrm{X})^p - 1$.

Corollary 4.2. *The functor*

$$\left\{ \begin{array}{c} \textit{continuous actions of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) \textit{ on} \\ \textit{an } \mathbb{F}_p\textit{-vector space of dimension } d \end{array} \right\} \to \left\{ \begin{array}{c} \textit{étale } \varphi\textit{-modules on} \\ \textit{an } \mathrm{E}\textit{-vector space of dimension } d \end{array} \right\}$$

$$\mathrm{U} \mapsto (\mathrm{U} \otimes_{\mathbb{F}_p} \bar{\mathrm{E}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}})}$$

is an equivalence of categories.

Proof: By Theorem 3.1 and Theorem 4.1. □

5. Lifting from $\mathbb{F}_p$ to $\mathbb{Z}_p$

Let

$$H := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) \quad \text{and} \quad \Gamma := \mathrm{Gal}(\mathbb{Q}_{p^{-\infty}}/\mathbb{Q}_p),$$

and let

$$\chi \colon \Gamma \xrightarrow{\sim} \mathbb{Z}_p^*$$
$$\sigma \mapsto \text{ the unique } x \text{ such that } \epsilon^\sigma = \epsilon^x \text{ for every unit root } \epsilon \text{ of } p\text{-power order}$$

be the *cyclotomic character*. In this section, we will

(i) lift the vector spaces from $\mathbb{F}_p$ to $\mathbb{Z}_p$,
(ii) extend the absolute Galois group of $\mathbb{Q}_{p^{-\infty}}$ to that of $\mathbb{Q}_p$.

(Finally we extend the coefficients from $\mathbb{Z}_p$ to a finite extension $\mathbf{K}$ of $\mathbb{Q}_p$.)

5.1. Coefficient rings. Let $\mathbf{k}$ be a field of positive characteristic. A *Cohen ring of* $\mathbf{k}$ is a complete discrete valuation ring such that

- its maximal ideal is generated by p,
- its characteristic is 0, and
- its residue field is $\mathbf{k}$.

Let $\mathcal{O}_{\mathcal{E}}$ be the Cohen ring of $E = \mathbb{F}_p((X))$ given by

$$\mathcal{O}_{\mathcal{E}} := \text{ all } \sum_{n \in \mathbb{Z}} a_n t^n \text{ in } \mathbb{Z}_p[[X, 1/X]] \text{ such that } a_{-1}, a_{-2}, \ldots \to 0$$

and let $\mathcal{O}_{\overline{\mathcal{E}}}$ be the Cohen ring of the separable closure $\bar{E}$ of E given by

$$\mathcal{O}_{\overline{\mathcal{E}}} := \text{ the } p\text{-adic completion of the maximal unramified extension of } \mathcal{O}_{\mathcal{E}} \, .$$

Let

$$\mathcal{E} := \mathcal{O}_{\mathcal{E}}[1/p] \quad \text{and} \quad \overline{\mathcal{E}} := \mathcal{O}_{\overline{\mathcal{E}}}[1/p].$$

The action of the absolute Galois group H given by all E-algebra automorphisms on $\bar{E}$ induces an action of H by $\mathcal{O}_{\mathcal{E}}$-algebra automorphisms on $\mathcal{O}_{\overline{\mathcal{E}}}$ and $\mathcal{E}$-algebra automorphisms on $\overline{\mathcal{E}}$. Let φ and Γ operate on $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{E}$ by

$$(5.1) \qquad \varphi(X) := (1+X)^p - 1 \quad \text{and} \quad X^\gamma = (1+X)^{\chi(\gamma)} - 1 := \sum_{n \in \mathbb{N}} \binom{\chi(\gamma)}{n} X^n$$

where $\chi \colon \Gamma \xrightarrow{\sim} \mathbb{Z}_p^*$ is the cyclotomic character.

5.2. Topology. Let us define the canonical topology on $\mathcal{O}_{\mathcal{E}}$ and on every finite module over $\mathcal{O}_{\mathcal{E}}$.

Definition. A *discrete filtration* on a ring R is a descending filtration $\mathcal{R}(i)$ by subrings indexed by $\mathbb{Z}$ such that for all i, j in $\mathbb{Z}$

$$\mathcal{R}(i)\mathcal{R}(j) \subseteq \mathcal{R}(i+j) \quad \text{and} \quad \mathcal{R}(i) + \mathcal{R}(j) \subseteq \mathcal{R}(\min\{i, j\}).$$

For every n in $\mathbb{N}$, there is a canonical ring morphism

$$\pi_n \colon \mathcal{O}_{\mathcal{E}} \to \mathbb{Z}_p/p^n\mathbb{Z}_p[[X]][1/X].$$

We equip the ring $\mathbb{Z}/p^n\mathbb{Z}[[X]][1/X]$ with the discrete filtration

$$\overline{\mathcal{R}}(n, \bullet) := X^{\bullet} \cdot \mathbb{Z}/p^n\mathbb{Z}[[X]]$$

and the ring $\mathcal{O}_{\mathcal{E}}$ for every n in $\mathbb{N}$ with a discrete filtration

$$\mathcal{R}(n, \bullet) := \pi_n^{-1}(\overline{\mathcal{R}}(n, \bullet)).$$

The *weak topology on* $\mathcal{O}_{\mathcal{E}_K}$ is the topology given by

$$\{\mathcal{R}(n, i) : n \in \mathbb{N}, i \in \mathbb{Z}\}$$

as neighborhood basis around 0 and turns $\mathcal{O}_{\mathcal{E}}$ into a topological ring. Explicitly

$$\mathcal{R}(n, i) = X^i\mathbb{Z}_p[[X]] + p^n\mathcal{O}_{\mathcal{E}}.$$

Definition. Let R be a ring and $\mathcal{R}$ a discrete filtration on R. A *discrete filtration* on a module M over R is a filtration $\mathcal{M}$ of $M(i)$ by modules over $\mathcal{R}(i)$ such that for all i, j in $\mathbb{Z}$

$$\mathcal{R}(i)\mathcal{M}(j) \subseteq \mathcal{M}(i+j) \quad \text{and} \quad \mathcal{M}(i) + \mathcal{M}(j) \subseteq \mathcal{M}(\min\{i, j\}).$$

Let M be a module over $\mathcal{O}_{\mathcal{E}}$. For every n in $\mathbb{N}$, there is a canonical module morphism

$$\pi_n \colon M \to M/p^nM.$$

We equip a module $\overline{\mathbf{M}}$ over $\mathbb{Z}/p^n\mathbb{Z}[[X]][1/X]$ with the discrete filtration

$$\overline{\mathcal{M}}(n, \bullet) := X^{\bullet} \cdot \overline{\mathbf{M}}$$

and a module M over $\mathcal{O}_{\mathcal{E}}$ for every n in $\mathbb{N}$ with a discrete filtration

$$\mathcal{M}(n, \bullet) := \pi_n^{-1}(\overline{\mathcal{M}}(n, \bullet))$$

for i in $\mathbb{Z}$. The *weak topology on* M is the topology given by

$$\{\mathcal{M}(n, i) : n \in \mathbb{N}, i \in \mathbb{Z}\}$$

as neighborhood basis around 0 and turns M into a topological module over $\mathcal{O}_{\mathcal{E}}$. Explicitly, the neighborhood basis around 0 on a finite module over $\mathcal{O}_{\mathcal{E}}$ can be given by the notion of a finitely generated module over $\mathcal{O}_{\mathcal{E}^+}$ of maximal rank T: then

$$\mathcal{M}(n, i) = X^iT + p^nM$$

and the topology is independent of the choice of T.

5.3. φ, Γ-modules.

Definition. Let R be topological ring and φ a morphism of R and let Γ act continuously on R. An *étale* φ, Γ*-module over* R is a finite module over R with commuting semilinear actions of a morphism φ and the group Γ such that φ is étale and Γ acts continuously.

Because

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)/\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{-\infty}}) = \mathrm{Gal}(\mathbb{Q}_{p^{-\infty}}/\mathbb{Q}_p) \xrightarrow{\sim} \Gamma,$$

Corollary 4.2 informs:

Corollary 5.1 ([**FO14**, Theorem 4.23]). *The functor*

$$\left\{ \begin{matrix} \textit{continuous actions of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \textit{on a } \mathbb{Z}_p\textit{-module of rank } d \end{matrix} \right\} \to \left\{ \begin{matrix} \textit{étale } \varphi, \Gamma\textit{-modules} \\ \textit{over } \mathcal{O}_{\mathcal{E}} \textit{ of rank } d \end{matrix} \right\}$$
$$\mathrm{U} \mapsto \mathrm{D(U)} := (\mathrm{U} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\overline{\mathcal{E}}})^{\mathrm{H}}$$

is an equivalence of categories.

An *étale* φ, Γ*-module over* $\mathcal{E}$ is the base extension from $\mathcal{O}_{\mathcal{E}}$ to $\mathcal{E}$ of an étale φ, Γ-module over $\mathcal{O}_{\mathcal{E}}$. By inverting p and observing that, if the compact group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on a finite-dimensional vector space V then there is always a lattice over $\mathbb{Z}_p$ inside V that it stabilizes, we obtain:

Corollary 5.2. *The functor* D

$$\left\{ \begin{matrix} \textit{continuous actions of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \textit{ on} \\ \textit{a } \mathbb{Q}_p\textit{-vector space of dimension } d \end{matrix} \right\} \to \left\{ \begin{matrix} \textit{étale } \varphi, \Gamma\textit{-modules} \\ \textit{over } \mathcal{E} \textit{ of dimension } d \end{matrix} \right\}$$
$$\mathrm{U} \mapsto \mathrm{D(U)} := (\mathrm{U} \otimes_{\mathbb{Q}_p} \overline{\mathcal{E}})^{\mathrm{H}}$$

is an equivalence of categories.

Since

$$\log \colon 1 + p\mathbb{Z}_p \xrightarrow{\sim} \begin{cases} p\mathbb{Z}_p, & \text{for } p > 2, \\ \mathbb{Z}/2\mathbb{Z} \times 2\mathbb{Z}_2, & \text{for } p = 2. \end{cases}$$

and thus

$$\Gamma \xrightarrow{\sim} \mathbb{Z}_p^* \xrightarrow{\sim} \mu_p \times (1 + p\mathbb{Z}_p) \xrightarrow{\sim} \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p,$$

the action of φ and Γ on a φ, Γ-module is (for $p > 2$) by continuity determined by the two matrices given by that of φ and that of a generator of the pro-cyclic (for $p > 2$) group Γ.

5.4. Extending coefficients. Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_{\mathbf{K}}$ its ring of integers. Put $\mathcal{O}_{\mathcal{E}_{\mathbf{K}}} := \mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{K}}$ and $\mathcal{E}_{\mathbf{K}} := \mathcal{E} \otimes_{\mathbb{Q}_p} \mathbf{K}$, and define $\mathcal{O}_{\overline{\mathcal{E}}_{\mathbf{K}}}$ and $\overline{\mathcal{E}}_{\mathbf{K}}$ likewise. If $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts linearly on a finite module U over $\mathcal{O}_{\mathbf{K}}$ (or $\mathbf{K}$) then

$$\mathrm{D(U)} := (\mathrm{U} \otimes_{\mathcal{O}_{\mathbf{K}}} \mathcal{O}_{\overline{\mathcal{E}}_{\mathbf{K}}})^{\mathrm{H}} \quad \text{or} \quad \mathrm{D(U)} := (\mathrm{U} \otimes_{\mathbf{K}} \overline{\mathcal{E}}_{\mathbf{K}})^{\mathrm{H}}$$

is an étale φ, Γ-module over $\mathcal{O}_{\mathcal{E}_{\mathbf{K}}}$ (or $\mathcal{E}_{\mathbf{K}}$), and the functor D is again an equivalence of categories between continuous actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on finitely generated modules over $\mathcal{O}_{\mathbf{K}}$ (or $\mathbf{K}$) and étale φ, Γ-module over $\mathcal{O}_{\mathcal{E}_{\mathbf{K}}}$ (or $\mathcal{E}_{\mathbf{K}}$).

6. Action of the mirabolic subgroup

Put $\mathbb{Z}_p^\bullet = p^{\mathbb{N}}\mathbb{Z}_p^*$. Let

$$\mathrm{M}_0 := \begin{pmatrix} 1 & \mathbb{Z}_p \\ & \mathbb{Z}_p^\bullet \end{pmatrix} \quad \text{and} \quad \mathrm{M} := \begin{pmatrix} 1 & \mathbb{Q}_p \\ & \mathbb{Q}_p^* \end{pmatrix}.$$

Then M_0 is a monoid and M is a group, the *mirabolic subgroup* of $\mathrm{GL}_2(\mathbb{Q}_p)$. Given a φ, Γ-module D, we will first combine the actions of φ and Γ on D into an action of M_0 on D. Then we extend it to one of all of M on $\varprojlim_\psi \mathrm{D}$ by the action ψ.

6.1. Action of the compact mirabolic subgroup. To define an action of M_0, we must define the actions of p, $\mathbb{Z}_p^*$ and $\mathbb{Z}_p$ on D: We

- let $\mathbb{Z}_p^\bullet = p^{\mathbb{N}}\mathbb{Z}_p^*$ act on D by the actions
  - of $\chi\colon \Gamma \xrightarrow{\sim} \mathbb{Z}_p^*$ (via the cyclotomic character) on D, and
  - of $p = \phi$ on D;
- let $\mathbb{Z}_p$ act on D by putting, for a in $\mathbb{Z}_p$,

$$(1+\mathrm{X})^a := \sum_n \binom{a}{n} \mathrm{X}^n$$

  and letting $(1+\mathrm{X})^a$ in $\mathcal{O}_{\mathbf{K}}[[\mathrm{X}]]$ (which is included in $\mathcal{E}$) act on D by scalar multiplication.

6.2. Action of the mirabolic subgroup. We extend the action of M_0 on $\varprojlim_\psi \mathrm{D}$ to M. For this, we note that

$$\begin{pmatrix} 1 & \mathbb{Q}_p \\ & \mathbb{Q}_p^* \end{pmatrix} = \left\langle \begin{pmatrix} 1 & \mathbb{Z}_p \\ & p^{\mathbb{Z}}\mathbb{Z}_p^* \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}, \mathrm{M}_0 \right\rangle,$$

and that it therefore suffices to define the action of p^{-1} on $\varprojlim_\psi \mathrm{D}$. By definition of $\varprojlim_\psi \mathrm{D}$, we find that φ is invertible by ψ, and we let p^{-1} act on D by ψ.

Part 3. The treillis of a crystalline Galois representation

To extend the action of M on

$$\varprojlim_{\psi} \mathrm{D} = \{ \text{ all } (x_n) \in \mathrm{D}^{\mathbb{N}} : x_n = \psi x_{n+1} \}$$

to an action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we must restrict it to the submodule of all bounded sequences. This submodule of bounded sequences is most explicitly described by the sequences of the bounded submodule given by the *treillis* of D.

7. Construction

Let D be an étale φ, Γ-module. We will

(i) define a section ψ of φ,
(ii) define a "locally convex" topology on D that allows for the notion of boundedness, and
(iii) describe

$$\varprojlim_{\psi}{}^{\mathrm{b}} \mathrm{D} := \{ \text{ all bounded sequences in } \varprojlim_{\psi} \mathrm{D}\}$$

where $\varprojlim_{\psi} \mathrm{D}$ is the projective limit running over $\mathbb{N}$ whose transition maps are all given by ψ. For this,

(a) we will define a submodule $\mathrm{D}^{\sharp}(\mathrm{T})$ stable under ψ such that

$$\varprojlim_{\psi}{}^{\mathrm{b}} \mathrm{D} = (\varprojlim_{\psi} \mathrm{D}^{\sharp}(\mathrm{T})) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K},$$

(b) and finally describe $\mathrm{D}^{\sharp}(\mathrm{T})$ as submodule of a two-dimensional module over the ring of power series over $\mathbf{K}$ that converge on the open unit disc in $\mathbb{C}_p$.

7.1. The section ψ on a φ-module. Let R be a ring of Laurent series in X. The algebra endomorphism φ on R given by $\mathrm{X} \mapsto (1+\mathrm{X})^p - 1$ is injective but not surjective; we have

$$\mathrm{R} = \varphi(\mathrm{R}) \oplus (1+\mathrm{X})\varphi(\mathrm{R}) \oplus \cdots \oplus (1+\mathrm{X})^{p-1}\varphi(\mathrm{R}).$$

Define the section ψ of φ by $\psi = \varphi^{-1} \circ \pi_0$ where $\pi_0 \colon \mathrm{R} \to \varphi(\mathrm{R})$. Because φ commutes with the action of Γ on R so does ψ.

Likewise, if D is a φ-module over R (that is, a finite free module D over R with an endomorphism φ of D that is semilinear for φ and is injective), then the module morphism φ on D is injective but not surjective; we have

$$\mathrm{D} = \varphi(\mathrm{D}) \oplus (1+\mathrm{X})\varphi(\mathrm{D}) \oplus \cdots \oplus (1+\mathrm{X})^{p-1}\varphi(\mathrm{D}).$$

Define the section ψ of φ by $\psi = \varphi^{-1} \circ \pi_0$ where $\pi_0 \colon \mathrm{D} \to \varphi(\mathrm{D})$. Likewise, because φ commutes with the action of Γ on D, so does ψ.

7.2. Boundedness on a finite free module over $\mathcal{E}$. We define boundedness with respect to the weak topology on $\mathcal{O}_{\mathcal{E}}$, then on $\mathcal{E}$ and finally on finite modules over $\mathcal{E}$.

Boundedness on $\mathcal{E}$. Let $\mathcal{R}$ be the discrete filtration on $\mathcal{O}_{\mathcal{E}}$. A subset S of $\mathcal{O}_{\mathcal{E}}$ is *bounded* (for the weak topology) if S is bounded for every discrete filtration $\mathcal{R}(n)$. That is, for every n in $\mathbb{N}$ there is i in $\mathbb{Z}$ such that $\mathcal{R}(n, i) \supseteq \mathrm{S}$. A subset S of $\mathcal{E}$ is *bounded* if

- the subset S is bounded for the p-adic topology, that is, there is n in $\mathbb{Z}$ such that $p^n\mathrm{S} \subseteq \mathcal{O}_{\mathcal{E}}$, and
- the subset $p^n\mathrm{S}$ of $\mathcal{O}_{\mathcal{E}}$ is bounded for the weak topology.

Modules over $\mathscr{E}$. Let M be a finite module over $\mathscr{E}$ and L a submodule over $\mathscr{O}_{\mathscr{E}}$ of M such that $L \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{E} \xrightarrow{\sim} M$. A subset S of L is *bounded* (for the weak topology) if S is bounded for every discrete filtration $\mathscr{M}(n)$. That is, for every n in $\mathbb{N}$ there is i in $\mathbb{Z}$ such that $\mathscr{M}(n,i) \supseteq S$. A subset S of M is *bounded* if

- the subset S is bounded for the p-adic topology, that is, there is n in $\mathbb{Z}$ such that $p^n S \subseteq L$ and
- the subset $p^N S$ of L is bounded for the weak topology.

This definition of boundedness is independent of the choice of L.

7.3. The treillis on which ψ is surjective. Let T be a finite free $\mathscr{O}_{\mathbf{K}}$-module on which $\mathscr{G}_{\mathbb{Q}_p}$ acts continuously and D(T) its corresponding étale φ, Γ-module over $\mathscr{O}_{\mathscr{E}_{\mathbf{K}}}$; let $V = T \otimes_{\mathscr{O}_{\mathbf{K}}} \mathbf{K}$ be the associated finite-dimensional $\mathbf{K}$ vector space on which $\mathscr{G}_{\mathbb{Q}_p}$ acts continuously and $D(V) = D(T) \otimes_{\mathscr{O}_{\mathbf{K}}} \mathbf{K}$ its corresponding étale φ, Γ-module over $\mathscr{E}_{\mathbf{K}}$. Put

$$\mathscr{O}_{\mathscr{E}}^{+} := \mathbb{Z}_p[[X]] \quad \text{and} \quad \mathscr{O}_{\mathscr{E}}^{+} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]],$$

and let accordingly $\mathscr{O}_{\mathscr{E}_{\mathbf{K}}^{+}}$ and $\mathscr{E}_{\mathbf{K}}^{+}$ be the tensor products of $\mathscr{O}_{\mathscr{E}}^{+}$ and $\mathscr{E}^{+}$ with $\mathscr{O}_{\mathbf{K}}$ over $\mathbb{Z}_p$.

DEFINITION. Let $\mathscr{D}$ be a finitely generated module over $\mathscr{O}_{\mathscr{E}}$. A *treillis* $\mathscr{T}$ of $\mathscr{D}$ is a module over $\mathscr{O}_{\mathscr{E}^{+}}$ such that, putting $\overline{\mathscr{T}} := \mathscr{T}/p\mathscr{T}$ and $\overline{\mathscr{M}} := \mathscr{D}/p\mathscr{D}$,

$$\overline{\mathscr{T}} \otimes_{\mathbb{F}_p[[X]]} \mathbb{F}_p((X)) = \overline{\mathscr{M}}$$

and for all n in $\mathbb{N}$, the module $\mathscr{T}/p^n\mathscr{T}$ is finitely generated over $\mathscr{O}_{\mathscr{E}^{+}}$. A *treillis* of a finitely generated module D over $\mathscr{E}$ is a treillis of a p-adic lattice $\mathscr{D}$ of D, that is, of a module $\mathscr{D}$ over $\mathscr{O}_{\mathscr{E}}$ such that $D = \mathscr{D} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

This does not imply that $\mathscr{T}$ is finitely generated over $\mathscr{O}_{\mathscr{E}^{+}}$ (see for example that at the beginning of [**Col10c**, Section II.7]), though the treillises that we will encounter are all finitely generated.

Proposition 7.1. *Let* D *be an étale φ-module over $\mathscr{O}_{\mathscr{E}}$. There is a unique treillis* $D^{\sharp}$ *inside* D *such that*

- $\psi(D^{\sharp}) = D^{\sharp}$, *and*
- *for all x in* D *and k in $\mathbb{N}$, there is* N *such that $\psi^n(x)$ in $D^{\sharp} + p^k D$ for all $n \geq N$.*

Moreover, let T *be a finite free $\mathscr{O}_{\mathbf{K}}$-module on which $\mathscr{G}_{\mathbb{Q}_p}$ acts continuously. Let $V = T \otimes_{\mathscr{O}_{\mathbf{K}}} \mathbf{K}$ and let* D(T) *be its corresponding étale φ, Γ-module over $\mathscr{O}_{\mathscr{E}_{\mathbf{K}}}$. If* V *is irreducible and of dimension* > 1, *then the unique treillis* $D^{\sharp}(T)$ *inside* D(T) *is already determined by* $\psi(D^{\sharp}(T)) = D^{\sharp}(T)$.

PROOF: The existence and uniqueness of a treillis such that

- $\psi(D^{\sharp}) = D^{\sharp}$, and
- for all x in D and k in $\mathbb{N}$, there is N such that $\psi^n(x)$ in $D^{\sharp} + p^k D$ for all $n \geq N$

is proved in [**Col10c**, Proposition II.4.2]. Moreover, by [**Col10c**, Section II.4 and II.5] there is inside D a smallest and largest treillis on which ψ is surjective. If $D = D(T)$, then they coincide by [**Col10c**, Proposition II.5.19 and Remarque II.2.4] if and only if $V^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ab}})} = 0$. The latter condition holds because V is irreducible and of dimension > 1. □

Proposition 7.2. *The inclusion* $D^\sharp(T) \hookrightarrow D(V)$ *induces an isomorphism of topological* $\mathbf{K}$*-vector spaces*

$$(\varprojlim_{\psi} D^\sharp(T)) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K} \xrightarrow{\sim} \varprojlim_{\psi}{}^{\mathrm{b}} D(V).$$

Proof: If $x = (x_m)$ in $\varprojlim^{\mathrm{b}} D(V)$ then, up to multiplication by a scalar, x_0, x_1, … in D(T).

Let m in $\mathbb{N}$. By definition of $D^\sharp(T)$, for every k in $\mathbb{N}$, for sufficiently large n, we have $x_m = \psi^n(x_{m+n})$ in $D^\sharp(T) + p^k D(T)$. Because this holds for all k in $\mathbb{N}$, we conclude x_m in $D^\sharp(T)$. Therefrom the surjectivity ([**BB10**, Proposition 2.3.6]). □

7.4. Describing the treillis through the Wach module. Let T be a finite $\mathcal{O}_{\mathbf{K}}$-module and let $\mathcal{G}_{\mathbb{Q}_p}$ act continuously on T. Put $V := T \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K}$.

If V is crystalline (and its filtration jump indices are nonnegative) then there is a distinguished treillis N(T) of D(V) that

- is stable under Γ and the operation on $N(T)/XN(T)$ is trivial, and
- for which there is h in $\mathbb{N}$ such that $X^{-h}D^+(T) \subseteq N(T) \subseteq D^+(T)$ (for a module $D^+(T)$ to be defined below).

Let $t = \log(1+X)$ and let $\mathcal{R}^+_{\mathbf{K}}$ be the topological $\mathbf{K}$-algebra of all power series over $\mathbf{K}$ that converge on the open unit disc of $\mathbb{C}_p$. We will first describe the $\mathcal{O}^+_{\mathcal{E}}$-module $X^{-h}N(T)$ by singling it out from $D_{\mathrm{cris}}(V) \otimes_{\mathbf{K}} \mathcal{R}^+_{\mathbf{K}}[1/t]$ by a growth and a filtration condition on its coefficients with respect to a basis of $D_{\mathrm{cris}}(V)$. In particular this description will show that $X^{-h}N(T)$ is stable under ψ. Because $\psi(N(T)) \supseteq N(T)$ (as N(T) is stable under φ) and $X^{-h}N(T) \supseteq \psi^n(N(T))$ for all n in $\mathbb{N}$, there is by noetherianity n_0 in $\mathbb{N}$ such that $\psi^{n_0}(N(T)) = \psi^{n_0+1}(N(T))$; thus, by Proposition 7.1, $\psi^{n_0}(N(T)) = D^\sharp(T)$. Thus, by Proposition 7.2, we conclude

$$\varprojlim_{\psi}{}^{b} D(V) = (\varprojlim_{\psi} D^\sharp(T)) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K} = (\varprojlim_{\psi} N(T)) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K}.$$

Coefficient rings linking p-adic Hodge theory and φ, Γ-modules. We will construct coefficient rings that admit morphisms into the rings defined in p-adic Hodge theory and those over which φ, Γ-modules are defined.

We defined before

$$\widetilde{E}^+ = \varprojlim_{n} \overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p \quad \text{and} \quad \widetilde{\mathcal{E}}^+ = Q(W(\widetilde{E}^+))$$

and

$$\widetilde{E} := Q(\widetilde{E}^+) \quad \text{and} \quad \widetilde{\mathcal{E}} := Q(W(\widetilde{E})).$$

The projective limit topology on $\widetilde{\mathrm{E}}^+$ is equivalently given by the valuation

$$v_{\widetilde{\mathrm{E}}^+}(x) := v_{\mathscr{O}_{\mathbb{C}_p}}(\lim_{n\to\infty} \widehat{x}_n^{p^n})$$

and which extends multiplicatively to a valuation $v_{\widetilde{\mathrm{E}}}$ on $\widetilde{\mathrm{E}}$. We have $\widetilde{\mathscr{E}} = \{\sum_{n>>-\infty} p^n[x_n] : x_n \in \widetilde{\mathrm{E}}\}$ and we use $v_{\widetilde{\mathrm{E}}}$ to single out, for every real number $r \geq 0$, the *overconvergent* subring

$$\widetilde{\mathscr{E}}^{\dagger,r} := \left\{ \sum_{n>>-\infty} p^n[x_n] \in \widetilde{\mathscr{E}} : n + (p-1)/(pr) \cdot v_{\widetilde{\mathrm{E}}}(x_n) \to \infty \right\}.$$

Put

$$\widetilde{\mathscr{E}}^{\dagger} := \bigcup_{r>0} \widetilde{\mathscr{E}}^{\dagger,r}.$$

Let ϵ be a p^∞-th root of unity in $\widetilde{\mathrm{E}}$ (for example, $\epsilon = (1, \epsilon_1, \ldots)$ such that $\epsilon_1 \neq 1$) and put $\mathrm{X} := \epsilon - 1$. We recall the subfields E and $\bar{\mathrm{E}}$ of $\widetilde{\mathrm{E}}$ given by

$$\mathrm{E} := \mathbb{F}_p((\mathrm{X})) \quad \text{and} \quad \bar{\mathrm{E}} := \text{ separable closure of } \mathrm{E}\ ;$$

and we recall that inside $\widetilde{\mathscr{E}}$ (putting $\mathrm{X} := [\epsilon] - 1$) there are: the quotient field of the Cohen ring of E,

$$\mathscr{E} = \text{ the } p\text{-adic completion of } \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p((\mathrm{X})),$$

and the quotient field of the Cohen ring of $\bar{\mathrm{E}}$,

$$\overline{\mathscr{E}} = \text{ the } p\text{-adic completion of the maximal unramified extension of } \mathscr{E}\ .$$

Put

$$\mathscr{E}^+ := \widetilde{\mathscr{E}}^+ \cap \mathscr{E}, \quad \mathscr{E}^{\dagger,r} := \widetilde{\mathscr{E}}^{\dagger,r} \cap \mathscr{E} \quad \text{and} \quad \mathscr{E}^{\dagger} := \widetilde{\mathscr{E}}^{\dagger} \cap \mathscr{E}.$$

and likewise

$$\overline{\mathscr{E}}^+ := \widetilde{\mathscr{E}}^+ \cap \overline{\mathscr{E}}, \quad \overline{\mathscr{E}}^{\dagger,r} := \widetilde{\mathscr{E}}^{\dagger,r} \cap \overline{\mathscr{E}} \quad \text{and} \quad \overline{\mathscr{E}}^{\dagger} := \widetilde{\mathscr{E}}^{\dagger} \cap \overline{\mathscr{E}}$$

Explicitly

$$\mathscr{E} = \{\sum_{n\in\mathbb{Z}} a_n \mathrm{X}^n : \{a_1, a_2, \ldots\} \text{ bounded, and } a_{-1}, a_{-2}, \ldots \to 0\}$$

and $\mathscr{E}^\dagger$ is the subfield of all power series $f(\mathrm{X}) = \sum_{n\in\mathbb{Z}} a_n \mathrm{X}^n$ such that

- $\{a_1, a_2, \ldots\}$ is bounded, and
- there is r in $[0, 1[$ such that $a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots$ converges for all x in $\mathbb{C}_p$ with $r \leq |x| < 1$.

and

$$\mathscr{E}^+ = \{\sum_{n\in\mathbb{N}} a_n \mathrm{X}^n : \{a_1, a_2, \ldots\} \text{ bounded } \} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{X}]],$$

that is, a_{-1}, a_{-2}, ... all vanish. Finally we put $\widetilde{\mathscr{E}}_{\mathbf{K}} := \widetilde{\mathscr{E}} \otimes_{\mathbb{Q}_p} \mathbf{K}$ and analogously for $\mathscr{E}$, $\overline{\mathscr{E}}$, $\mathscr{E}^\dagger$ and $\mathscr{E}^+$. The morphism φ and the topological group Γ act on all rings $\mathscr{E}$, $\overline{\mathscr{E}}$, $\mathscr{E}^\dagger$ and $\mathscr{E}^+$ (and their tensor products with $\mathbf{K}$ over $\mathbb{Q}_p$) by Equation (5.1).

Overconvergent and finite-height φ, Γ*-modules.* Let V be a p-adic Galois representation. The Galois group H acts on $\overline{\mathscr{E}}_{\mathbf{K}}$, $\overline{\mathscr{E}}^{\dagger}_{\mathbf{K}}$ and $\overline{\mathscr{E}}^{+}_{\mathbf{K}}$. We recall

$$\mathrm{D}(\mathrm{V}) = (\mathrm{V} \otimes_{\mathbf{K}} \overline{\mathscr{E}}_{\mathbf{K}})^{\mathrm{H}},$$

which is a module over $\mathscr{E}_{\mathbf{K}}$ (because $\bar{\mathrm{E}}^{\mathrm{H}} = \mathrm{E}$ and $\overline{\mathscr{E}}^{\mathrm{H}} = \mathscr{E}$). We put likewise

$$\mathrm{D}^{\dagger}(\mathrm{V}) := (\mathrm{V} \otimes_{\mathbf{K}} \overline{\mathscr{E}}^{\dagger}_{\mathbf{K}})^{\mathrm{H}} \quad \text{and} \quad \mathrm{D}^{+}(\mathrm{V}) := (\mathrm{V} \otimes_{\mathbf{K}} \overline{\mathscr{E}}^{+}_{\mathbf{K}})^{\mathrm{H}}$$

and, because $\overline{\mathscr{E}}^{\dagger\mathrm{H}} = \mathscr{E}^{\dagger}$ and $\overline{\mathscr{E}}^{+\mathrm{H}} = \mathscr{E}^{+}$,

$$\mathrm{D}^{\dagger}(\mathrm{V}) \text{ is a module over } \mathscr{E}^{\dagger}_{\mathbf{K}} \quad \text{and} \quad \mathrm{D}^{+}(\mathrm{V}) \text{ is a module over } \mathscr{E}^{+}_{\mathbf{K}}.$$

Definition of the Wach module. A crystalline p-adic Galois representation is *positive* if all filtration jump indices of $\mathrm{D}_{\mathrm{cris}}(\mathrm{V})$ are nonnegative.

THEOREM 7.3. *Let* V *be a* p*-adic Galois representation. If* V *is crystalline and positive then there is a unique module* $\mathscr{E}^{+}_{\mathbf{K}}$*-module* N(V) *that*

- *fulfills* $\mathrm{N}(\mathrm{V}) \otimes_{\mathscr{E}^{+}_{\mathbf{K}}} \mathscr{E}_{\mathbf{K}} \xrightarrow{\sim} \mathrm{D}(\mathrm{V})$ *as* φ, Γ*-modules,*
- *is stable under* Γ *and* Γ *acts trivially on* $\mathrm{N}(\mathrm{V})/\mathrm{X} \cdot \mathrm{N}(\mathrm{V})$*, and*
- *for which there is* h *in* $\mathbb{N}$ *such that* $\mathrm{X}^{h} \cdot \mathrm{D}^{+}(\mathrm{V}) \subseteq \mathrm{N}(\mathrm{V}) \subseteq \mathrm{D}^{+}(\mathrm{V})$.

Moreover N(V) *is stable under* φ.

PROOF: By [**Col99**, Théorème 1], if V is crystalline then D(V) is *of finite height*, that is, there is a submodule D^{+} over $\mathscr{E}^{+}_{\mathbf{K}}$ inside D(V) that is stable under φ and Γ and such that

$$(*) \qquad \mathrm{D}^{+} \otimes_{\mathscr{E}^{+}_{\mathbf{K}}} \mathscr{E}_{\mathbf{K}} \xrightarrow{\sim} \mathrm{D}(\mathrm{V}).$$

By [**Fon90**, Section B2.1], there is a submodule over $\mathscr{E}^{+}_{\mathbf{K}}$ inside D(V) that fulfills $(*)$ if and only if there is a submodule over $\mathscr{E}^{+}_{\mathbf{K}}$ inside $\mathrm{D}^{+}(\mathrm{V})$ that fulfills $(*)$.

By [**Wac96**, A5], if V is crystalline (and of finite height) then there is a submodule N(V) of $\mathrm{D}^{+}(\mathrm{V})$ that

1. satisfies $(*)$, and
2. is stable under Γ and Γ acts trivially on $\mathrm{N}(\mathrm{V})/\mathrm{X} \cdot \mathrm{N}(\mathrm{V})$.

By [**Ber04**, Section II.1] there is a unique such module N(V) such that

3. there is h in $\mathbb{N}$ such that $\mathrm{X}^{h} \cdot \mathrm{D}^{+}(\mathrm{V}) \subseteq \mathrm{N}(\mathrm{V})$.

Because the smallest $\mathscr{E}^{+}$-module that includes N(V) and $\varphi\mathrm{N}(\mathrm{V})$ fulfills again Conditions 1. – 3., by uniqueness $\varphi\mathrm{N}(\mathrm{V}) \subseteq \mathrm{N}(\mathrm{V})$. □

The Wach module over the Amice Ring. The ring $\widetilde{\mathscr{E}}^{\dagger,r}$ carries for every $s \geq r$ a valuation

$$v_s\left(\sum_n [x_n] p^n\right) := \min\{v_{\widetilde{\mathrm{E}}}(x_{-n}) - rn : n \in \mathbb{N}\} \cup \{v_{\widetilde{\mathrm{E}}}(x_n) + sn : n \in \mathbb{N}\}.$$

Let

$$\widetilde{\mathscr{R}}^{r} := \text{ completion of } \widetilde{\mathscr{E}}^{\dagger,r} \text{ for the Fréchet topology given by } \{v_s : s \geq r\}$$

and

$$\widetilde{\mathcal{R}}^{+,r} := \text{ completion of } \widetilde{\mathcal{E}}^{+} \text{ for the Fréchet topology given by } \{v_s : s \geq r\}$$

and, inside $\widetilde{\mathcal{R}}^{+,r}$,

- let $\overline{\mathcal{R}}^{r}$ be the closure of $\overline{\mathcal{E}}^{\dagger,r}$ and let $\overline{\mathcal{R}}^{+,r}$ be the closure of $\overline{\mathcal{E}}^{+}$, and
- let $\mathcal{R}^{r}$ be the closure of $\mathcal{E}^{\dagger,r}$ and let $\mathcal{R}^{+,r}$ be the closure of $\mathcal{E}^{+}$.

Put

$$\widetilde{\mathcal{R}} := \bigcup_{r>0} \widetilde{\mathcal{R}}^{r} \quad \text{and} \quad \widetilde{\mathcal{R}}^{+} := \bigcup_{r>0} \widetilde{\mathcal{R}}^{+,r},$$

$$\overline{\mathcal{R}} := \bigcup_{r>0} \overline{\mathcal{R}}^{r} \quad \text{and} \quad \overline{\mathcal{R}}^{+} := \bigcup_{r>0} \overline{\mathcal{R}}^{+,r},$$

and

$$\mathcal{R} := \bigcup_{r>0} \mathcal{R}^{r} \quad \text{and} \quad \mathcal{R}^{+} := \bigcup_{r>0} \mathcal{R}^{+,r}.$$

Let $\widetilde{\mathcal{R}}^{r}_{\mathbf{K}}$ and $\widetilde{\mathcal{R}}^{+,r}_{\mathbf{K}}$ denote the tensor products of $\widetilde{\mathcal{R}}^{r}$ and $\widetilde{\mathcal{R}}^{+,r}$ with $\mathbf{K}$ over $\mathbb{Q}_p$, and analogously for $\overline{\mathcal{R}}^{r}_{\mathbf{K}}, \overline{\mathcal{R}}^{+,r}_{\mathbf{K}}$ and $\mathcal{R}^{r}_{\mathbf{K}}, \mathcal{R}^{+,r}_{\mathbf{K}}$, and their unions $\widetilde{\mathcal{R}}_{\mathbf{K}}, \widetilde{\mathcal{R}}^{+}_{\mathbf{K}}$, $\overline{\mathcal{R}}_{\mathbf{K}}, \overline{\mathcal{R}}^{+}_{\mathbf{K}}$ and $\mathcal{R}_{\mathbf{K}}, \mathcal{R}^{+}_{\mathbf{K}}$ over all $r > 0$.

Let $[p^{-r}, 1[$ be the annulus of all x in $\mathbb{C}_p$ with $p^{-r} \leq |x| < 1$. Under the identification $[\varepsilon] - 1 \mapsto X$,

$$\mathcal{R}^{r}_{\mathbf{K}} := \{ \text{ all } f(X) \text{ in } \mathbf{K}[[X, 1/X]] \text{ that converge on } [p^{-r}, 1[\},$$

$$\mathcal{R}^{+}_{\mathbf{K}} := \{ \text{ all } f(X) \text{ in } \mathbf{K}[[X]] \text{ that converge on the open unit disc } \}$$

and $\mathcal{R}_{\mathbf{K}}$ is the *Robba ring* of all $\sum_{n \in \mathbb{Z}} a_n X^n$ with entries in $\mathbf{K}$ that converge on some annulus up to the boundary of the open unit disc of $\mathbb{C}_p$. Whereas the power series in $\mathcal{E}^{\dagger,r}_{\mathbf{K}}$ and $\mathcal{E}^{+}_{\mathbf{K}}$ converge and are bounded, those in $\mathcal{R}^{\dagger,r}_{\mathbf{K}}$ and $\mathcal{R}^{+}_{\mathbf{K}}$ only converge but may be unbounded.

The morphism φ and the topological group Γ act continuously on $\widetilde{\mathcal{R}}^{r}_{\mathbf{K}}$, $\widetilde{\mathcal{R}}^{+}_{\mathbf{K}}$ and $\widetilde{\mathcal{R}}_{\mathbf{K}}$ (as well as $\overline{\mathcal{R}}^{r}_{\mathbf{K}}$, $\overline{\mathcal{R}}^{+}_{\mathbf{K}}$ and $\overline{\mathcal{R}}_{\mathbf{K}}$ and $\mathcal{R}^{r}_{\mathbf{K}}$, $\mathcal{R}^{+}_{\mathbf{K}}$ and $\mathcal{R}_{\mathbf{K}}$) by Equation (5.1).

Let V be a finite-dimensional $\mathbf{K}$-vector space on which $\mathcal{G}_{\mathbb{Q}_p}$ acts continuously.

Theorem 7.4. *If* V *is crystalline positive then as* φ*-modules*

$$D_{\mathrm{cris}}(V) = \left(N(V) \otimes_{\mathcal{E}^{+}_{\mathbf{K}}} \mathcal{R}^{+}_{\mathbf{K}}\right)^{\Gamma} \quad \textit{and} \quad D_{\mathrm{cris}}(V) \otimes_{\mathbf{K}} \mathcal{R}^{+}_{\mathbf{K}} \hookrightarrow N(V) \otimes_{\mathcal{E}^{+}_{\mathbf{K}}} \mathcal{R}^{+}_{\mathbf{K}}.$$

Proof: By [**Ber02**, Proposition 3.7], if V is crystalline and positive then, as φ-modules,

$$(*) \quad D_{\mathrm{cris}}(V) = (D^{\dagger}(V) \otimes_{\mathcal{E}^{\dagger}} \mathcal{R})^{\Gamma} \quad \text{and} \quad D_{\mathrm{cris}}(V) \otimes_{\mathbb{Q}_p} \mathcal{R} \xrightarrow{\sim} D^{\dagger}(V) \otimes_{\mathcal{E}^{\dagger}_{\mathbf{K}}} \mathcal{R}.$$

If V is crystalline positive then, by [**Ber04**, Proposition II.2.1],

$$D_{\mathrm{cris}}(V) \hookrightarrow N(V) \otimes_{\mathcal{E}^{+}} \mathcal{R}^{+}.$$

We conclude by $(*)$ and taking the tensor product over $\mathbb{Q}_p$ by $\mathbf{K}$. □

Let V be a crystalline Galois representation over $\mathbf{K}$. The *lowest filtration jump* of V is the highest h in $\mathbb{Z}$ such that $\mathrm{Fil}^h \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) = \mathrm{D}_{\mathrm{cris}}(\mathrm{V})$.

Proposition 7.5 ([**Ber04**, Proposition II.2.1]). *If* V *is crystalline positive and* h *its lowest filtration jump then*

$$\mathrm{N}(\mathrm{V}) \otimes_{\mathscr{E}_{\mathbf{K}}} 1/\mathrm{X}^h \cdot \mathscr{R}^+_{\mathbf{K}} \subseteq \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} 1/t^h \cdot \mathscr{R}^+_{\mathbf{K}}.$$

PROOF: By [**Ber02**, Proposition 4.12],

- for all radii $r < 1$, the subring

$$\mathscr{R}^{+,r}_{\mathbf{K}} := \left\{ f(x) \in \mathbf{K}[[\mathrm{X}]] : f(x) \text{ converges for all } x \text{ in } \mathbb{C}_p \text{ with } |x| \leq r \right\}$$

  of $\mathscr{R}^+_{\mathbf{K}}$ is a principal ideal domain, and
- consequently an Elementary Divisor Theorem over $\mathscr{R}^+_{\mathbf{K}}$ holds. That is, given a morphism between finite free modules over $\mathscr{R}^+_{\mathbf{K}}$ there are a basis of its domain and a basis of its codomain such that it is given by a diagonal matrix (and whose entries are called the *elementary divisors*).

Let d be the dimension of V and let $\delta_1, \ldots, \delta_d$ be the elementary divisors of the inclusion of free modules

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes \mathscr{R}^+_{\mathbf{K}} \subseteq \mathrm{N}(\mathrm{V}) \otimes_{\mathscr{E}^+_{\mathbf{K}}} \mathscr{R}^+_{\mathbf{K}}.$$

Let h be the lowest filtration jump of $\mathrm{D}_{\mathrm{cris}}(\mathrm{V})$. By [**BB10**, Théorème 3.2.2], the *divisor* of $\delta_1, \ldots, \delta_d$ (= its zeroes counted with their multiplicities) is included in that of $(t/\mathrm{X})^h$ (where $t/\mathrm{X} = \log(1+\mathrm{X})/\mathrm{X} = 1 - \mathrm{X}/2 + \mathrm{X}^2/3 - \cdots$), and thence $\delta_1, \ldots, \delta_d$ divides $(t/\mathrm{X})^h$ by Weierstrass division. □

Put $r_0 = 1/(p-1)$ and $r_n = p^{n-1}/(p-1)$. The Frobenius $\cdot^p$ on $\widetilde{\mathrm{E}}^+$, by definition of $\widetilde{\mathrm{E}}^+$ a topological ring automorphism, gives by functoriality of the Witt vectors (and of the quotient field) a topological ring automorphism φ on $\widetilde{\mathscr{E}}$. More exactly:

$$\varphi^n(\widetilde{\mathscr{E}}^{\dagger,r_0}) = \widetilde{\mathscr{E}}^{\dagger,r_n},$$

and therefore a topological ring isomorphism

$$\varphi^{-n} \colon \widetilde{\mathscr{E}}^{\dagger,r_n} \xrightarrow{\sim} \widetilde{\mathscr{E}}^{\dagger,r_0}.$$

For every $s \geq r$, the ring morphism φ^{-1} is uniformly continuous for v_s and therefore extends to a morphism of topological rings

$$\varphi^{-n} \colon \widetilde{\mathscr{R}}^{r_n} \to \widetilde{\mathscr{R}}^{r_0}.$$

For n in $\mathbb{N}$, let

$$\iota_n := \iota_0 \circ \varphi^{-n} : \widetilde{\mathscr{R}}^{+,r_n} \hookrightarrow \mathbb{B}^+_{\mathrm{dR}}$$

be the composition of

- the restriction of φ^{-n} onto $\widetilde{\mathscr{R}}^{+,r_n}$ with

- the embedding of topological rings

$$\widetilde{\mathcal{R}}^{+,r_0} = \mathbb{B}^+_{\max} \overset{\iota_0}{\hookrightarrow} \mathbb{B}^+_{\mathrm{dR}},$$

which extends after inverting t (and where we recall $\mathbb{B}_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}[1/t]$) to a morphism

$$\iota_n : \widetilde{\mathcal{R}}^{+,r_n}[1/t] \hookrightarrow \mathbb{B}_{\mathrm{dR}}$$

(and likewise after taking the tensor product over $\mathbb{Q}_p$ with $\mathbf{K}$). Let V be a p-adic Galois representation over $\mathbf{K}$. If V is crystalline positive then by Theorem 7.4

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathcal{R}^+_{\mathbf{K}} \hookrightarrow \mathrm{N}(\mathrm{V}) \otimes_{\mathcal{E}^+_{\mathbf{K}}} \mathcal{R}^+_{\mathbf{K}},$$

and after base extension

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}} \hookrightarrow \mathrm{N}(\mathrm{V}) \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}}.$$

Because

$$\mathrm{N}(\mathrm{V}) \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}} \subseteq \mathrm{D}^+(\mathrm{V}) \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}} \hookrightarrow \mathrm{V} \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}},$$

(where the embedding on the right-hand side is given by [the flat scalar extension by $\cdot \otimes_{\overline{\mathcal{E}}^+_{\mathbf{K}}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}}$ of] the embedding of $\mathcal{E}^+_{\mathbf{K}}$-modules

$$\mathrm{D}^+(\mathrm{V}) \otimes_{\mathcal{E}^+_{\mathbf{K}}} \overline{\mathcal{E}}^+_{\mathbf{K}} = (\mathrm{V} \otimes_{\mathbf{K}} \overline{\mathcal{E}}^+_{\mathbf{K}})^{\mathrm{H}} \otimes_{\mathcal{E}^+_{\mathbf{K}}} \overline{\mathcal{E}}^+_{\mathbf{K}} \hookrightarrow (\mathrm{V} \otimes_{\mathbf{K}} \mathcal{E}^+_{\mathbf{K}}) \otimes_{\mathcal{E}^+_{\mathbf{K}}} \overline{\mathcal{E}}^+_{\mathbf{K}} = \mathrm{V} \otimes_{\mathbf{K}} \overline{\mathcal{E}}^+_{\mathbf{K}},$$

whose injectivity follows by $\overline{\mathcal{E}}^{+\,\mathrm{H}}_{\mathbf{K}} = \mathcal{E}^+_{\mathbf{K}}$ from [**FO14**, Proof of Theorem 2.13.(1)]), we obtain

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}} \hookrightarrow \mathrm{V} \otimes_{\mathbf{K}} \widetilde{\mathcal{R}}^{+,r_n}_{\mathbf{K}}.$$

After scalar restriction to $\mathcal{R}^+_{\mathbf{K}}$ and applying ι_n,

$$\iota_n \colon \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathcal{R}^+_{\mathbf{K}} \hookrightarrow \mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}_{\mathrm{dR}/\mathbf{K}}.$$

If V is crystalline positive and h its lowest filtration jump, then by Proposition 7.5

$$\mathrm{N}(\mathrm{V}) \otimes_{\mathcal{E}_{\mathbf{K}}} \mathrm{X}^{-h} \cdot \mathcal{R}^+_{\mathbf{K}} \subseteq \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} t^{-h} \cdot \mathcal{R}^+_{\mathbf{K}},$$

and more exactly:

Proposition 7.6. *If* V *is crystalline positive and h its lowest filtration jump, then*

$$\mathrm{N}(\mathrm{V}) \otimes_{\mathcal{E}^+_{\mathbf{K}}} \mathrm{X}^{-h} \cdot \mathcal{R}^+_{\mathbf{K}} = \{x \in \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} (1/t)^h \cdot \mathcal{R}^+_{\mathbf{K}} : \iota_1(x), \iota_2(x), \ldots \in \mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}\}$$

Moreover $\mathrm{N}(\mathrm{V}) \otimes_{\mathcal{E}^+_{\mathbf{K}}} \mathrm{X}^{-h} \cdot \mathcal{R}^+_{\mathbf{K}}$ *is stable under* ψ.

Proof: Because

$$\iota_n \colon \mathcal{R}^{+,r_n} \xrightarrow{\sim} \mathcal{R}^{+,r_0} \hookrightarrow \mathbb{B}^+_{\mathrm{dR}}$$

and

$$\mathcal{R}^{+,r_n} \cap \mathcal{R}^+[1/t] = \mathcal{R}^+[\varphi^n(\mathrm{X})/t],$$

we have

$$\bigcap_{n \geq 1} \left(\iota_n^{-1}(\mathcal{R}^{+,r_0}) \cap \mathcal{R}^+[1/t] \right) = \bigcap_{n \geq 1} \mathcal{R}^+[\varphi^n(\mathrm{X})/t] = \mathcal{R}^+[1/\mathrm{X}].$$

In particular, because $\iota_n(r)$ in $\mathbb{B}^+_{\mathrm{dR}}$ if and only if r in $\iota_n^{-1}(\mathscr{R}^{+,r_0})$,

$$\{r \in t^{-h} \cdot \mathscr{R}^+ : \iota_n(r) \text{ in } \mathbb{B}^+_{\mathrm{dR}} \text{ for all } n \geq 1\} = \mathrm{X}^{-h}\mathscr{R}^+.$$

By Theorem 7.4 and Proposition 7.5,

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} t^{-h} \cdot \mathscr{R}^+_{\mathbf{K}} = \mathrm{N}(\mathrm{V}) \otimes_{\mathscr{E}_{\mathbf{K}}} t^{-h} \cdot \mathscr{R}^+_{\mathbf{K}}$$

and, because V is crystalline,

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathbb{B}_{\mathrm{dR}/\mathbf{K}} = \mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}_{\mathrm{dR}/\mathbf{K}}$$

Therefore, after a choice of basis,

$$\mathrm{N}(\mathrm{V}) \otimes_{\mathscr{E}^+} \mathrm{X}^{-h} \cdot \mathscr{R}^+_{\mathbf{K}} = \{x \in \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} t^{-h} \cdot \mathscr{R}^+_{\mathbf{K}} : \iota_n(x) \in \mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}} \text{ for all } n \geq 1\}.$$

Finally, $\mathrm{N}(\mathrm{V}) \otimes_{\mathscr{E}^+_{\mathbf{K}}} \mathrm{X}^{-h} \cdot \mathscr{R}^+_{\mathbf{K}}$ is stable under ψ, because we have

- that $\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} t^{-h} \cdot \mathscr{R}^+_{\mathbf{K}}$ is stable under ψ, and
- that, for every n in $\mathbb{N}$, if $\iota_{n+1}(f)$ in $\mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$ then $\iota_n(\psi(f))$ in $\mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$.

For details, see [**BB10**, Proposition 3.3.3]. □

The Wach module. We single out the Wach module by conditions on the growth of the power series coefficients towards the boundary of the open unit ball of $\mathbb{C}_p$:

Definition. Let $r \geq 0$ be a rational number. Let $\|\cdot\|_r : \mathscr{R}^+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be given by

$$\|\sum_{n \in \mathbb{N}} a_n \mathrm{X}^n\|_r := \sup\{|a_n|/n^r : n \in \mathbb{N}\}.$$

A power series f in $\mathscr{R}^+$ is *of order* r if $\|f\|_r < \infty$ and $\|\cdot\|_r$ is a norm on all power series in $\mathscr{R}^+$ of order r.

For example, the order of $t = \log(1+\mathrm{X}) = \mathrm{X} - \mathrm{X}^2/{+}\mathrm{X}^3/3 - \cdots$ is 1. The notion of order and the norm $\|\cdot\|_r$ extends canonically onto $\mathscr{R}^+[1/t]$ and, for a tuple $r_1, \ldots, r_d$, to a notion of order and norm $\|\cdot\|_{r_1,\ldots,r_d}$ on a free module of rank d over $\mathscr{R}^+[1/t]$.

Let us give an alternative characterization of the notion of order: For positive $\rho < 1$, for example $\rho = r_0$, let $\|\cdot\|_{\mathrm{D}(0,\rho)}$ be the norm on $\mathscr{R}^+$ given by the supremum on the closed disc $\mathrm{D}(0,\rho)$ inside $\mathbb{C}_p$ of radius ρ. This norm extends canonically to every finite free module over $\mathscr{R}^+$.

Let f in $\mathscr{R}^+$. Then f is of order r if and only if, for every positive $\rho < 1$, the set $\{\|\psi^n f\|_{\mathrm{D}(0,\rho)}/p^{nr} : n \text{ in } \mathbb{N}\}$ is bounded ([**Col03**, Corollaire V.3.20]). Consequently, let φ on $\mathscr{R}^+ \oplus \cdots \oplus \mathscr{R}^+$ have eigenvalues $1/\alpha_1, \ldots, 1/\alpha_d$ with $r_1 = -v(\alpha_1) \geq 0, \ldots, r_d = -v(\alpha_d) \geq 0$. Then f in $\mathscr{R}^+ \oplus \cdots \oplus \mathscr{R}^+$ is of order $r_1, \ldots, r_d$ if and only if, for every positive $\rho < 1$, the set $\{\|\psi^n f\|_{\mathrm{D}(0,\rho)} : n \text{ in } \mathbb{N}\}$ is bounded. Because $\mathrm{N}(\mathrm{V})$ has rank d over $\mathscr{E}^+$, the ring of elements of order 0 in $\mathscr{R}^+$, we obtain:

Proposition 7.7. *If* V *is crystalline positive and* h *its lowest filtration jump and* $r_1 = -v(\alpha_1), \ldots, r_d = -v(\alpha_d)$ *where* $\alpha_1, \ldots, \alpha_d$ *are the eigenvalues of* φ*, then*

$$X^{-h} \cdot N(V) = \text{ the set of all } f = (1/t)^h f_1 + \cdots + (1/t)^h f_d \text{ in } D_{cris}(V) \otimes_{\mathbf{K}} t^{-h} \cdot \mathcal{R}^+_{\mathbf{K}}$$

such that

- $\iota_1(f), \iota_2(f), \ldots$ *in* $V \otimes_{\mathbf{K}} \mathbb{B}^+_{dR/\mathbf{K}}$*, and*
- f_1 *is of order* $h + r_1, \ldots, f_d$ *is of order* $h + r_d$.

Moreover $1/X^h \cdot N(V)$ *is stable under* ψ.

For the stability under ψ, we use Proposition 7.6 and that the set $\{\psi^n f : n \in \mathbb{N}\}$ is stable under ψ (and thus in particular the boundedness of its norms).

By compactness of $\mathcal{G}_{\mathbb{Q}_p}$, let T be a finite free $\mathcal{O}_{\mathbf{K}}$-module of V stable under $\mathcal{G}_{\mathbb{Q}_p}$, and let D(T) be its corresponding étale φ, Γ-module over $\mathcal{O}_{\mathcal{E}}$. Put

$$N(T) := N(V) \cap D(T).$$

In particular N(T) is stable under φ, because N(V) (by Theorem 7.3) and D(T) are stable under φ.

Corollary 7.8. *If* V *is irreducible and of dimension* > 1 *then there is* n *in* $\mathbb{N}$ *such that* $D^\sharp(T) = \psi^n(N(T))$.

Proof: Because $\varphi(N(T)) \subseteq N(T)$ we have

$$\psi(N(T)) \supseteq N(T).$$

Because $X^{-h} \cdot N(T)$, like $X^{-h} N(V)$, is stable under ψ, the sequence

$$\psi(N(T)) \subseteq \psi^2(N(T)) \subseteq \ldots$$

is bounded above by $X^{-h} \cdot N(T)$. Thus, because $\mathcal{E}^+$ is noetherian and N(T) finitely generated over $\mathcal{O}_{\mathcal{E}^+}$, there is n in $\mathbb{N}$ such that $\psi^n(N(T)) = \psi^{n+1}(N(T))$. Therefore $N = \psi^n(N(T))$ is a module over $\mathcal{O}_{\mathcal{E}^+}$ such that $\psi(N) = N$. Thus $N = D^\sharp(T)$ by Proposition 7.1. □

Corollary 7.9. *We have as* φ, Γ*-modules*

$$(\varprojlim_{\psi} N(T)) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K} = (\varprojlim_{\psi} D^\sharp(T)) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K} \xrightarrow{\sim} \varprojlim_{\psi}{}^{b} D(V).$$

Proof: By Proposition 7.2 and Corollary 7.8. □

Let finally V be crystalline negative, that is, all filtration jump indices of $D_{cris}(V)$ are non-positive. Then all preceding results apply to the crystalline positive p-adic Galois representation $V(-h) = V \otimes \chi^{-h}$ for h the lowest filtration jump index of $D_{cris}(V)$. We have, as sets,

- that $N(V(-h)) = X^h \cdot N(V)$, and
- that $D_{cris}(V(-h)) = t^h \cdot D_{cris}(V)$.

In particular,

$$\varprojlim_n{}^{\mathrm{b}}\, \mathrm{N}(\mathrm{V}) = \varprojlim_n{}^{\mathrm{b}}\, \mathrm{X}^{-h} \cdot \mathrm{N}(\mathrm{V}(-h))$$

and $\mathrm{X}^{-h} \cdot \mathrm{N}(\mathrm{V}(-h))$ as subset of

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{V}(-h)) \otimes_{\mathbf{K}} t^{-h} \cdot \mathscr{R}^+_{\mathbf{K}} = \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathscr{R}^+_{\mathbf{K}}$$

is given in Proposition 7.7. Moreover ([**BB10**, Proposition 3.3.8]), the p-adic topology on $\mathrm{X}^{-h}\mathrm{N}(\mathrm{V}(-h))$ inside $\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathscr{R}^+_{\mathbf{K}}$ is given by the norm

$$\|w_\alpha e_\alpha + w_\beta e_\beta\| = \max\{\|w_\alpha\|_{v(\alpha)}, \|w_\beta\|_{v(\beta)}\}.$$

We obtain:

Corollary 7.10. *Let* V *be a crystalline p-adic Galois representation that is*

- *of dimension* 2*,*
- *absolutely irreducible,*
- *negative and*
- *such that the automorphism* φ *of its corresponding filtered* φ*-module* $\mathrm{D}_{\mathrm{cris}}(\mathrm{V})$ *has distinct eigenvalues, say* $1/\alpha$ *and* $1/\beta$ *for* α *and* β *in* $\mathbf{K}$.

Let e_α *and* e_β *be the eigenvectors of* φ *in* $\mathrm{D}_{\mathrm{cris}}(\mathrm{V})$. *Then the sequence* $(w_n = w_{\alpha,n}e_\alpha + w_{\beta,n}e_\beta : n \in \mathbb{N})$ *of entries in* $\mathscr{R}^+_{\mathbf{K}} \otimes_{\mathbf{K}} \mathrm{D}_{\mathrm{cris}}(\mathrm{V})$ *is in*

$$\varprojlim_\psi{}^{\mathrm{b}}\, \mathrm{D}(\mathrm{V})$$

if and only if, for all $n \geq 0$*,*

- $w_{\alpha,n}$ *is of order* $v(\alpha)$ *and* $w_{\beta,n}$ *is of order* $v(\beta)$*, and* $\{\|w_{\alpha,n}\|_{v(\alpha)}\}$ *and* $\{\|w_{\beta,n}\|_{v(\beta)}\}$ *are bounded,*
- $\iota_1(w_n), \iota_2(w_n), \ldots$ *in* $\mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$*, and*
- $\psi(w_{\alpha,n+1}) = 1/\alpha \cdot w_{\alpha,n}$ *and* $\psi(w_{\beta,n+1}) = 1/\beta \cdot w_{\beta,n}$.

Let us make $\iota_m \colon \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathscr{R}^+_{\mathbf{K}} \to \mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$ explicit: Under the isomorphism

$$\mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}} = \mathrm{Fil}^0 \left(\mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathbb{B}_{\mathrm{dR}/\mathbf{K}}\right),$$

the morphism

$$\iota_m \colon \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathscr{R}^+_{\mathbf{K}}[1/t] \to \mathrm{D}_{\mathrm{cris}}(\mathrm{V}) \otimes_{\mathbf{K}} \mathbf{K}_m[[t]],$$

where we

- let ζ_{p^m} be a root of unity of order p^m,
- put $\mathrm{L}_m := \mathbf{K} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^m})$, and
- because $\mathbb{B}^+_{\mathrm{dR}}$ is complete for the ideal generated by t, it includes the subring $\mathbf{K}_m[[t]]$,

is given by

$$\iota_m = \varphi^{-m} \otimes \iota_m,$$

where

- the $\mathbf{K}$-vector space automorphism φ^{-1} of $D_{cris}(V)$ is the inverse of φ, and
- the $\mathbf{K}$-algebra morphism $\iota_m \colon \mathscr{R}^+_{\mathbf{K}} \to \mathbf{K}_m[[t]]$ is given by

$$\iota_m(X) = \zeta_{p^m} \exp(t/p^m) - 1.$$

Let $-h$ be the lowest filtration jump index in $D_{cris}(V)$. Because

$$\mathrm{Fil}^i D_{cris}(V) = \mathbf{K} \cdot (e_\alpha + e_\beta) \quad \text{for } i = -h+1, \ldots, 0$$

and the filtration of $\mathbb{B}^+_{dR/\mathbf{K}}$ is given by the fractional ideals generated by powers of t or, equivalently, of $\zeta_{p^m} \exp(t/p^m) - 1$, we have

$$\iota_m(w_\alpha e_\alpha + w_\beta e_\beta) \text{ in } \mathrm{Fil}^0(D_{cris}(V) \otimes_{\mathbf{K}} \mathbf{K}_m((t)))$$

(where the right-hand side carries the tensor product filtration) if and only if

$$\alpha^m \iota_m(w_\alpha) - \beta^m \iota_m(w_\beta) \text{ in } t^h \mathbf{K}_m[[t]]$$

Part 4. Amice transform: From power series to function spaces

8. Amice Transform

8.1. Amice Transform on $\mathbb{Z}_p$. Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_{\mathbf{K}}$ its ring of integers. Let $\mathscr{C}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbf{K}})$ be all continuous functions $f \colon \mathbb{Z}_p \to \mathcal{O}_{\mathbf{K}}$, endowed with the supremum norm, and let $\mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbf{K}})$ be its topological dual of all continuous (for the supremum norm) linear maps $\mu \colon \mathscr{C}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbf{K}}) \to \mathcal{O}_{\mathbf{K}}$, endowed with the operator norm.

Continuous Functions. Every continuous function $f \colon \mathbb{Z}_p \to \mathcal{O}_{\mathbf{K}}$ is uniformly approximated by locally constant functions f_n in $\mathcal{O}_{C_p}[\mathbb{Z}/p^n\mathbb{Z}]$; dually, the natural map

$$\mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbf{K}}) \xrightarrow{\sim} \mathcal{O}_{\mathbf{K}}[[\mathbb{Z}_p]]$$

is an isomorphism of topological $\mathcal{O}_{\mathbf{K}}$-algebras, where

- the left-hand side is equipped with the convolution product, and
- the right-hand side is the completed group algebra $\varprojlim \mathcal{O}_{\mathbf{K}}[\mathbb{Z}/p^n\mathbb{Z}]$ with the projective limit topology.

The topological group $\mathbb{Z}_p$ is generated by a single element, say $\gamma = \mathbf{1}$, yielding the *Iwasawa isomorphism* of topological algebras

$$\begin{aligned} \mathcal{O}_{\mathbf{K}}[[\mathbb{Z}_p]] &\xrightarrow{\sim} \mathcal{O}_{\mathbf{K}}[[X]] \\ \gamma &\mapsto X - 1. \end{aligned}$$

The composed isomorphism

$$\begin{aligned} \mathscr{D}^0(\mathbb{Z}_p, \mathcal{O}_{\mathbf{K}}) &\xrightarrow{\sim} \mathcal{O}_{\mathbf{K}}[[X]] \\ \delta_1 - \delta_0 &\mapsto X \end{aligned}$$

sends X to the evaluation measure $\delta_1 - \delta_0$ given by $f \mapsto f(1) - f(0)$.

Locally Analytic Functions. This isomorphism extends to the dual of all locally analytic functions: Let $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ be the Fréchet space of all $\mathbb{Q}_p$-locally analytic functions $f\colon \mathbb{Z}_p \to \mathbf{K}$.

THEOREM ([**Ami64**, THÉORÈME 10.1]). *The morphism of topological* $\mathbf{K}$*-algebras*

$$\mathrm{A}\colon\ \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \mathscr{R}^+$$
$$\delta_1 - \delta_0 \mapsto \mathrm{X}$$

between

- *all continuous linear maps* $\mu\colon\ \mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}$, *and*
- *all power series over* $\mathbf{K}$ *that converge on the open unit disc of* $\mathbb{C}_p$

is an isomorphism.

8.2. Amice Transform on $\mathbb{Q}_p$. Let us

- define the continuous linear forms on all r-times differentiable functions of compact support on $\mathbb{Q}_p$, and
- identify them under the Amice transform with a projective limit of normed vector spaces of power series:

The endomorphism $f \mapsto f(p\cdot)$ on $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ induces an endomorphism on $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$, and in particular maps $\delta_1 - \delta_0$ to $\delta_1^p - \delta_0$. Therefore, under the Amice transform $\mathrm{X} \mapsto \delta_1 - \delta_0$, we obtain

$$\begin{array}{ccc} \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) & \longrightarrow & \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) \\ \downarrow & & \downarrow \\ \mathscr{R}^+ & \xrightarrow{\ \varphi\ } & \mathscr{R}^+ \end{array}$$

where

- the bottom arrow φ maps X to $(\mathrm{X}+1)^p - 1$, and
- the upper arrow is induced from the endomorphism $p\cdot$ on $\mathbb{Z}_p$; that is, it precomposes $\mu\colon\ \mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}$ in the left-hand side with the endomorphism $f \mapsto f(p\cdot)$ on $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$.

The section ψ of φ then, under the isomorphism $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} \mathscr{R}^+$, identifies with the endomorphism of $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ given by precomposition with the endomorphism

$$f \mapsto \mathbf{1}_{p\mathbb{Z}_p} f(\cdot/p)$$

on $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$, where $\mathbf{1}_{p\mathbb{Z}_p}$ is the indicator function on $\mathbb{Z}_p$ with support $p\mathbb{Z}_p$, that is, it takes the value 1 on $p\mathbb{Z}_p$ and vanishes everywhere else. We obtain:

$$\begin{array}{ccc} \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) & \longrightarrow & \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K}) \\ \downarrow & & \downarrow \\ \mathscr{R}^+ & \xrightarrow{\ \psi\ } & \mathscr{R}^+ \end{array}$$

Because $\mathbb{Q}_p = \bigcup_n p^{-n}\mathbb{Z}_p$, therefore

$$\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) = \bigcup_n \mathscr{C}^{\mathrm{la}}(p^{-n}\mathbb{Z}_p, \mathbf{K}) = \varinjlim_n \mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$$

where

$$\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) := \{ \text{ all locally analytic functions } f : \mathbb{Q}_p \to \mathbf{K} \text{ of compact support } \},$$

and the transition maps of the injective limit running over $\mathbb{N}$ are all given by $f \mapsto 1_{p\mathbb{Z}_p} f(\cdot/p)$. Therefore, dually under the Amice transform

$$\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \xrightarrow{\sim} \varprojlim_{\psi} \mathscr{R}^+$$

where the transition maps of the projective limit running over $\mathbb{N}$ are all given by ψ.

We express the action of M on the φ, Γ-module $\mathbb{D}$ under the Amice transform. For this, we

1. express the action of M_0 and ψ on the φ, Γ-module $\mathscr{R}^+$ under the Amice transform;
2. use
   - this description of the action of M_0 and ψ on $\mathscr{R}^+$, and
   - that of $\mathbb{D} \otimes_{\mathscr{E}^+} \mathscr{R}^+ = \varprojlim_{\psi} \mathrm{D}_{\mathrm{cris}} \otimes \mathscr{R}^+$ and $\mathrm{D}_{\mathrm{cris}} \otimes \mathscr{R}^+ = \mathscr{R}^+ \oplus \mathscr{R}^+$ in Corollary 7.10

   to express the action of M on $\mathbb{D}$ as submodule of $\mathbb{D} \otimes_{\mathscr{E}^+} \mathscr{R}^+$ under the Amice transform.

8.3. Action of M_0 and ψ on $\mathscr{R}^+$ under the Amice transform. Recall $\mathbb{Z}_p^\bullet = p^{\mathbb{N}}\mathbb{Z}_p^*$.

Proposition 8.1. *Under the Amice transform $\mathscr{A} : \mathscr{R}^+ \xrightarrow{\sim} \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$*

- *the action of $\mathbb{Z}_p$ on $\mathscr{R}^+$ by $(1+\mathrm{X})^a \cdot$ is on $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ given by the dual of*

$$f \mapsto f(a + \cdot),$$

- *that of $\mathbb{Z}_p^\bullet$ on $\mathscr{R}^+$ by $a \mapsto [(1+\mathrm{X})^a - 1]\cdot$ is on $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ given by the dual of*

$$f \mapsto f(a \cdot)$$

Proof: For a in $\mathbb{Z}_p$, let δ_a be the Dirac measure $\delta_a : f \mapsto f(a)$. Then $\mathscr{A}(\delta_a) = (1+\mathrm{X})^a$. Let F in $\mathscr{R}^+$ and let $\mu_{\mathrm{F}} = \mathscr{A}(\mathrm{F})$. By definition $\mu(f(\cdot + a)) = \mu * \delta_a(f)$ for every f in $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$, hence

$$\mathscr{A}((1+\mathrm{X}^a) \cdot \mathrm{F})(f) = \delta_a * \mu_{\mathrm{F}}(f) = \mu_{\mathrm{F}}(f(\cdot + a)).$$

Similarly for the action of $\mathbb{Z}_p^\bullet = p^{\mathbb{N}}\mathbb{Z}_p^*$ (for example cf. [**Col10a**, Section II.4]). □

8.4. Action of M on $\mathbb{D}$ under the Amice transform. The action of M_0 on $\mathscr{R}^+$ in Proposition 8.1 allows us to describe that of M on $\mathbb{D}$ as submodule of

$$\mathbb{D} \otimes_{\mathscr{E}^+} \mathscr{R}^+ = \varprojlim_{\psi} D_{\mathrm{cris}} \otimes \mathscr{R}^+ = \varprojlim_{\psi} \mathscr{R}^+ \oplus \mathscr{R}^+.$$

Recall that φ on $\mathbb{D}$ is invertible by ψ and that, by assumption, there is a basis of $D_{\mathrm{cris}}(V)$ such that φ is a diagonal matrix of eigenvalues α^{-1} and β^{-1} for distinct α and β in $\mathbf{K}$.

Proposition 8.2. *Under the Amice transform*

$$\mathscr{A}\colon \mathbb{D} \otimes \mathscr{R}^+ \xrightarrow{\sim} \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$$

the action of M *on* $\mathbb{D}$ *decomposes into*

- *that of a in $\mathbb{Z}_p$ on $\mathbb{D}$ by scalar multiplication with $(1+X)^a$, which is under the Amice transform inside $\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ given by the dual of*
$$f \mapsto f(a + \cdot),$$
- *that of $\mathbb{Z}_p^*$ on $\mathbb{D}$ by Γ, which is under the Amice transform inside $\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ given by the dual of*
$$[\mu + \nu] f \mapsto [\mu + a^{-(k-2)} \cdot \nu] f(a \cdot), \quad \textit{and}$$
- *that of p on $\mathbb{D}$ by ϕ, which is under the Amice transform inside $\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ given by*
$$[\mu + \nu] f \mapsto [\alpha^{-1}\mu + \beta^{-1}\nu] f(p \cdot).$$

Proof: The group $\mathbb{Z}_p$ operates by scalar multiplication (as embedded into $\mathscr{O}_{\mathbf{K}}[[X]]$) on D(V) and is therefore under the Amice transform given by that on $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ in Proposition 8.1.

Let v and w be a basis of eigenvectors in $D_{\mathrm{cris}}(V)$ for φ (of eigenvalues α^{-1} and β^{-1}).

For the action of Γ under the Amice transform, we observe:

- that the group $\Gamma \xrightarrow{\sim} \mathbb{Z}_p^*$ operates on D(V) semilinearly, and there is a basis of D(V) such that every γ in Γ is given by
$$\begin{pmatrix} 1 & \\ & \gamma^{-(k-2)} \end{pmatrix},$$
- that the action of Γ on $\mathscr{O}(B)$ is by Proposition 8.1 under the Amice transform given by the dual of $f \mapsto f(\cdot \gamma)$ on $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$.

For the action of φ under the Amice transform, we observe:

- that there is a basis of D(V) such that φ is given by
$$\begin{pmatrix} \alpha^{-1} & \\ & \beta^{-1} \end{pmatrix},$$

- that the action of ϕ on $\mathcal{O}(B)$ is by Proposition 8.1 under the Amice transform given by the dual of $f \mapsto f(\cdot p)$ on $\mathcal{D}^{la}(\mathbb{Z}_p, \mathbf{K})$.

□

9. The open cell of the locally analytic parabolic induction

Let $G = GL_2(\mathbb{Q}_p)$. We denote

- by B and $\overline{B}$ the subgroup of all upper respectively lower triangular matrices in G, and
- by N and $\overline{N}$ be the subgroups of B and $\overline{B}$ of all unipotent upper respectively lower triangular matrices, and
- by T the common subgroup of all diagonal matrices.

(Observe that T acts on N by conjugation, that is, ${}^t n = tnt^{-1}$ and $n^t = t^{-1}nt$.) Let $\chi\colon T \to \mathbf{K}^*$ be a locally analytic character. It extends uniquely to $\overline{B}$ (because $\overline{B} = TN$, the image is commutative and N is the commutator of $\overline{B}$). Let $i^{la}(\chi)$ be the *locally analytic parabolic induction* of χ, the $\mathbf{K}[G]$-module

$$i^{la}(\chi) := (\mathbf{K}[G] \otimes_{\mathbf{K}[\overline{B}]} \mathbf{K})^{la}.$$

where we denote

- by $\cdot^{la}$ all locally analytic vectors (that is, all vectors whose orbit map is locally analytic), and
- where the action of $\overline{B}$ on $\mathbf{K}$ is given by a locally algebraic character $\chi\colon \overline{B} \to \mathbf{K}^*$.

Explicitly

$$i^{la}(\chi) := \{f : G \to \mathbf{K} : f \text{ locally analytic and } f(\bar{b}\cdot) = \chi(\bar{b})f \text{ for all } \bar{b} \text{ in } \overline{B}\}$$

and on which G acts by $f^g = f(\cdot g)$. Let

$$i^{la}(\chi)(N) := \{f \in i^{la}(\chi) \text{ of support in } \bar{B}N\},$$

a $\mathbf{K}[B]$-module.

Let Z be the center of G. Because $B = MZ$, by fixing a character $\zeta\colon Z \to \mathbf{K}^*$ of G and letting z in Z act by scalar multiplication with $\zeta(z)$, the action of M on $\mathbb{D} \otimes \mathcal{R}^+$ extends to B. The B-action on M determines a locally algebraic character χ such that, as $\mathbf{K}[B]$-modules under the Amice transform,

$$\mathcal{A}\colon \mathbb{D} \otimes_{\mathcal{E}^+} \mathcal{R}^+ \xrightarrow{\sim} i^{la}(\chi_1)(N)^* \oplus i^{la}(\chi_w)(N)^*$$

where $\chi_1 = \chi$ and χ_w a twist of χ (a product of χ by a certain locally constant *modulus* character δ), and $\cdot^*$ is the continuous dual.

Then there is exactly one choice of(locally algebraic) character $\zeta\colon Z \to \mathbf{K}^*$ for which the action of B on

$$\mathcal{A}\colon \mathbb{D} \hookrightarrow i^{la}(\chi_1)(N)^* \oplus i^{la}(\chi_w)(N)^*$$

extends to an action of G on $\mathbb{D}$. In this case, putting $\chi_1 = \chi$,

$$\mathcal{A}\colon \mathbb{D} \xrightarrow{\sim} i^{\mathrm{lr}}(\chi)^*$$

where we denote

- by $i^{\mathrm{lr}}(\chi) = (\mathbf{K}[\mathrm{G}] \otimes_{\mathbf{K}[\overline{\mathrm{B}}]} \mathbf{K})^{\mathrm{lr}}$ all locally algebraic (rational) vectors of $\mathbf{K}[\mathrm{G}] \otimes_{\mathbf{K}[\overline{\mathrm{B}}]} \mathbf{K}$, and
- by $i^{\mathrm{lr}}(\chi)^*$ the continuous dual of $i^{\mathrm{lr}}(\chi)$ (where continuous means continuous for a certain *universal unitary norm*, to be defined).

9.1. The action of the Borel subgroup as locally analytic parabolic induction. We describe the action of B on $\mathbb{D}$, under the Amice transform, by that of B on the locally analytic induction $i^{\mathrm{la}}(\chi)(\mathrm{N})$ for a suitable locally analytic character χ.

Lemma ([Nag15, Lemma 3.1]). *Let*

$$\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathrm{N}) = \{ \textit{ all functions } f\colon \mathrm{N} \to \mathbf{K} \textit{ locally analytic of compact support } \}$$

and let B *act on* $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathrm{N})$ *by*

$$f^b = \chi(b) f(\cdot^t n) \quad \textit{for } b = tn \textit{ with } t \in \mathrm{T}, n \in \mathrm{N}.$$

Then the map

$$\begin{aligned} i^{\mathrm{la}}(\chi)(\mathrm{N}) &\xrightarrow{\sim} \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathrm{N}) \\ f &\mapsto f_{|\mathrm{N}} \end{aligned}$$

is an isomorphism of $\mathbf{K}[\mathrm{B}]$*-modules.*

Under the identification $\mathrm{N} = \mathbb{Q}_p$ we obtain, by definition:

Corollary 9.1. *Let*

$$\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p) = \{ \textit{ all functions } f\colon \mathbb{Q}_p \to \mathbf{K} : f \textit{ locally analytic of compact support } \}$$

and let $\mathrm{B} = \mathrm{TN}$ *act on* $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p)$ *by*

- $f^t = \chi(t) f((d/a)\cdot)$ *for all* $t = \begin{pmatrix} a & \\ & d \end{pmatrix}$ *in* T, *and*
- $f^n = f(\cdot + n)$ *for all* $n \in \mathrm{N}$.

Then

$$\begin{aligned} i^{\mathrm{la}}(\chi)(\mathrm{N}) &\xrightarrow{\sim} \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p) \\ f &\mapsto f_{|\mathbb{Q}_p}, \end{aligned}$$

9.2. The Amice transform of $\mathbb{D}$ as locally analytic parabolic induction. We

- let $\Psi\colon \mathrm{T} \to \mathbf{K}^*$ be the dominant algebraic character (with respect to $\mathbf{B}$) given by
$$\Psi\begin{pmatrix} x & \\ & y \end{pmatrix} = y^{-(k-2)};$$
- let $\theta\colon \mathrm{T} \to \mathbf{K}^*$ be the unramified algebraic character determined by
$$\theta\begin{pmatrix} p & \\ & 1 \end{pmatrix} = \alpha \quad \text{and} \quad \theta\begin{pmatrix} 1 & \\ & p \end{pmatrix} = p\beta.$$

Let $\delta_{\mathrm{B}}\colon \mathrm{B} \to \mathbb{Q}$ be the *modulus character* on B given by precomposition of the projection $\mathrm{B} \twoheadrightarrow \mathrm{T}$ with the character $\delta_{\mathrm{B}}\colon \mathrm{T} \to p^{\mathbb{Z}}$ defined by $\delta_{\mathrm{B}}(t) := |\det \mathrm{Ad}_{\mathfrak{n}}(t)|$; here $\mathfrak{n}$ is the Lie-group of N and Ad the adjoint operation of T on N through conjugation. Explicitly, for a compact open subgroup N_0 of N and t stabilizes N_0, then
$$\delta_{\mathrm{B}}(t) = 1/\#[\mathrm{N}_0 : \mathrm{N}_0^t].$$
Put $w := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Then w acts on T by conjugation (from the right) and consequently on characters of T. We have $\operatorname{im} \delta_{\mathrm{P}}/\delta_{\mathrm{P}}^w \subseteq p^{2\mathbb{Z}}$, and conclude that there is a well-defined unramified character $(\delta_{\mathrm{P}}/\delta_{\mathrm{P}}^w)^{1/2}\colon \mathrm{T} \to \mathbf{K}^*$. Put $\theta_w := \theta^w(\delta_{\bar{\mathrm{B}}}/\delta_{\bar{\mathrm{B}}}^w)^{1/2}\colon \mathrm{T} \to \mathbf{K}^*$; explicitly,
$$\theta_w\begin{pmatrix} p & \\ & 1 \end{pmatrix} = \beta \quad \text{and} \quad \theta_w\begin{pmatrix} 1 & \\ & p \end{pmatrix} = p\alpha.$$
Put
$$\chi_1 := \theta\Psi \quad \text{and} \quad \chi_w = \theta_w\Psi.$$
Let $i^{\mathrm{la}}(\chi_1)(\mathrm{N})^*$ and $i^{\mathrm{la}}(\chi_w)(\mathrm{N})^*$ be the continuous duals of $i^{\mathrm{la}}(\chi_1)(\mathrm{N})$ and $i^{\mathrm{la}}(\chi_w)(\mathrm{N})$. Let $\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ be the continuous dual of the $\mathbf{K}[\mathrm{B}]$-module $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ whose action by B is given by χ_1 and χ_w (as in Corollary 9.1).

Corollary 9.2. *There is an isomorphism of* $\mathbf{K}[\mathrm{M}]$*-modules*
$$\mathbb{D} \otimes_{\mathscr{E}^+} \mathscr{R}^+ \xrightarrow{\sim} \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \xrightarrow{\sim} i^{\mathrm{la}}(\chi_1)(\mathrm{N})^* \oplus i^{\mathrm{la}}(\chi_w)(\mathrm{N})^*.$$

Proof: By Corollary 9.1, the restriction map $f \mapsto f_{\restriction \mathbb{Q}_p}$
$$i^{\mathrm{la}}(\chi)(\mathrm{N}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$$
is an isomorphism of $\mathbf{K}[\mathrm{B}]$-modules. Dually, we obtain an isomorphism of $\mathbf{K}[\mathrm{B}]$-modules
$$\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \xrightarrow{\sim} i^{\mathrm{la}}(\chi_1)(\mathrm{N})^* \oplus i^{\mathrm{la}}(\chi_w)(\mathrm{N})^*.$$
After restricting the action of B to M, the Amice transform $\mathscr{A}$ gives by Proposition 8.2 the claimed isomorphism between $\mathbf{K}[\mathrm{M}]$-modules. □

We note that the isomorphism between $\mathbf{K}[\mathrm{M}]$-modules holds for all characters χ_1 and χ_w such that
$$\chi_1\begin{pmatrix} p & \\ & 1 \end{pmatrix} = \beta \quad \text{and} \quad \chi_w\begin{pmatrix} 1 & \\ & p \end{pmatrix} = p\alpha.$$

10. Fractional non-Archimedean differentiability

We show that the condition of order r on $\mathscr{R}^+$ corresponds under the isomorphisms

$$\varprojlim_{\psi} \mathscr{R}^+ \xrightarrow{\sim} \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \xrightarrow{\sim} i^{\mathrm{la}}(\chi)(\mathrm{N})^*$$

- on $\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ to a continuity condition on all locally polynomial functions $f : \mathbb{Q}_p \to \mathbf{K}$ (of degree $\leq r$) for a norm of r-times differentiable functions (defined next), and
- on $i^{\mathrm{la}}(\chi)(\mathrm{N})^*$ to a continuity condition on all locally polynomial functions $f : \mathbb{Q}_p \to \mathbf{K}$ (of degree $\leq r$) for the "smallest" unitary norm of the $\mathbf{K}[\mathrm{B}]$-module $i^{\mathrm{la}}(\chi)(\mathrm{N})$ (defined afterward).

Let in this section $\mathbf{K}$ be a non-Archimedean field, that is, there is an absolute value $|\cdot|$ on $\mathbf{K}$ that is

- *non-Archimedean*, that is, $|x + y| \leq \max\{|x|, |y|\}$ for all x, y in $\mathbf{K}$,
- *nontrivial*, that is, there is x in $\mathbf{K}$ such that $|x| \neq 0, 1$, and
- whose induced topology turns $\mathbf{K}$ into a complete field.

Given a real number $r \geq 0$, we shall define r-fold differentiability over $\mathbf{K}$. First we decompose $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ into its integer part $\nu \in \mathbb{N}$ and its fractional part $\rho \in [0, 1[$. Then we define ν-fold differentiability by iteratively taking partial difference quotients, and ρ-fold differentiability by a strengthened Hölder-continuity condition. Finally, an r-times differentiable function is a ν-times differentiable function such that each of its partial difference quotients is ρ-times differentiable.

10.1. $\mathscr{C}^\nu$-functions for a natural number ν.

Pathologies under the Archimedean approach. To see that the Archimedean derivative does not suffice to describe differentiability of a function over a non-Archimedeanly valued domain, in particular in higher degrees, we exhibit a function f that is

- infinitely often Archimedeanly differentiable, but its Taylor polynomial expansion of degree greater than 1 does not converge, and
- is injective, but its derivative is zero everywhere.

(At the other extreme, there is also a function whose derivative is everywhere invertible, but nowhere injective. See [**Sch84**, Example 26.6].) Let

$$f : \mathbb{Z}_p \to \mathbb{Z}_p \qquad (10.1)$$
$$\sum_{n \in \mathbb{N}} a_n p^n \mapsto \sum_{n \in \mathbb{N}} a_n p^{2n}.$$

Because $|f(x + h) - f(x)| = |h|^2$, it is a differentiable function whose derivative f' vanishes everywhere and is thus infinitely often Archimedeanly differentiable.

However, its Taylor polynomial expansion up to degree 2 does not converge, for example at $a = 0$. That is, let

$$T(h) = f(0) + f'(0)h + f''(0)h^2 = 0$$

be the Taylor polynomial of f at 0, up to degree 2, and

$$R(h) = f(h) - T(h) = f(h)$$

its rest term. Then $|R(h)|/|h|^2 = 1$ for every h in the domain. In particular, if $h \to 0$, then $|R(h)|/|h|^2 \not\to 0$.

Lack of the Mean-Value Theorem. These pathologies are excluded if we assume that a function $f : X \to \mathbf{K}$ over an open subset X of $\mathbf{K}$ is *non-Archimedeanly* (or *strictly*) continuously differentiable. That is, its *differential*

$$f^{]1[}(x, y) = \frac{f(x) - f(y)}{x - y} \tag{10.2}$$

defined for all distinct x and y in X, extends to a continuous function over all of $X \times X$.

For a real-valued function over the real numbers, the mean-value theorem shows that the non-Archimedean and Archimedean differentiability condition are equivalent. In a way, the non-Archimedean differentiability condition is more natural than the Archimedean one: If we use the non-Archimedean differentiability definition then general facts in Archimedean Calculus like

(i) the local invertibility around a point where the derivative is invertible
(ii) the existence of the Taylor polynomial, and
(iii) the completeness of the normed space of differentiable functions

follow from the definition, whereas if we use the Archimedean differentiability definition then they are proved by a detour either via the mean-value theorem (like in (i)) or via the fundamental theorem of calculus (like in (ii) and (iii)).

Coordinate-free approach. Let us recall *non-Archimedean continuous* (or *strict*) differentiability. Let V and $\mathbf{E}$ be two $\mathbf{K}$-Banach spaces, X an open subset of V. The function $f : X \to \mathbf{E}$ is *continuously differentiable* or $\mathscr{C}^1$ at a point a in X if there is a continuous $\mathbf{K}$-linear map $A: V \to \mathbf{E}$ such that for every $\epsilon > 0$, there is a neighborhood U of a such that for, all $x + h, x \in U$,

$$\|f(x + h) - f(x) - A(h)\| \leq \epsilon \|h\|.$$

This condition is stricter than the Archimedean differentiability condition, because here the offset h and the expansion point x varies, there h varies but x is fixed.

Coordinate-wise approach. We want to define two-fold (and eventually ν-fold) differentiability by applying strict differentiability to the differential (and eventually iterate).

Pathology (10.1) showed that the Archimedean derivative does not yield a coherent theory of non-Archimedean Calculus. Instead, similar to Definition

(10.2) in one variable, we define a differential $f^{]1[}$ that computes all the difference quotients around x.

Let V be a **K**-vector space of dimension d and f a function over an open subset of V. We will define a differential of f that takes a point x together with a set of points in X around x whose differences span V and returns a linear map that approximates f. To formulate it conveniently, we introduce coordinates on V by choosing an ordered basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$ of V.

Definition. Let V be a finite-dimensional **K**-vector space and X an open subset of V and $f\colon \mathrm{X} \to \mathbf{E}$. The differential $f^{]1[}(x+h, x)$ of f at $x+h, x$ in X with $h \in \mathbf{K}^{*d}$ is the **K**-linear map A determined by

$$\mathrm{A}\boldsymbol{e}_k := \frac{f(x + h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1} + h_k\boldsymbol{e}_k) - f(x + h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1})}{h_k}$$

for $k = 1, \ldots, d$. The function f is a $\mathscr{C}^1$*-function* if $f^{]1[}$ extends to a continuous function $f^{[1]}\colon \mathrm{X} \times \mathrm{X} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$.

Example. Let $f\colon \mathrm{X}' \times \mathrm{X}'' \to \mathbf{K}$ with $\mathrm{X}', \mathrm{X}'' \subset \mathbf{K}$ open. Then

$$f^{]1[} = (f^{]1,0[}, f^{]0,1[})$$

with

$$f^{]1,0[}(x+h, x) = \frac{f(x'+h', x'') - f(x)}{h'}, \quad f^{]0,1[}(x+h, x) = \frac{f(x', x''+h'') - f(x)}{h''}$$

for $x+h, x \in \mathrm{X}' \times \mathrm{X}''$ with $h = (h', h'') \in \mathbf{K}^* \times \mathbf{K}^*$.

Iterated differentiability. Let $f \in \mathscr{C}^1(\mathrm{X}, \mathbf{E})$. Let us compare the domain and codomain of $f^{[1]}$ to those of f. The domain $\mathrm{X}^{[1]} := \mathrm{X} \times \mathrm{X}$ of $f^{[1]}$ is included in the finite-dimensional **K**-vector space $\mathrm{V}^{[1]} = \mathrm{V} \times \mathrm{V}$ with a canonical ordered basis, like the domain X of f, and the codomain $\mathbf{E}^{[1]} := \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ of $f^{[1]}$ is a **K**-Banach space, like the codomain **E** of f. We may therefore iterate the non-Archimedean differentiability definition by applying it to $f^{[1]}$, that is, f is in $\mathscr{C}^2(\mathrm{X}, \mathbf{E})$ if, first $f^{[1]}$ exists, and second its differential

$$f^{]2[} = (f^{[1]})^{]1[}\colon (\mathrm{X}^{[1]})^{]1[} \to (\mathbf{E}^{[1]})^{[1]}$$

extends to a continuous function $f^{[2]}$ over $\mathrm{X}^{[2]} := (\mathrm{X}^{[1]})^{[1]}$ (with values in $\mathbf{E}^{[2]} := (\mathbf{E}^{[1]})^{[1]}$).

Definition. Let $f\colon \mathrm{X} \to \mathbf{E}$ be a function over an open subset X of the finite-dimensional **K**-vector space V with values in the K-Banach space **E**. Let ν be a natural number. The function f is a $\mathscr{C}^{\nu+1}$*-function*

- if f is a $\mathscr{C}^{\nu}$-function, and
- if $\mathfrak{X} = \mathrm{X}^{[\nu]}, \mathfrak{E} = \mathbf{E}^{[\nu]}$ and $\mathfrak{f} = f^{[\nu]}$, then $\mathfrak{f}^{]1[}$ extends to a continuous function $\mathfrak{f}^{[1]}\colon \mathfrak{X} \times \mathfrak{X} \to \mathrm{Hom}_{\mathbf{K}}(\mathfrak{E}, \mathbf{E})$.

10.2. $\mathscr{C}^\rho$**-functions for** $\rho \in [0,1[$. Let $\rho \in [0,1[$. Roughly, ρ-fold differentiability is stricter Hölder-continuity.

Definition 10.1. Let X and $\mathbf{Y}$ be metric spaces with metrics d and $\mathbf{d}$. Let A be a subset of X and $f\colon A \to \mathbf{Y}$. Let a be a point in X. The function f is $\mathscr{C}^\rho$ at a if for every $\varepsilon > 0$ there is a neighborhood $U \ni a$ in X such that

$$\mathbf{d}(f(x), f(y)) \leq \varepsilon \cdot d(x,y)^\rho \quad \text{for all } x, y \in U \cap A.$$

The function f is a *$\mathscr{C}^\rho$-function* if f is $\mathscr{C}^\rho$ at all points $a \in A$. Let $\mathscr{C}^\rho(A, \mathbf{Y})$ denote the set of all $\mathscr{C}^\rho$-functions $f\colon A \to \mathbf{Y}$.

For later use, we record that if a is in the boundary of A inside X and $\mathbf{Y}$ is complete, then f extends uniquely to a:

Proposition 10.2 ([**Nag11**, Proposition 1.6]). *Let* X *be a metric space and let* A *be a subset of* X*, let* $\mathbf{Y}$ *a complete metric space and* $f\colon A \to \mathbf{Y}$*. If* B *denotes the set of* $\mathscr{C}^\rho$*-points of* f *included in the closure of* A *inside* X*, then* f *extends to a unique* $\mathscr{C}^\rho$*-function over* B.

10.3. $\mathscr{C}^r$**-functions for a real number** $r \geq 0$. Henceforth we fix a real number $r \geq 0$ and its decomposition

$$r = \nu + \rho$$

into • an integral part $\nu := \lfloor r \rfloor \in \mathbb{N}$, and • a fractional part $\rho := \{r\} \in [0,1[$.

Definition. Let $f\colon X \to \mathbf{E}$ be a function over an open subset X of a finite-dimensional $\mathbf{K}$-vector space with values in $\mathbf{E}$. The function f is a *$\mathscr{C}^r$-function* if f is a $\mathscr{C}^\nu$-function, and $f^{[\nu]}$ is a $\mathscr{C}^\rho$ function.

The symmetry of the differential allows us to reduce, for increasing degree of differentiability ν, the exponential growth in the number of variables of the total differential $f^{[\nu]}$ to a linear growth in the number of variables of certain partial differentials $f^{[\boldsymbol{n}]}$ of total degree ν. For a symmetric function, partial differentiability in one, say its first, coordinate is equivalent to total differentiability, reducing an exponential growth of parameters to a linear one: Let $X' \times X''$ be an open subset of $V := \mathbf{K} \times \mathbf{K}$ and let $F\colon X' \times X'' \to \mathbf{E}$ be a symmetric function. Then $F \in \mathscr{C}^1$ if for all $x' + h, x' \in X', x'' \in X''$ with $h \in \mathbf{K}^*$, the linear map $F^{]1,0[}(x'+h, x'; x'')$ in $\mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$ defined by

$$h \mapsto f(x'+h, x'') - f(x', x'')$$

extends to a continuous function $F^{[1,0]}\colon (X' \times X') \times X'' \to \mathbf{E}$. The following definition of two-fold differentiability is that given in [**Sch84**, Section 28].

Example. Let $f \in \mathscr{C}^1(X, \mathbf{E})$. Then $f^{[1]} \in \mathscr{C}^1(X \times X, \mathbf{E})$ if and only if $f^{]2[}$ defined by

$$f^{]2[}(x,y,z) = \frac{f^{[1]}(x,z) - f^{[1]}(y,z)}{x - y}$$

for all distinct $x, z \in \mathrm{X}$ and $y \in \mathrm{X}$ extends to a continuous function $f^{[2]} \colon (\mathrm{X} \times \mathrm{X}) \times \mathrm{X} \to \mathbf{E}$.

Following [**Sch84**, Section 29 ff.], let us define iterated divided differences of functions of one variable:

Definition. Let X be a subset of **K**. For $\nu \in \mathbb{N}$ put

$$\mathrm{X}^{[\nu]} = \mathrm{X}^{\{0,\dots,\nu\}} \quad \text{and} \quad \mathrm{X}^{]\nu[} := \{(x_0, \dots, x_\nu) : x_i = x_j \text{ only if } i = j\}.$$

Let $f \colon \mathrm{X} \to \mathbf{K}$. The *ν-th difference quotient* $f^{]\nu[} \colon \mathrm{X}^{]\nu[} \to \mathbf{K}$ of f is inductively given by $f^{]0[} := f$ and for n in $\mathbb{N}$ and $(x_0, \dots, x_n) \in \mathrm{X}^{]n[}$ by

$$f^{]n[}(x_0, \dots, x_n) := \frac{f^{]n-1[}(x_0, x_2, \dots, x_n) - f^{]n-1[}(x_1, x_2, \dots, x_n)}{x_0 - x_1}.$$

Definition 10.3. Let X be a subset of **K** and let **E** be a **K**-Banach space.

- Let a in X. The function $f \colon \mathrm{X} \to \mathbf{E}$ is $\mathscr{C}^r$ at a if $f^{]\nu[} \colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a, \dots, a)$ in $\mathrm{X}^{[\nu]}$.
- The function f is a *$\mathscr{C}^r$-function* if f is $\mathscr{C}^r$ at all a in X.
- Let $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ denote the set of all $\mathscr{C}^r$-functions $f \colon \mathrm{X} \to \mathbf{E}$.

Let X be an open subset of **K** and $f \colon \mathrm{X} \to \mathbf{E}$. Let a in X. We let

$$\mathrm{D}^n f(a) := \text{ the } n\text{-th derivative of } f \text{ at } a$$

that is, the unique value to which $f^{]n[}$ extends and is $\mathscr{C}^\rho$ at $\vec{a}$ by Proposition 10.2. If $f^{(n)}$ is the n-th ordinary, Archimedean, derivative of f then $n!\,\mathrm{D}^n f = f^{(n)}$.

The locally convex topology of $\mathscr{C}^r$-functions. Given a real valued function f over a compact topological space, let $\|f\|_{\sup}$ denote the supremum norm of f.

Definition. Let X and **Y** be metric spaces with metric d and **d** and $f \colon \mathrm{X} \to \mathbf{Y}$. The ρ-th differential $|f^{]\rho[}|$ of f is defined by

$$|f^{]\rho[}|(x, y) = \frac{\mathbf{d}(f(x), f(y))}{\mathrm{d}(x, y)^\rho} \quad \text{for all distinct } x, y \text{ in X}.$$

The function $f \colon \mathrm{X} \to \mathbf{Y}$ is a $\mathscr{C}^\rho$-function if and only if $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|$ on all of $\mathrm{X} \times \mathrm{X}$ that vanishes on the diagonal of $\mathrm{X} \times \mathrm{X}$ (and which is *unique* provided X is free of isolated points).

Definition. Let X be compact and free of isolated points. The norm $\|\cdot\|_{\mathscr{C}^\rho}$ over $\mathscr{C}^\rho(\mathrm{X}, \mathbf{E})$ is defined by

$$\|f\|_{\mathscr{C}^\rho} = \max\{\|f\|_{\sup}, \|\,|f^{[\rho]}|\,\|_{\sup}\}.$$

If a function f is r-times differentiable, then its divided difference $f^{]\nu[}$ extends to a $\mathscr{C}^\rho$-functions. As in in [**Sch84**, Section 29] for r in $\mathbb{N}$, we will use the $\mathscr{C}^\rho$-norm of the partial derivatives of f to define the $\mathscr{C}^r$-norm of f.

Proposition ([**Nag11**, Proposition 2.8]). *Let* X *be an open subset of* **K** *and* $f\colon X\to\mathbf{E}$. *The function* f *is* r*-times differentiable if and only if the function* $f^{]\nu[}\colon X^{]\nu[}\to\mathbf{E}$ *extends to a* $\mathscr{C}^{\rho}$*-function* $f^{[\nu]}\colon X^{[\nu]}\to\mathbf{E}$ *(and which is* unique *when* X *is free of isolated points).*

Moreover, if $s\leq r$ and f is a $\mathscr{C}^r$-function, then f is a $\mathscr{C}^s$-function ([**Nag11**, Lemma 2.3]). Therefore we may define:

Definition. Let X be a compact open subset of **K**. The *norm* $\|\cdot\|_{\mathscr{C}^r}$ over $\mathscr{C}^r(X,\mathbf{E})$ is defined by

$$\|f\|_{\mathscr{C}^r}=\max\{\|f^{[n]}\|_{\sup}:n=0,\dots,\nu\}\cup\{\|f^{[\nu]}\|_{\mathscr{C}^{\rho}}\}$$

- The normed **K**-vector space $\mathscr{C}^r(X,\mathbf{E})$ is complete and a normed **K**-algebra if **E** is a normed **K**-algebra.
- Let $f\colon X\to Y$ be either a $\mathscr{C}^r$-function, if $r\geq 1$, or a locally Lipschitz function otherwise. The precomposition operator

$$\mathscr{C}^r(Y,\mathbf{E})\to\mathscr{C}^r(X,\mathbf{E})$$
$$f\mapsto f\circ g$$

  is well defined and continuous (by [**Nag11**, Proposition 3.23 and 3.24]).

10.4. Locally polynomial functions. Let X be an open subset of **K**.

Definition. A function $f\colon X\to\mathbf{E}$ is *locally analytic* if for every point a in X there is

- a ball B included in X around a, and
- a power series $F(X)=a_0+a_1X+a_2X^2+\cdots$

such that $f(a+x)=F(x)$ for all x in B.

The following natural observation rests on on the completeness of **E** (or, consequently, that of $\mathscr{C}^r(X,\mathbf{E})$):

Proposition 10.4 ([**Nag11**, Proposition 3.18]). *A locally analytic function* $f\colon X\to\mathbf{E}$ *is arbitrarily often differentiable.*

Proof: By completeness, a series in $\mathscr{C}^r(X,\mathbf{E})$ converges if and only if its entries converge to zero. Therefore, if the function f is analytic over the open ball B, say of radius ϵ, given by the power series $\sum a_i x^i$, then f is r-times differentiable if and only if $a_i x^i$ converges to zero in $\mathscr{C}^r(B,\mathbf{E})$. That is, if and only if $\|x^i\|_{\mathscr{C}^r}\leq\epsilon^i$. □

Definition. A function $f\colon X\to\mathbf{E}$ is *locally polynomial* of degree ν if for every point a in X there is

- a ball B included in X around a, and
- a polynomial $F(X)=a_0+a_1X+\cdots+a_nX^n$

such that $f(a+x) = F(x)$ for all x in B.

Every locally polynomial function is locally analytic and, by Proposition 10.4, every locally analytic function is arbitrarily often differentiable. Similar to [**Sch78**, Theorem 8.22] for r in $\mathbb{N}$, every continuous linear form is already determined by its values on all locally polynomial functions of degree at most ν, that is:

Proposition 10.5 ([**Nag11**, Proposition 3.30]). *The locally polynomial functions of maximal degree* ν *are dense inside* $\mathscr{C}^r(X, \mathbf{K})$.

Because every locally constant function is a limit of polynomial functions in $\mathscr{C}^r(X, \mathbf{K})$, the following variant obtains:

Corollary 10.6 ([**Nag11**, Corollary 3.32]). *The polynomial functions are dense inside* $\mathscr{C}^r(X, \mathbf{K})$.

10.5. The Amice transform. Let $\mathbf{K}$ be a non-Archimedean field over $\mathbb{Q}_p$ and $\mathcal{O}_\mathbf{K}$ its valuation ring over $\mathbb{Z}_p$. The continuous linear forms

- on $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ correspond to all power series that are bounded, and
- on $\mathscr{C}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$ correspond to all power series that converge on B.

Let $r \geq 0$. We will establish the counterpart of these correspondences for all r-times differentiable functions. Put

$$\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) := \{ \text{ all } r\text{-times differentiable functions } f\colon \mathbb{Z}_p \to \mathbf{K}\},$$

and

$$\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K}) := \{ \text{ all continuous linear maps } \nu\colon \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K}\}.$$

Let

$$d^r(\mathbb{N}, \mathbf{K}) := \{ \text{ all } \sum a_n X^n \text{ in } \mathbf{K}[[X]] \text{ such that } \{|a_n|/n^r\} \text{ bounded } \}.$$

THEOREM 10.7 ([**NAG11**, COROLLARY 3.49]). *The algebra morphism*

$$\delta_1 - \delta_0 \mapsto X$$

continuously extends to an isomorphism of topological $\mathbf{K}$*-vector spaces*

$$\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K}) \xrightarrow{\sim} d^r(\mathbb{N}, \mathbf{K})$$

10.6. Order as degree of differentiability. We will translate the conditions on $\mathbb{D}$ found in Corollary 7.10 under the Amice transform $\mathscr{A}$ to conditions on

$$\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}).$$

Let $r := v(\alpha)$. We will show that F in $\mathscr{R}^+$ is of order r if and only if the continuous linear map $\mathscr{A}(F)\colon \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \to \mathbf{K}$ extends to a continuous linear map on all of $\mathscr{C}^r_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ (for the topology given by the $\mathscr{C}^r$-norm).

Let

$$\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K}) := \{ \text{ all maps } \mu\colon \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) \to \mathbf{K} \text{ that are linear continuous } \}$$

and endow it with the operator norm. By Theorem 10.7

$$\mathscr{A}\colon\ d^r(\mathbb{N},\mathbf{K}) \xrightarrow{\sim} \mathscr{D}^r(\mathbb{Z}_p,\mathbf{K})$$

The vector space $d^r(\mathbb{N},\mathbf{K})$ is included in $\mathscr{R}^+$ and stable under ψ. Because under the Amice transform φ is given by the dual of $f \mapsto f(\cdot p)$, this isomorphism extends to an isomorphism

$$(\varprojlim_{\alpha\cdot\psi} d^r(\mathbb{N},\mathscr{O}_{\mathbf{K}})) \otimes_{o_{\mathbf{K}}} \mathbf{K} \xrightarrow{\sim} (\varprojlim \mathscr{D}^r(\mathbb{Z}_p,\mathscr{O}_{\mathbf{K}})) \otimes_{o_{\mathbf{K}}} \mathbf{K}.$$

where every transition map of the projective limit running over $\mathbb{N}$ is induced by the endomorphism $p^{-1}\cdot$ on $\mathbb{Z}_p$, that is, it precomposes μ with the endomorphism of $\mathscr{C}^r(\mathbb{Z}_p,\mathscr{O}_{\mathbf{K}})$ given by $f \mapsto f(p^{-1}\cdot)$. Put

$$\mathscr{C}^{\mathrm{lp}\le r}(\mathbb{Z}_p,\mathbf{K}) := \{ \text{ all } f\colon \mathbb{Z}_p \to \mathbf{K} \text{ locally polynomial of degree} \le r\}$$

and

$$\mathscr{C}^{\mathrm{lr}\le r}_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K}) := \{ \text{ all } f\colon \mathbb{Q}_p \to \mathbf{K} \text{ loc. pol. of degree} \le r \text{ and of compact support}\}.$$

Proposition 10.8. *Let* $\mathrm{F}+\mathrm{G}$ *be in* $\mathscr{R}^+\oplus\mathscr{R}^+$. *Then the pair of power series* $\mathrm{F}+\mathrm{G}$ *satisfies*

- *that* F *is of order* r, *and*
- *that* G *is of order* s,

if and only if, putting $\mu=\mathscr{A}(\mathrm{F})$ *and* $r := v(\alpha)$ *with* $r = n+\rho$, *and* $\nu = \mathscr{A}(\mathrm{G})$ *and* $s := v(\beta)$ *with* $s = m+\sigma$, *the pair of distributions* $\mu+\nu$ *satisfies*

- *that* μ *is continuous on* $\mathscr{C}^{\mathrm{lr}\le n}_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K})$ *for* $\|\cdot^{[n]}\|_{\mathscr{C}^\rho}$, *and*
- *that* ν *is continuous on* $\mathscr{C}^{\mathrm{lr}\le m}_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K})$ *for* $\|\cdot^{[m]}\|_{\mathscr{C}^\sigma}$.

Proof: By Proposition 10.5 $\mathscr{C}^{\mathrm{lp}\le r}(\mathbb{Z}_p,\mathbf{K})$ is dense inside $\mathscr{C}^r_{\mathrm{cp}}(\mathbb{Z}_p,\mathbf{K})$. Therefore

$$\mathscr{D}^r(\mathbb{Z}_p,\mathbf{K}) = \{ \text{ all } \mu\colon\ \mathscr{C}^{\mathrm{lp}\le r}(\mathbb{Z}_p,\mathbf{K}) \to \mathbf{K} \text{ linear continuous for } \|\cdot\|_{\mathscr{C}^r}\}.$$

Consequently

$$\mathscr{D}^r_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K})) = \{ \text{ all } \mu\colon\ \mathscr{C}^{\mathrm{lr}\le r}_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K}) \to \mathbf{K} \text{ linear continuous for } \|\cdot\|_{\mathscr{C}^r}\}.$$

We obtain

$$(\varprojlim \mathscr{D}^r(\mathbb{Z}_p,\mathscr{O}_{\mathbf{K}})) \otimes_{o_{\mathbf{K}}} \mathbf{K} \xrightarrow{\sim} \mathscr{D}^r_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K}).$$

Finally, because every f in $\mathscr{C}^{\mathrm{lr}}_{\mathrm{cp}}(\mathbb{Q}_p,\mathbf{K})$ vanishes outside of its support, we have

$$\|f\|_{\mathscr{C}^r} = \|f^{[\nu]}\|_{\mathscr{C}^\rho}.$$

11. The universal unitary norm on the locally algebraic parabolic induction

We will express the conditions on $\mathbb{D}$ found in Corollary 7.10, under the Amice transform $\mathscr{A}$, as conditions on the $\mathbf{K}[\mathrm{B}]$-representation

$$i^{\mathrm{la}}(\chi_1)(\mathrm{N})^* \oplus i^{\mathrm{la}}(\chi_w)(\mathrm{N})^*.$$

11.1. The greatest unitary norm. Let G be a group, let V be a **K**-vector space and let G act on V. Let $\mathcal{O}_{\mathbf{K}}$ be the ring of integers of **K**.

Definition.

- A seminorm $\|\cdot\|$ on V is *unitary* if $\|g\cdot\| = \|\cdot\|$ for every g in G. An equivalence class of seminorms is *unitary* if it contains a unitary norm.
- Let N′ and N″ be equivalence classes of seminorms. Then N′ is *greater* than N″ if for every seminorm $\|\cdot\|'$ in N′ and every seminorm $\|\cdot\|''$ in N″ there is a constant $C > 0$ such that $C\|\cdot\|' \geq \|\cdot\|''$.
- If there is a greatest among all equivalence classes of unitary seminorms, then it is unique and called the *universal* unitary seminorm. For example, we see below that it exists when V is finitely generated as module over **K**[G].

Definition. A *lattice* $\mathfrak{L}$ of a **K**-vector space V is an $\mathcal{O}_{\mathbf{K}}$-submodule such that for every $v \in V$ there is λ in $\mathbf{K}^*$ such that λv in $\mathfrak{L}$.

The notions of a commensurability class of lattices and an equivalence class of seminorms on a non-Archimedean vector space are equivalent; that is, the following assignments induce mappings between all equivalence classes of seminorms and all commensurability classes of lattices that are inverse to each other:

- Every lattice $\mathfrak{L}$ of a vector space V gives rise to a seminorm $\|\cdot\|_{\mathfrak{L}}$ on V given by

$$\|v\|_{\mathfrak{L}} := \sup |\{ \text{ all } \lambda \in \mathbf{K}^* \text{ such that } \lambda v \in \mathbf{K}\}|, \quad \text{and}$$

- every seminorm gives rise to a lattice given by its closed unit ball.

By definition, a lattice $\mathfrak{L}$ need not be free and may even coincide with its surrounding vector space V, just as its corresponding seminorm $\|\cdot\|$ need not be 0 solely on the 0 vector and may even be 0 everywhere. In fact, (if V is countably infinite-dimensional then) $\mathfrak{L}$ is free if and only if $\|\cdot\|$ is a norm ([**Sch02**, Proposition 10.4]). Moreover, let $\|\cdot\|'$ and $\|\cdot\|''$ be two norms on a common **K**-vector space with corresponding lattices $\mathfrak{L}'$ and $\mathfrak{L}''$. Then $\|\cdot\|'$ and $\|\cdot\|''$ are equivalent if and only if $\mathfrak{L}'$ and $\mathfrak{L}''$ are *commensurable*, that is, there are λ' and λ'' in **K** such that

$$\mathfrak{L}' \subseteq \lambda'' \cdot \mathfrak{L}'' \quad \text{and} \quad \mathfrak{L}'' \subseteq \lambda' \cdot \mathfrak{L}'.$$

Also, the above notions for seminorms (unitarity, one equivalence class of seminorms being greater than another, and universality) correspond to the following notions for lattices:

- An equivalence class of norms is unitary if and only if its corresponding commensurability class contains a lattice stable under G.
- Let N′ and N″ be equivalence classes of norms with corresponding commensurability classes of lattices $\mathfrak{L}'$ and $\mathfrak{L}''$. Then N′ is greater

than N'' if and only if $\mathfrak{L}'$ is *smaller* than $\mathfrak{L}''$, that is, for every lattice $\mathfrak{L}'$ in $\mathfrak{L}'$ and every lattice $\mathfrak{L}''$ in $\mathfrak{L}''$, there is λ in $\mathcal{O}_{\mathbf{K}}$ such that $\lambda \mathfrak{L}' \subseteq \mathfrak{L}''$.

- An equivalence class of norms is the universal unitary norm if and only if its corresponding lattice is the smallest commensurability class of all lattices that are stable under G.

In particular, since the module generated by a set X is the smallest module that contains X:

Proposition 11.1. *If* V *is a finitely generated* $\mathbf{K}[G]$*-module, that is, there are* $v_1, \dots, v_n$ *in* V *such that* $V = \mathbf{K}[G]v_1 + \cdots + \mathbf{K}[G]v_n$*, then its universal unitary lattice* $\mathfrak{L}$ *is given by the smallest* $\mathcal{O}_{\mathbf{K}}[G]$*-module of* V *that contains* $v_1, \dots, v_n$*, that is,*

$$\mathfrak{L} = \mathcal{O}_{\mathbf{K}}[G]v_1 + \cdots + \mathcal{O}_{\mathbf{K}}[G]v_n.$$

11.2. The universal unitary norm on the open cell. We link the $\mathcal{C}^r$-norm on $\mathcal{C}^{\mathrm{lr}\le r}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ and the unitary universal norm on $i^{\mathrm{lr}}(\chi)(N)$. We recall that we

- let $\Psi\colon T \to \mathbf{K}^*$ be the dominant algebraic character (with respect to $\mathbf{B}$) given by
$$\Psi\begin{pmatrix} x & \\ & y \end{pmatrix} = y^{-(k-2)}, \text{ and}$$
- let $\theta\colon T \to \mathbf{K}^*$ be the unramified algebraic character determined by
$$\theta\begin{pmatrix} p & \\ & 1 \end{pmatrix} = \alpha \quad \text{and} \quad \theta\begin{pmatrix} 1 & \\ & p \end{pmatrix} = p\beta.$$

and

$$\chi := \theta\Psi.$$

Then we had:

Proposition 11.2. *The isomorphism of groups* $N = \mathbb{Q}_p$ *induces an isomorphism of* $\mathbf{K}[B]$*-modules*

$$i^{\mathrm{lr}}(\chi)(N) = \mathcal{C}^{\mathrm{lp}\le(k-2)}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$$

where $B = TN$ *acts on* $\mathcal{C}^{\mathrm{lp}\le k-2}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ *by*

- $f^t = \chi(t) f((d/a)\cdot)$ *for all* $t = \begin{pmatrix} a & \\ & d \end{pmatrix}$ *in* T*, and*
- $f^n = f(\cdot + n)$ *for all* $n \in N$.

Proof: We have

$$i^{\mathrm{lr}}(\chi) = i^{\mathrm{lc}}(\theta) \otimes_{\mathbf{K}} i^{\mathrm{alg}}(\Psi)$$

where $\cdot^{\mathrm{lc}}$ are all locally constant and $\cdot^{\mathrm{alg}}$ are all algebraic vectors of the abstract parabolic induction $\mathrm{ind}_B^G \theta\Psi$. The unique irreducible algebraic representation $i^{\mathrm{alg}}(\Psi)$ of highest weight Ψ has as a basis all functions $f\colon \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbf{K}$ that are given by products of $k-2$ factors in the two coordinate functions of the upper row of the 2×2-matrices and their determinant function. The products in this basis restrict on N to the monomial functions $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & X^{k-2} \\ & 1 \end{pmatrix}$. □

Proposition 11.3. *We have*

$$\mathscr{C}_{\mathrm{cp}}^{\mathrm{lp}\le k-2}(\mathbb{Q}_p, \mathbf{K}) = \mathbf{K}[\mathrm{B}] \cdot (\mathbf{1}_{\mathbb{Z}_p} \cdot x^{k-2}).$$

In particular, the universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$*-module* $i(\chi)(\mathrm{N})$ *is generated by* $1_{\mathbb{Z}_p} \cdot x^{k-2}$.

Proof: Let

$$t_0 = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$$

By Proposition 11.2

$$t_0^n \cdot \mathbf{1}_{\mathbb{Z}_p} = \chi(t_0^n) \cdot \mathbf{1}_{p^n\mathbb{Z}_p}$$

Therefore $\mathrm{T} \cdot \mathbf{1}_{\mathbb{Z}_p}$ contains the indicator functions of a neighborhood basis of 0 in $\mathbb{Z}_p$. Because N translates the support, therefore $\mathrm{B} \cdot \mathbf{1}_{\mathbb{Z}_p}$ contains the indicator functions of a neighborhood basis of $\mathbb{Z}_p$. That is, $\mathrm{B} \cdot \mathbf{1}_{\mathbb{Z}_p}$ generates the $\mathbf{K}$-vector space of all locally constant functions of compact support.

For every neighborhood U around 1 inside N (such as $\mathrm{U} = \mathrm{N}_0^{t_0^n}$) the translates $\mathrm{U} \cdot x^{k-2}$ generate the $\mathbf{K}$-vector space of all polynomials of degree at most $k-2$. Therefore $\mathrm{B} \cdot x^{k-2}\mathbf{1}_{\mathbb{Z}_p}$ generates the $\mathbf{K}$-vector space of all locally polynomial functions of degree at most $k-2$ of compact support. (See [**Nag15**, Corollary 3.5] for a general proof.) □

Put

$$f_0 := \mathbf{1}_{\mathbb{Z}_p} \cdot x^{k-2}.$$

A lattice $\mathfrak{L}$ is in the smallest commensurability class of all lattices that contain all $b \cdot f_0$ for b in B if and only if its corresponding seminorm $\|\cdot\|$ is in the greatest equivalence class of all seminorms that satisfy $\|b \cdot f_0\| \le \|f_0\|$ for all b in B. We conclude:

Corollary 11.4. *There is a unitary norm on the* B*-representation* $i^{\mathrm{lr}}(\chi)(\mathrm{N})$ *if and only if there is a norm* $\|\cdot\|$ *on* $i^{\mathrm{lr}}(\chi)(\mathrm{N})$ *and a constant* $\mathrm{C} > 0$ *such that*

$$\|b \cdot f_0\| \le \mathrm{C}$$

for all b *in* B.

Let $r := v(\chi(\begin{pmatrix} p & \\ & 1 \end{pmatrix})) = v(\chi_1(p))$. By Proposition 11.3, the universal unitary lattice inside $i(\chi)(\mathrm{N})$ is generated by $1_{\mathbb{Z}_p}x^{k-2}$. Thus, by Corollary 11.4, the universal unitary norm is (up to equivalence) the greatest norm $\|\cdot\|$ on $\mathscr{C}_{\mathrm{cp}}^{\mathrm{lp}\le k-2}(\mathbb{Q}_p, \mathbf{K})$ that is

- invariant under translation, and
- for which there is a constant $\mathrm{C} > 0$ such that $\|1_{p^n\mathbb{Z}_p}x^{k-2}\| \le \mathrm{C} \cdot p^{(r-(k-2))n}$ for all $n \in \mathbb{Z}$.

Recall that, given ν in $\mathbb{N}$,

$$\mathbb{Q}_p^{[\nu]} = \mathbb{Q}_p^{\{0,\dots,\nu\}} \quad \text{and} \quad \mathbb{Q}_p^{]\nu[} := \{(x_0, \dots, x_\nu) \in \mathbb{Q}_p^{[\nu]} : x_i = x_j \text{ only if } i = j\};$$

that for $f\colon \mathbb{Q}_p \to \mathbf{K}$, its ν-th divided difference

$$f^{]\nu[}\colon \mathbb{Q}_p^{]\nu[} \to \mathbf{K}$$

is inductively given by $f^{]0[} := f$ and for $n \in \mathbb{N}$ by

$$f^{]n[}(x_0, \dots, x_n) := \frac{f^{]n-1[}(x_0, x_2, \dots, x_n) - f^{]n-1[}(x_1, x_2, \dots, x_n)}{x_0 - x_1},$$

and finally

$$|f^{]\nu[}|^{]\rho[} := \frac{|f^{]\nu[}(x) - f^{]\nu[}(y)|}{|x-y|^\rho}$$

for distinct x and y in $\mathbb{Q}_p^{]\nu[}$. If f is $\mathscr{C}^r$ then $|f^{[\nu]}|^{]\rho[}$ extends to a continuous function

$$|f^{[\nu]}|^{[\rho]}\colon \mathbb{Q}_p^{[\nu]} \times \mathbb{Q}_p^{[\nu]},$$

and if f is of compact support then

$$\| |f^{[\nu]}|^{[\rho]} \|_{\sup}$$

is well-defined. Moreover, there is a constant $C \geq 1$ and y in $\mathbb{Q}_p$ such that for all x in the support of f, we have $|x-y| = 1/C$ and $f(y) = 0$. Hence

$$\frac{|f(x) - f(y)|}{|x-y|} = C \cdot |f(x)|$$

and we conclude, by definition of $f^{[n]}$ for $n = 0, \dots, \nu$,

$$(*) \qquad \|f\|_{\mathscr{C}^r} = \|f^{[\nu]}\|_{\mathscr{C}^\rho}.$$

We can now prove:

Proposition 11.5. *Let $k \in \mathbb{N}$ and $r \geq 0$ such that $r \leq k-2$. The universal unitary norm on the $\mathbf{K}[\mathrm{B}]$-module $\mathscr{C}_{\mathrm{cp}}^{\mathrm{lr} \leq k-2}(\mathbb{Q}_p, \mathbf{K})$ is given $\|\cdot\|_{\mathscr{C}^r}$.*

Proof: We first have to show that, with the B-action as given in Proposition 11.2, the norm $\|\cdot^{[\nu]}\|_{\mathscr{C}^\rho}$ is

(i) invariant under translation, and
(ii) there is a constant $C > 0$ such that $\|1_{p^n\mathbb{Z}_p} x^{k-2}\|_{\mathscr{C}^r} \leq C \cdot p^{(r-(k-2))n}$ for all $n \in \mathbb{Z}$.

Ad (i): By definition $\|\cdot^{[\nu]}\|_{\mathscr{C}^\rho}$ is translation invariant.
Ad (ii): For $r \leq 1$ and for all $n \in \mathbb{N}$,

$$\|1_{p^n\mathbb{Z}_p} x^{k-2}\|_{\mathscr{C}^r} \leq \frac{|p^{(k-2)n} - 0|^r}{|p^n - p^{n-1}|^r} = C \cdot p^{(1-(k-2))n}$$

where $C := p^{-1} > 0$, and the case of $r > 1$ follows by induction. (See [Nag15, Lemma 8.3] for a general statement.)

We finally have to show that

$$\|\cdot\| := \|\cdot^{[v]}\|_{\mathscr{C}^\rho}$$

is the greatest norm on $\mathscr{C}^{\mathrm{lp} \le k-2}(\mathbb{Q}_p, \mathbf{K})$ that satisfies (i) and (ii). For this, it suffices to show that there is a subset of $B \cdot f_0$ which is

- orthogonal (that is, $\|\sum \lambda_b b\| = \max\{\|\lambda_b b\|\}$ for all scalars λ_b), and
- which topologically spans $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ for $\|\cdot^{[v]}\|_{\mathscr{C}^\rho}$ or equivalently, by Equation (∗), for $\|\cdot\|_{\mathscr{C}^r}$.

The *van der Put-basis*,

$$\{e_n^i := 1_{p^{l(n)}\mathbb{Z}_p} x^i(\cdot - n) : (n, i) \in \mathbb{N} \times \{0, \dots, k-1\}\},$$

where $l(0) = 0$ and $l(n) = \lfloor \log_p(n) \rfloor$ for $n > 0$, forms by [**Nag14**, Theorem 3.8] an orthogonal basis of $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ with $\|e_n^i\| = p^{(r-i)l(n)}$. In particular, it is orthogonal and topologically spans $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$. □

Part 5. Intertwining

We will show that the three conditions conditions of Corollary 7.10 on $\mathbb{D}$ inside $\varprojlim_\psi \mathscr{R}^+ \oplus \mathscr{R}^+$ characterize, under the Amice transform, the continuous dual of $i^{\mathrm{lr}}(\chi)$ for the universal unitary lattice. That is, all linear maps $\mu\colon i^{\mathrm{lr}}(\chi) \to \mathbf{K}$ that are bounded on the universal unitary lattice of the $\mathbf{K}[G]$-module $i^{\mathrm{lr}}(\chi)$. Then properties of $\mathbb{D}$ transfer to the universal unitary completion of $i^{\mathrm{lr}}(\chi)$, for example that it is nonzero and irreducible (if $\mathbb{D}$ is).

12. The injection into $V \otimes \mathbb{B}^+_{\mathrm{dR}}$ as intertwining condition

We will regard the embedding condition, that is, if $v = (F, G)$ in $\mathbb{D}$ then $\iota_1(v), \iota_2(v), \dots$ in $V \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$ and show that, under the Amice transform $\mathscr{A}$, it permits to glue the continuous linear maps $\mu = \mathscr{A}(F)$ and $\nu = \mathscr{A}(G)$ on $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ to a linear map on the embedding topological vector space $i^{\mathrm{la}}(\chi)$.

12.1. The intertwiner. Let $\theta\colon \bar{B} \to \mathbf{K}^*$ be a character and θ_w as defined at the end of Section 9. Then θ is *regular* if $\theta_w \neq \theta$.

THEOREM 12.1. *If $\theta\colon T \to \mathbf{K}^*$ is regular then there is a nonzero morphism of $\mathbf{K}[G]$-modules, unique up to multiplication by a scalar,*

$$i^{\mathrm{lc}}(\theta_w) \to i^{\mathrm{lc}}(\theta).$$

PROOF: By [**Car79**, Theorem 3.5] and Frobenius reciprocity. □

Such a morphism between the inductions from a parabolic subgroup, such as $\overline{B}$, to G of twists (by the Weyl group) of a given character, such as θ, is called an *intertwining operator.*

Corollary 12.2. *If* $\theta\colon \mathrm{T} \to \mathbf{K}^*$ *is regular then there is a nonzero morphism of* $\mathbf{K}[\mathrm{G}]$*-modules, unique up to multiplication by a scalar,*

$$\mathrm{T}_w : i^{\mathrm{lr}}(\chi) \to i^{\mathrm{lr}}(\chi_w).$$

PROOF: This morphism is the tensor product of

- the intertwining operator on the smooth part, and
- the identity morphism on the algebraic part. □

Proposition 12.3 ([**BB10**, Lemme 5.2.3]). *The pair of (sequences of) power series* $v = \mathrm{F} + \mathrm{G}$ *in* $\varprojlim_\psi \mathscr{d}^r \oplus \mathscr{d}^s$ *satisfies* $\iota_1(v), \iota_2(v), \ldots$ *in* $\mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$ *if and only if the pair of distributions*

$$\mathscr{A}(v) = \mu + \nu\colon i^{\mathrm{lr}}(\chi)(\mathrm{N}) \oplus i^{\mathrm{lr}}(\chi_w)(\mathrm{N}) \to \mathbf{K}$$

satisfies

$$\nu = \mu \circ \mathrm{T}_w \quad \textit{on} \quad i^{\mathrm{lr}}(\chi)(\mathrm{N}) \cap \mathrm{T}_w^{-1}\left(i^{\mathrm{lr}}(\chi_w)(\mathrm{N})\right)$$

(for an appropriate choice of intertwining operator T_w *among all its scalar multiples).*

12.2. The universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$-module $i^{\mathrm{lr}}(\chi)$. Let $\mathrm{N}(\mathrm{T})$ be the normalizer of T in G. Let $\mathrm{W} = \mathrm{N}(\mathrm{T})/\mathrm{T}$ be the Weyl group of G and let us fix $\{1, w\}1$ as set of representatives of W. For each v in W, we define the $\mathbf{K}$-linear morphism, given by restriction (and translation),

$$\iota_v : i^{\mathrm{lc}}(\theta)(\mathrm{N}w) \hookrightarrow \mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$$
$$f \mapsto f_{\restriction \mathrm{N}w}(\cdot v^{-1}).$$

Because T stabilizes N by conjugation and W stabilizes T by conjugation, T stabilizes $\mathrm{N}w$ by conjugation for every v. Therefore T stabilizes $i(\theta)(\mathrm{N}w)$ for every v in W and thus, via ι_v in W. There is t_0 in T such that $\mathrm{N}_0^{t_0}$ is properly included in N_0 for every open subset N_0 of N. For v in W, put $t_v = t_0^v$. Let $\mathbf{1}_{\mathrm{N}_0}\colon \mathbb{N} \to \mathbf{K}$ be the indicator function of N_0. For every v in W, under the action of T on $\mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ under ι_v, the supports of

$$\{\mathbf{1}^t_{\mathrm{N}_0} : t \in t_v^{\mathbb{N}}\}$$

constitute a neighborhood basis of 1.

Let $\mathscr{V}$ be a basis of neighborhoods of 1 in G. A subset S of $i(\theta)$ generates $i(\theta)$ as $\mathbf{K}$-vector space if S contains for every g in a section of $\overline{\mathrm{P}}\backslash\mathrm{G}$ and every V in $\mathscr{V}$ a function whose support is $\mathrm{V}g$. For example, by the Bruhat decomposition ([**Bor91**, IV.14.12]), $\mathrm{N} \cup \{w\}$ is a section of $\overline{\mathrm{P}}\backslash\mathrm{G}$.

Because N (and its right-translate $\mathrm{N}w$) are open in $\overline{\mathrm{P}}\backslash\mathrm{G}$ and the translated restriction ι_w of G to $\mathrm{N}w$ preserves the inclusion of supports, the set of supports of a set of functions S in $\mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ is a neighborhood basis inside N if and only if the set of supports of the preimages of S under restriction is a neighborhood basis inside G. We conclude:

Lemma 12.4. *Let $\mathcal{V}$ be a basis of neighborhoods of* 1 *inside* N. *A subset* S *of* $i(\theta)$ *generates* $i(\theta)$ *as* **K***-vector space if* S *contains,*

- *for every n in* N *and every* V *in* $\mathcal{V}$*, the preimage (under ι_1) of a function whose support is* Vn*, and*
- *for every* V *in* $\mathcal{V}$*, the preimage (under ι_w) of a function whose support is* V.

Let I be the standard *Iwahori subgroup*, the preimage in $\mathrm{GL}_2(\mathbb{Z}_p)$ of the subgroup of all upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_p)$. Let $\Phi\colon \mathrm{G} \to \mathbf{K}$ be given by

$$\Phi(\bar{b}in) = \theta(\bar{b}) \quad \text{for } \bar{b} \in \overline{\mathrm{B}}, i \in \mathrm{I} \text{ and } n \in \mathrm{N}.$$

Because θ is trivial on $\mathrm{I} \cap \overline{\mathrm{P}}$, this function is well-defined. Put

$$\Phi_1 := \Phi \quad \text{and} \quad \Phi_w := \Phi(\cdot w),$$

which restrict in $\mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ to the indicator function $\mathbf{1}_{\mathrm{N}_0}$ of N_0. Because

- by Lemma 12.4 the $\mathbf{K}[\mathrm{B}]$-module $i^{\mathrm{lc}}(\theta)$ is generated by ϕ_1 and ϕ_w, and
- by [**Cas80**, Theorem 3.4], there are c_1 and nonzero c_w in **K** such that $\mathrm{T}_w(\phi_1) = c_1\phi_1 + c_w\phi_w$ where ϕ_1,

the $\mathbf{K}[\mathrm{B}]$-module $i(\theta)$ is generated by ϕ_1 and $\mathrm{T}_w(\phi_w)$. We obtain the epimorphism of $\mathbf{K}[\mathrm{B}]$-modules

$$i^{\mathrm{lr}}(\chi)(\mathrm{N}) \oplus i^{\mathrm{lr}}(\chi_w)(\mathrm{N}) \overset{\mathrm{id}+\mathrm{T}_w}{\twoheadrightarrow} i^{\mathrm{lr}}(\chi).$$

Consequently, the universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$-module $i^{\mathrm{lr}}(\chi)$ is generated by the preimages of $\mathbf{1}_{\mathbb{Z}_p}x^{k-2}$ under the isomorphisms of $\mathscr{C}^{\mathrm{lp}\leq k-2}_{\mathrm{cp}}(\mathbb{Q}_p)$ with $i^{\mathrm{lr}}(\chi)(\mathrm{N})$ and $i^{\mathrm{lr}}(\chi_w)(\mathrm{N})$. Thus for the continuous dual,

$$i^{\mathrm{lr}}(\chi)^* \hookrightarrow i^{\mathrm{lr}}(\chi)(\mathrm{N})^* \oplus i^{\mathrm{lr}}(\chi_w)(\mathrm{N})^*,$$

where continuity means that the continuous linear form is bounded on the universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$-modules $i^{\mathrm{lr}}(\chi)(\mathrm{N})$, $i^{\mathrm{lr}}(\chi_w)(\mathrm{N})$ and $i^{\mathrm{lr}}(\chi)^*$.

Corollary 12.5. *The pair of (sequences of) power series $v = \mathrm{F} + \mathrm{G}$ in $\varprojlim_{\psi} d^{r} \oplus d^{s}$ satisfies $\iota_1(v), \iota_2(v), \ldots$ in $\mathrm{V} \otimes_{\mathbf{K}} \mathbb{B}^+_{\mathrm{dR}/\mathbf{K}}$ if and only if the pair of distributions $\mathscr{A}(v) = (\mu, \nu)\colon i^{\mathrm{lr}}(\chi_1)(\mathrm{N}) \oplus i^{\mathrm{lr}}(\chi_w)(\mathrm{N}) \to \mathbf{K}$ satisfies*

$$\mu + \nu \circ \mathrm{T}_w \quad \textit{in } i^{\mathrm{lr}}(\chi)^*.$$

where the continuity condition on the dual is given by boundedness on the universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$*-module* $i^{\mathrm{lr}}(\chi)$.

Proof: Let (μ, ν) in

$$\mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{\mathrm{la}}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) = i^{\mathrm{la}}(\chi_1)(\mathrm{N}) \oplus i^{\mathrm{la}}(\chi_w)(\mathrm{N}).$$

If (μ, ν) in

$$\mathscr{D}^{r}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}^{r}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$$

then μ and ν are by Proposition 11.2 and Proposition 11.5 bounded on the universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$-module

$$\mathscr{C}_{\mathrm{cp}}^{\mathrm{lr}\leq r}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{C}_{\mathrm{cp}}^{\mathrm{lr}\leq r}(\mathbb{Q}_p, \mathbf{K}) = i^{\mathrm{lr}}(\chi_1)(\mathrm{N}) \oplus i^{\mathrm{lr}}(\chi_w)(\mathrm{N}).$$

We conclude by Proposition 12.3 and applying the above observation. □

12.3. The universal unitary lattice of the $\mathbf{K}[\mathrm{G}]$-module $i^{\mathrm{lr}}(\chi)$. If (μ, ν) in $\mathscr{D}_{\mathrm{cp}}^{r}(\mathbb{Q}_p, \mathbf{K}) \oplus \mathscr{D}_{\mathrm{cp}}^{r}(\mathbb{Q}_p, \mathbf{K})$ such that $v = \mu + \nu \circ \mathrm{T}_w$ in $i^{\mathrm{lr}}(\chi)^*$, then v is by Corollary 12.5 bounded on the universal unitary lattice of the $\mathbf{K}[\mathrm{B}]$-module

$$i^{\mathrm{lr}}(\chi).$$

The following proposition shows that the universal unitary lattice of the B-representation $i(\chi)$ is the universal unitary lattice of the G-representation $i(\chi)$:

Lemma 12.6. *Let* G *be a topological group and* V *a* $\mathbf{K}$*-linear* G*-representation. If* B *is a subgroup of* G *such that*

$$\mathrm{G} = \mathrm{BK},$$

for a compact subgroup K *of* G *and*

- *the group* G *is* locally profinite *(that is, every neighborhood of* 1 *contains a compact open subgroup), and*
- *the representation* V *is finitely generated over* $\mathbf{K}[\mathrm{G}]$ *and* locally finitely generated *over* $\mathbf{K}$ *(that is, for every* v *in* V*, there is a compact open subgroup* G_0 *of* G *such that* $\mathbf{K}[\mathrm{G}_0] \cdot v$ *is finitely generated over* K*),*

then the universal unitary lattice of the $\mathbf{K}[\mathrm{G}]$*-module* V *is given by every lattice that is finitely generated over* $\mathscr{O}_{\mathbf{K}}[\mathrm{B}]$.

Proof: By Proposition 11.1, the universal unitary lattice of V is given by any lattice finitely generated as an $\mathscr{O}_{\mathbf{K}}[\mathrm{G}]$-module. We hence have to show that the commensurability class of lattices finitely generated as an $\mathscr{O}_{\mathbf{K}}[\mathrm{G}]$-module equals that of lattices finitely generated as an $\mathscr{O}_{\mathbf{K}}[\mathrm{B}]$-module.

Let $\mathfrak{L} := \sum_{i \in \mathrm{I}} \mathscr{O}_{\mathbf{K}}[\mathrm{G}] v_i$ with I finite be such a lattice. Then $\sum_{i \in \mathrm{I}} \mathbf{K}[\mathrm{K}] \cdot v_i$ is a finite-dimensional $\mathbf{K}$-vector space: By assumption, there is a compact open subgroup $\mathrm{K}_0 \subseteq \mathrm{G}$ such that $\mathrm{V}_0 := \sum_i \mathbf{K}[\mathrm{K}_0] \cdot v_i$ is a finite-dimensional $\mathbf{K}$-vector space. By intersecting with K and possibly shrinking K_0, we can assume K_0 to be an open normal subgroup of K, so that the quotient K/K_0 is a finite group. Therefore $\sum_i \mathbf{K}[\mathrm{K}] v_i = \sum_{k \in \mathrm{K}/\mathrm{K}_0} (\sum_i \mathbf{K}[k\mathrm{K}_0] \cdot v_i)$ is finite dimensional.

We thus find the $\mathscr{O}_{\mathbf{K}}$-module $\sum_{i \in \mathrm{I}} \mathscr{O}_{\mathbf{K}}[\mathrm{K}] \cdot v_i$ to be finitely generated as a $\mathscr{O}_{\mathbf{K}}[\mathrm{K}]$-module and, since K is compact, also to be bounded. Therefore it is finitely generated as an $\mathscr{O}_{\mathbf{K}}$-module and hence finite free. Denote its basis by $\{v_j : j \in \mathrm{J}\}$ for a finite index set J. We can then observe $\mathfrak{L}$ to be a finitely generated $\mathcal{O}_{\mathbf{K}}[B]$-module through the following equality chain of module spans:

$$\begin{aligned}\mathfrak{L} = \sum_{i\in I} \mathcal{O}_{\mathbf{K}}[G]v_i &=< g \cdot v_i : g \in BK, i \in I >_{\mathcal{O}_{\mathbf{K}}\text{-mod.}} \\ &=< k \cdot v_i : k \in K, i \in I >_{\mathcal{O}_{\mathbf{K}}[B]\text{-mod.}} \\ &=< v_j : j \in J >_{\mathcal{O}_{\mathbf{K}}[B]\text{-mod.}} \qquad = \sum_{j\in J} \mathcal{O}_{\mathbf{K}}[B]v_j.\end{aligned}$$

Conversely, assume we are given a lattice $\mathfrak{L} = \sum_{i\in I} \mathcal{O}_{\mathbf{K}}[B]v_i \subseteq V$ with I finite. Then likewise $\sum_{i\in I} \mathcal{O}_{\mathbf{K}}[G]v_i = \sum_{j\in J} \mathcal{O}_{\mathbf{K}}[B]v_j$ with J finite. So by finiteness (and because $\mathfrak{L}$ is a lattice), we find $\{v_j\} \subseteq \Lambda \cdot \mathfrak{L}$ for some $\Lambda \in K$ and hence by B-stability of $\mathfrak{L}$, we find $G \cdot \{v_i\} \subseteq \Lambda \cdot \mathfrak{L}$. Therefore, putting $\tilde{\mathfrak{L}} = \sum_{i\in I} \mathcal{O}_{\mathbf{K}}[G]v_i$, we have

$$\mathfrak{L} \subseteq \tilde{\mathfrak{L}} \subseteq \Lambda \cdot \mathfrak{L}.$$

In other words $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ are commensurable. □

Corollary 12.7. *The universal unitary lattice of the locally algebraic* G*-representation* $i^{\mathrm{lr}}(\chi)$ *is given by every lattice that is finitely generated over* $\mathcal{O}_{\mathbf{K}}[B]$.

Proof: We check that the conditions of Lemma 12.6 apply to the $\mathbf{K}[G]$-module $V = i^{\mathrm{lr}}(\chi)$:

- If G is a connected reductive group and B a minimal parabolic subgroup of G, then by the *Iwasawa decomposition* there is a maximal compact open subgroup K of G such that

  $$G = BK.$$

  (For our choice of G and B, the maximal compact open subgroup $G = GL_2(\mathbb{Z}_p)$ satisfies this decomposition.)
- Every affine algebraic group G over a local field is locally compact and totally disconnected, equivalently, locally profinite.
- Lemma 12.4 in particular proves that $i(\chi)$ is finitely generated over $\mathbf{K}[G]$. Because $i^{\mathrm{lr}}(\chi)$ is locally algebraic, it is locally given by an algebraic, in particular by a finite dimensional representation. □

13. Conclusion

In this final section, after

- applying the Amice transform $\mathcal{A}(\mathbb{D})$ of $\mathbb{D}$, and
- taking the continuous dual (for the bounded-weak topology) of $\mathcal{A}(\mathbb{D}) = i^{\mathrm{lr}}(\chi)^*$,

we will obtain by (Schikhof) duality the universal unitary completion

$$U := \widehat{i^{\mathrm{lr}}(\chi)}$$

of $i^{\mathrm{lr}}(\chi)$. Let us observe that in the *p-adic* Langlands program, in contrast to the *local* Langlands program where the vector spaces take coefficients in $\mathbb{C}$, the M-representation U is obtained by a functor from the Galois representation T:

$$\mathrm{U} = \left(\mathscr{A}\left(\left(\varprojlim_{\psi} \mathrm{D}^{\#}(\mathrm{T})\right) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K}\right)\right)^{*}$$

13.1. The Universal Unitary completion as dual representation. We want to translate our findings on the dual $(\widehat{i^{\mathrm{lr}}(\chi)})^{*}$ to $\widehat{i^{\mathrm{lr}}(\chi)}$ via a duality statement:

Schikhof Duality (cf. [**Sch95**]*).* Let $\mathbf{K}$ be a non-Archimedean field and V be a non-Archimedean $\mathbf{K}$-Banach space. We denote by V^{*} its dual $\mathbf{K}$-Banach space of continuous $\mathbf{K}$-linear forms endowed with the supremum norm. The natural duality map

$$\begin{aligned} \mathrm{V} &\to (\mathrm{V}^{*})^{*} \\ v &\mapsto \mathrm{ev}_{v} : [\mathrm{V}^{*} \ni v^{*} \mapsto v^{*}(v) \in \mathbf{K}] \end{aligned}$$

is surjective only if V is finite dimensional. To see this, note that (for example, by [**Sch02**, Section 10]) any $\mathbf{K}$-Banach space has an orthogonal basis, that is, there is an index set X such that it is equivalent to

$$c_0(\mathrm{X}) = \{f : \mathrm{X} \to \mathbf{K} : \text{ For every } \varepsilon > 0, \text{ there are only finitely many } x \in \mathrm{X} \text{ such that } |f(x)| > \varepsilon\}.$$

This is the completion of the $\mathbf{K}$-vector space whose basis is indexed by X, that is, of

$$\mathbf{K}^{\oplus \mathrm{X}} := \{f : \mathrm{X} \to \mathbf{K} : \text{ It holds } f(x) = 0 \text{ for all but finitely many } x \in \mathrm{X}\}.$$

The continuous dual of $c_0(\mathrm{X})$ is given by the bounded functions $c_b(\mathrm{X})$ on X, that is,

$$c_b(\mathrm{X}) = \mathcal{O}_{\mathbf{K}}{}^{\mathrm{X}} \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K}$$

with its supremum norm. It does not include any dense $\mathbf{E}$-vector subspace of cardinality X with respect to the topology of uniform convergence.

Nevertheless the subspace $\mathbf{K}^{\oplus \mathrm{X}}$ is dense in $c_b(\mathrm{X})$ with respect to the weaker topology of *point-wise convergence*. Then the usual dual of bounded linear forms on this topological $\mathbf{K}$-vector space identifies with $c_0(\mathrm{X})$ again.

This way a duality holds by endowing the continuous dual with the topology of point-wise instead of uniform convergence:

Theorem ([**Sch95**, Theorem 4.6]). *The functors between the categories*

- *of all* $\mathbf{K}$*-Banach spaces* V *with continuous maps, and*
- *of all torsion-free* bounded *topological* $\mathcal{O}_{\mathbf{K}}$*-modules (tensor products of* $\mathbf{K}$ *with a torsionfree compact topological* $\mathcal{O}$*-module) with continuous linear morphisms*

given

- *by* $V \mapsto V^* = \{$ *all uniformly continuous linear* $f\colon V \to \mathbf{K}\}$ *with the topology of point-wise convergence, and*
- *by* $M \mapsto M' = \{$ *all point-wise continuous linear* $f\colon M \to \mathbf{K}\}$ *with the topology of uniform convergence.*

are quasi-inverse.

Corollary 13.1. *If we*

- *endow* $U^* := \widehat{i^{\mathrm{lr}}(\chi)}^*$ *with the topology of bounded point-wise convergence (that is, the initial topology of the inclusion of its unit ball (= all continuous linear forms that take values in* $\mathcal{O}_{\mathbf{K}}$*)), and*
- *let* U^{*d} *be the continuous dual of* U^* *for the norm topology,*

Then

$$U^{*d} \xrightarrow{\sim} U := \widehat{i^{\mathrm{lr}}(\chi)}.$$

13.2. Properties of the Universal Unitary completion.

Proposition. *Let* $\chi = \theta\Psi\colon T \to \mathbf{K}^*$ *be the product of*

- *an unramified character* θ *(that is,* θ *is trivial on the maximal compact open subgroup of* T*) which is regular (that is,* $\theta \neq \theta_w$*), and*
- *a dominant algebraic character* $\Psi\colon \begin{pmatrix} x & \\ & y \end{pmatrix} \mapsto x^k y^l$ *(that is,* $k \geq l$*).*

If

$$|\chi(t)| \leq 1 \quad \textit{and} \quad |\chi_w(t)| \leq 1 \textit{ for all } t \textit{ in } T,$$

then

$$\widehat{\mathrm{ind}_{\mathbf{B}}^{G}\, \chi^{\mathrm{lr}}} \neq 0.$$

Proof: Let $U := \widehat{\mathrm{ind}_{B}^{G}\, \chi^{\mathrm{lr}}}$. Because

$$U = \left(\mathscr{A}\left(\left(\varprojlim_{\psi} D^{\#}(T)\right) \otimes_{\mathcal{O}_{\mathbf{K}}} \mathbf{K}\right)\right)^*,$$

we have $U = 0$ if and only if $\varprojlim_{\psi} D^{\#}(T) = 0$. Because

$$\varprojlim_{\psi} D^{\#}(T) \supseteq D(V)^{\psi=\mathrm{id}},$$

it suffices to show that the right-hand side is nonzero. By [**Col99**, Proposition II.1], the right-hand side includes a basis of D(V), in particular is nonzero.

□

Proposition. *Let* $\chi = \theta\Psi\colon T \to \mathbf{K}^*$ *be the product of an unramified regular character* θ *and a dominant algebraic character* Ψ *such that*

$$|\chi(t)| \leq 1 \quad \textit{and} \quad |\chi_w(t)| \leq 1 \textit{ for all } t \textit{ in } T.$$

If θ *is not unitary (that is,* $|\theta| = 1$*), then the topological* $\mathbf{K}[M]$*-module*

$$\widehat{\mathrm{ind}_{\mathbf{B}}^{G}\, \chi^{\mathrm{lr}}}$$

is irreducible.

PROOF: Because θ is regular, that is, $\alpha \neq \beta$, and θ is not unitary, its corresponding φ, Γ-module is by Section 2 irreducible. Therefore the M-module $\mathbb{D} := \varprojlim_{\psi} D^{\#}(T)$ is by [**Col10c**, Corollary III.3.11] irreducible . We conclude that the topological $\mathbf{K}[M]$-module $U := \widehat{\mathrm{ind}_B^G \chi^{lr}}$, obtained by a fully faithful functor from $\mathbb{D}$, is irreducible. □

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