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Fractional differentiability and Unitarity on parabolic inductions

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ABSTRACT. Let G be a split connected p-adic reductive group and P a Borel group of G with unipotent radical N. Let χ be the product of a locally constant and an algebraic character of P. Let $I(\chi)$ be the locally algebraic induction of χ to G of locally algebraic functions $f : G \to \mathbf{K}$ and $I(\chi)(N)$ its P-subrepresentation of all functions in $I(\chi)$ that vanish outside the open Bruhat cell $N\bar{P}$ of G (where $\bar{P}$ is the Borel group opposite to P).

We give a necessary and sufficient condition on χ for the existence of a P-invariant norm $\|\cdot\|$ on $I(\chi)(N)$ by comparing $\|\cdot\|$ to a norm of $\boldsymbol{r}$-times partially differentiable functions in many variables for a suitable tuple $\boldsymbol{r}$ of nonnegative real numbers.

This question is informed by the p-adic Langlands program.

Contents

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Introduction

Let $\mathbf{F}$ be a p-adic number field, that is, a finite extension of $\mathbb{Q}_p$. Let n in $\mathbb{N}$ and $G = GL_n(\mathbf{F})$. The p-adic Langlands program envisions a bridge between

- continuous linear actions of the absolute Galois group of $\mathbf{F}$ on an n-dimensional p-adic vector space, and
- unitary continuous linear actions of G on (usually infinite-dimensional) p-adic normed spaces.

Examples of such actions of G are commonly constructed by *inducing* actions $\chi_1, \ldots, \chi_d$ of smaller general linear groups $M_1 = GL_{n_1}(\mathbf{F}), \ldots, M_d = GL_{n_d}(\mathbf{F})$ with $n_1 + \cdots + n_d = n$: Put $M = M_1 \times \cdots \times M_d$ and $\chi = \chi_1 \otimes \cdots \otimes \chi_d$. Let $\mathbf{K}$ be a complete extension of $\mathbf{F}$. Then χ is a $\mathbf{K}[M]$-module and its *induction* to G is the $\mathbf{K}[G]$-module

$$\operatorname{ind}_M^G \chi := \mathbf{K}[G] \otimes_{\mathbf{K}[M]} \chi.$$

We assume that $M_1 = \cdots = M_n = GL_1(\mathbf{F}) = \mathbf{F}^*$ and that $\chi_1, \ldots, \chi_n \colon \mathbf{F}^* \to \mathbf{K}^*$ is *locally algebraic*, that is, locally given by $\lambda \cdot^{\pm k}$ for some λ in $\mathbf{K}^*$ and k in $\mathbb{N}$.

Let $\bar{P}$ be the subgroup of G of all lower triangular matrices and $\bar{N}$ the subgroup of $\bar{P}$ of all matrices whose diagonal entries are all equal to 1 (and P and N the transposes of $\bar{P}$ and $\bar{N}$ given by upper triangular matrices). Then $\bar{P} = M\bar{N}$ and χ extends uniquely to $\bar{P}$ (as $[\bar{P}, \bar{P}] = \bar{N}$ and $\operatorname{im} \chi$ is abelian) by the projection $\bar{P} \twoheadrightarrow M$. The *locally algebraic induction* $I(\chi)$ is given by all vectors in $\operatorname{ind}_{\bar{P}}^G \chi$ whose orbit maps $g \mapsto gv$ are locally algebraic; explicitly

$$I(\chi) = \{\text{all locally algebraic } f \colon G \to \mathbf{K} \text{ such that } f(\bar{p}\cdot) = \chi(\bar{p}) f \text{ for all } \bar{p} \text{ in } \bar{P}\}$$

and G acts by right translation. Such locally algebraic inductions are under the p-adic Langlands program attached to *crystalline* Galois actions, the prototypic kind of p-adic Galois actions that operate on the cohomology of an algebraic variety (and those *geometric* p-adic Galois actions form practically all known examples of p-adic Galois actions).

The action of G on a normed space V with norm $\|\cdot\|$ is *unitary* if all operator norms of G on V are bounded by one and the same constant or, equivalently, if $\|\cdot\|$ (is equivalent to a norm which) satisfies $\|g\cdot\| = \|\cdot\|$ for all g in G.

We want to study the existence of a unitary norm $\|\cdot\|$ on $I(\chi)$.

For $G = GL_2(\mathbb{Q}_p)$ such a unitary norm on $I(\chi)$ was constructed as quotient norm of r-times differentiable functions over $N = \mathbb{Q}_p$ for a real number $r \geq 0$ ([**BB10**, Section 4]). In general, we observe that:

- The entire above discussion generalizes from GL_n to a connected split reductive group $\mathbf{G}$ over $\mathbf{F}$. Then P becomes a *Borel subgroup* of G and N the *unipotent radical* of P (and $\bar{P}$ and $\bar{N}$ the *opposites* of P and N).
- There is a G-invariant norm on V if and only if there is a P-invariant norm on V ([**Nag11**, Corollary 3.3]), because $G = KP$ with a compact subgroup K (such as $GL_n(\mathbb{Z}_p)$ if $G = GL_n(\mathbb{Q}_p)$), for which every norm can be made unitary by taking its supremum over all translates in K.

We show (Corollary 8.4) that the P-subrepresentation $I(\chi)(N)$ of $I(\chi)$ given by all functions that vanish outside the open Bruhat-cell $N\bar{P}$ has a unitary norm by comparing it to the norm $\|\cdot\|_{\mathscr{C}^{\boldsymbol{r}}}$ of $\boldsymbol{r}$-times partially differentiable functions for a suitable tuple of nonnegative real numbers $\boldsymbol{r}$.

Outline

In Part 0 we establish terminology: We let G be (the rational points of) a connected split reductive group over $\mathbf{F}$ and $\bar{P}$ a Borel subgroup of G. Let $\theta \colon \bar{P} \to \mathbf{K}^*$ be a locally constant character. Let $I(\theta) \otimes U$ be the *locally algebraic representation* (as defined in Section 2) given by the tensor product of the smooth principal series $I(\theta) = \mathrm{Ind}_{\bar{P}}^{G} \theta^{\mathrm{lc}}$ and an algebraic representation U.

Because G is reductive and $\operatorname{char} \mathbf{F} = 0$, we may assume U irreducible. Because G is split, U is parametrized by a *dominant* (see Section 2) algebraic character $\psi \colon \bar{P} \to \mathbf{K}^*$. Put $\chi := \theta\psi$ and $I(\chi) := I(\theta) \otimes U$. Then $I(\chi)$ consists of all *locally algebraic functions* $f \colon G \to \mathbf{K}$ in the abstract $\mathbf{K}$-linear principal series representation $\mathrm{Ind}_{\bar{P}}^{G} \chi$ on which G acts by right translation.

In Part 1 we identify the P-subrepresentation $I(\chi)(N)$ of all functions in $I(\chi)$ that vanish outside the open Bruhat cell $\bar{P}N$ with one of locally polynomial functions $f \colon N \to \mathbf{K}$ of compact support, and give a general criterion for the existence of a unitary norm on $I(\chi)(N)$.

In Part 2 we construct a norm $\|\cdot\|_{\mathscr{C}^{\boldsymbol{r}}}$ on $\boldsymbol{r}$-times partially differentiable functions in many variables over $\mathbf{F}$ for a tuple of nonnegative real numbers $\boldsymbol{r} \geq 0$ and verify that $\|\cdot\|_{\mathscr{C}^{\boldsymbol{r}}}$ satisfies a boundedness condition.

Part 3 gives a condition on χ for the existence of a unitary norm on $I(\chi)(N)$ that we prove necessary, and sufficient: For this we

- make the existence criterion for a unitary norm established in Part 1 explicit, and
- show that the boundedness condition on (a variant of) the $\|\cdot\|_{\mathscr{C}^{\boldsymbol{r}}}$ of Part 2 is the explicit description of that existence criterion.

Part 1, 2 and 3 revise parts of [**Nag11**, Chapter II] which gives a conditional criterion for the existence of a unitary norm on all of $I(\chi)$. The author seeks to lessen this condition using a different notion of fractional differentiability closer to that of [Col10] and [DI13].

In Section 9 we relate our construction to that in [BB10] for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Part 0. Terminology

Let $\mathbf{F}$ be a *p-adic number field*, that is, a finite extension of the p-adic numbers $\mathbb{Q}_p$, and let $\mathbf{o_F}$ be its ring of integers with maximal ideal $\mathfrak{m}_\mathbf{F}$ and residue field $\mathbf{k_F} = \mathbf{o_F}/\mathfrak{m}_\mathbf{F}$. Let $v_\mathbf{F}\colon \mathbf{F} \to \mathbb{Z} \cup \{\infty\}$ be the additive valuation on $\mathbf{F}$ and $|\cdot|_\mathbf{F}$ the absolute value on $\mathbf{F}$, defined by $|x|_\mathbf{F} := p^{-v_\mathbf{F}(x)}$. Let $\pi_\mathbf{F}$ be a uniformizer of $\mathbf{F}$, that is, $v_\mathbf{F}(\pi_\mathbf{F}) = 1$.

We will drop subscripts whenever confusion is unlikely.

1. The groups

An (*affine* or *linear*) *algebraic group* is an affine group scheme of finite type over a field. We assume throughout this article all algebraic groups defined over our fixed field $\mathbf{F}$. In particular, the coefficients of the group of rational points of our affine algebraic groups lie in $\mathbf{F}$. Let us denote

- an algebraic group by a boldface letter (such as $\mathbf{G}$),
- its rational points by the corresponding ordinary type letter (such as G), which, via the topology of $\mathbf{F}$, is a topological group, and
- its compact open subgroup of $\mathbf{o_F}$-points (if defined) by an additional subscript naught (such as G_0).

For example, if $\mathbf{M}$ is a split torus, then $\mathrm{M} = \mathbf{M}(\mathbf{F})$ and $\mathrm{M}_0 = \mathbf{M}(\mathbf{o_F})$ is the maximal compact open subgroup of M.

Throughout this article, we fix the following notations for a split connected reductive group $\mathbf{G}$ defined over $\mathbf{F}$:

- Henceforth
  - let $\bar{\mathbf{P}}$ be a Borel subgroup of $\mathbf{G}$,
  - let $\mathbf{P}$ denote the Borel subgroup opposite to $\bar{\mathbf{P}}$,
  - let $\mathbf{M}$ be a maximal split torus of $\bar{\mathbf{P}}$ and $\mathbf{P}$, and
  - let $\mathbf{N}$ respectively $\bar{\mathbf{N}}$ denote the unipotent radical of $\mathbf{P}$ respectively $\bar{\mathbf{P}}$.
- Let $\mathbf{Z} = \mathbf{Z_G}$ be the center of $\mathbf{G}$.
- The torus $\mathbf{M}$ normalizes $\mathbf{N}$ respectively $\bar{\mathbf{N}}$ by conjugation and $\mathbf{P} = \mathbf{N}\mathbf{M}$ and $\bar{\mathbf{P}} = \bar{\mathbf{N}}\mathbf{M}$. Let $\mathbf{N_G}(\mathbf{M})$ be the normalizer of $\mathbf{M}$ inside $\mathbf{G}$ and let $\mathrm{W} = \mathbf{N_G}(\mathbf{M})/\mathbf{C_G}(\mathbf{M})$ be the *Weyl group* of $\mathbf{G}$.
- Let K be a special, good, maximal compact open subgroup in $\mathrm{G} = \mathbf{G}(\mathbf{F})$ such that its Iwahori subgroup $\bar{\mathrm{B}} \subseteq \mathrm{K}$ is of the same type as $\bar{\mathbf{P}}$. Let B be the Iwahori subgroup opposite to $\bar{\mathrm{B}}$.
- The choice of the maximal $\mathbf{F}$-split torus $\mathbf{M}$ determines a (relative) root system Φ. By [Bor91, Proposition 21.9], there is for each $\alpha \in \Phi$ a

unique root subgroup, denoted $\mathbf{N}_\alpha$, that is normalized by $\mathbf{M}$ and on which $\mathbf{M}$ acts through the adjoint action by the character α.

The choice of the Borel group $\mathbf{P}$ determines a basis Δ of simple roots in Φ. Then $\Phi = (\sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \cdot \alpha \cap \Phi) \dot{\cup} (\sum_{\alpha \in \Delta} \mathbb{Z}_{\leq 0} \cdot \alpha \cap \Phi)$ and $\alpha \in \Phi$ is *positive*/*negative* (or $\alpha \gtrless 0$) if it lies in the left/right-hand segment.

2. The representations

A G-*representation* is a vector space V together with a linear action of a group G. All vector spaces, if not mentioned otherwise, will be defined over the field $\mathbf{K}$; in particular those that G acts on.

- A representation V of a *locally profinite* group G, that is, locally compact and totally disconnected, is *smooth* if the natural map $G \times V \to V$ is continuous for the discrete topology on V or, equivalently, if every vector is *smooth*, that is, its stabilizer $\{g \in G : gv = v\}$ is open.
- A representation of (the rational points of) an algebraic group G on a finite-dimensional $\mathbf{F}$-vector space V is *algebraic* if the natural map $G \times V \to V$ is given by (the rational points of) a morphism between affine $\mathbf{F}$-schemes $\mathbf{G} \times \mathbf{V} \to \mathbf{V}$ (where $\mathbf{V}$ is the affine $\mathbf{F}$-scheme defined by $\mathbf{V}(R) = V \otimes_{\mathbf{F}} R$ for every $\mathbf{F}$-algebra R). That is, a representation of an algebraic group G on a finite-dimensional vector space V is algebraic if the action of G on V is given by a rational function in the coordinate entries of G and V.

The two notions of a smooth and algebraic representation combine for a p-adic algebraic group G (which is algebraic and locally profinite), as follows:

- An action of G on V is *smooth* respectively *algebraic* if every vector is smooth respectively algebraic for G, where a vector v in V is *smooth* respectively *algebraic* if the orbit map $o_v : g \mapsto g \cdot v$ is locally constant respectively algebraic on G.
- A G-representation V is *locally algebraic* if every vector is locally algebraic, where a vector $v_0 \in V$ is *locally algebraic* if there is a finite-dimensional $\mathbf{K}$-vector subspace V_0 that contains v_0 and a compact open subgroup G_0 of G such that the natural map $G_0 \times V_0 \to V_0$ is the restriction of an algebraic representation of G ([**Eme11**, Corollary 4.2.9 ff.]).

The tensor product $V \otimes U$ of a smooth representation V with an algebraic representation U is always locally algebraic; vice versa, every irreducible locally algebraic representation arises this way ([**ST01**, Appendix]).

Let $\mathbf{G}$ be a split connected reductive group over $\mathbf{F}$. Let $\chi \colon M \to \mathbf{K}^*$ be a character. By precomposition with the projection $\bar{P} \twoheadrightarrow M$, it induces a character $\chi \colon \bar{P} \to \mathbf{K}^*$. We can then construct the $\mathbf{K}$-linear G-representation

$$\mathrm{Ind}_{\bar{P}}^{G} \chi := \{f \colon G \to \mathbf{K} : f(\bar{p}g) = \chi(\bar{p}) \cdot f(g) \quad \text{for all } \bar{p} \in \bar{P}, g \in G\},$$

on which G acts by right translation, denoted by $f^g := f(\cdot g)$ or $g \cdot f$. In correspondence with the induced character χ, we look at the subrepresentation of all locally constant, algebraic and locally algebraic vectors:

- Let $\theta \colon \mathrm{M} \to \mathbf{K}^*$ be a *smooth* character, that is, trivial on a compact open subgroup M. The *smooth principal series* is the smooth G-representation

$$\mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \theta^{\mathrm{lc}} := \{ \text{ all smooth vectors of } \mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \theta \}.$$

  It is nonzero and, as G acts by translation, consists of all locally constant functions in $\mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \theta$.
- Let $\psi \colon \mathrm{M} \to \mathbf{K}^*$ be an *algebraic* character, that is, it is given by (the rational points of) a morphism between $\mathbf{F}$-schemes $\mathbf{M} \to \mathbb{G}_m$ into the multiplicative group $\mathbb{G}_m$; it is *dominant* if $\langle \psi, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Phi^+$ (where $\langle \psi, \check{\alpha} \rangle := \psi \circ \check{\alpha} \in \mathrm{Aut}(\mathbb{G}_{\mathrm{m}}) = \mathbb{Z}$).

  Let U be an irreducible algebraic G-representation. By the classification of all irreducible algebraic representations of a split connected reductive group, there is a unique one-dimensional subspace fixed by P whose algebraic character $\psi \colon \mathrm{P} \to \mathbf{F}^*$ is the *highest weight* of U; conversely, for every dominant algebraic character ψ, there is a unique irreducible algebraic G-representation U_ψ of highest weight ψ. If $\mathbf{F}$ has characteristic 0 then

$$\mathrm{U}_\psi = \mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \psi^{\mathrm{alg}} := \{ \text{ all algebraic vectors of } \mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \psi \}.$$

  It is nonzero and, as G acts by translation, consists of all rational functions in $\mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \theta$. Because every algebraic representation of a split reductive group splits (into irreducible representations), we assume that ψ is dominant.
- Let $\chi \colon \mathrm{M} \to \mathbf{K}^*$ be a *locally algebraic* character, that is, it is the product $\chi = \theta\psi$ of a locally constant character θ and an algebraic character ψ. The *locally constant principal series* $\mathrm{I}(\chi)$ is $\mathrm{I}(\chi) = \mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \theta^{\mathrm{lc}} \otimes_{\mathbf{K}} \mathrm{U}_\psi$. Equivalently

$$\mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \chi^{\mathrm{la}} := \{ \text{ all locally algebraic vectors of } \mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \chi \}.$$

  It is nonzero and, as G acts by translation, consists of all locally rational functions in $\mathrm{Ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \chi$.

Part 1. The open cell as a representation of P

Let $\mathbf{G}$ be a split connected reductive group over $\mathbf{F}$ and let $\theta \colon \mathrm{M} \to \mathbf{K}^*$ be a locally constant character.

3. The open cell as representation of the Borel group

The *support* of a function f is the closure of all points where f does not vanish. There is a well-defined notion of support on the flag variety $\mathfrak{F} := \bar{\mathrm{P}}\backslash\mathrm{G}$ for functions in $\mathrm{I}(\theta)$ because $\bar{\mathrm{P}}$ acts on $\mathrm{I}(\theta)$ from the left by multiplication with invertible scalars. Thus N is an open subset of $\mathfrak{F}$ as image of the immersion $\mathrm{N} \hookrightarrow \mathrm{G} \twoheadrightarrow \bar{\mathrm{P}}\backslash\mathrm{G}$.

Let $\mathrm{I}(\theta)(\mathrm{N})$ denote the **K**-vector subspace of all functions in $\mathrm{I}(\theta)$ whose *support* lies in the open subset N of $\mathfrak{F}$. The support of such a function, as closed subset of the compact topological space $\mathfrak{F}$, is compact.

Let $\mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ be all locally constant functions $f\colon \mathrm{N} \to \mathbf{K}$ of compact support. For a compact open subset U of N, let $\mathbf{1}_{\mathrm{U}}$ be the *indicator function* of U defined by $\operatorname{supp} \mathbf{1}_{\mathrm{U}} = \mathrm{U}$ and $\mathbf{1}_{\mathrm{U}}(x) = 1$ for all x in U.

Lemma 3.1. *The restriction $f \mapsto f_{\restriction \mathrm{N}}$ is an isomorphism between* **K***-vector spaces*

$$\mathrm{Ind}^{\mathrm{G}}_{\bar{\mathrm{P}}}\, \theta^{\mathrm{lc}}(\mathrm{N}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K}).$$

Proof: The mapping is surely injective. We must show that it is surjective: Since $1 \in \mathrm{N}$ has a neighborhood basis of compact open subgroups, the **K**-vector space $\mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ is generated by all indicator functions $\mathbf{1}_{\mathrm{N}_c n}$ for $n \in \mathrm{N}$ and $\{\mathrm{N}_c\}$ a neighborhood basis of compact open subgroups in N. We want to construct their preimages. Fix n in N.

By [Cas95, Proposition 1.4.4], the compact open subgroups I of G with Iwahori factorization $\mathrm{I} = \mathrm{I}_{\bar{\mathrm{P}}}\mathrm{I}_{\mathrm{N}}$ with $\mathrm{I}_{\bar{\mathrm{P}}} = \mathrm{I} \cap \bar{\mathrm{P}}$ and $\mathrm{I}_{\mathrm{N}} = \mathrm{I} \cap \mathrm{N}$ are a neighborhood basis of the identity. Choose $\mathrm{I}_{\bar{\mathrm{P}}}$ so small that θ is trivial on it.

Because θ is trivial on $\mathrm{I}_{\bar{\mathrm{P}}}$ and I is a group, the function f defined by $\operatorname{supp} f = \bar{\mathrm{P}}\mathrm{I}$ and $f(\bar{p}i) = \theta(\bar{p})$ for $\bar{p} \in \bar{\mathrm{P}}, i \in \mathrm{I}$ is well-defined. By construction, f is constant on all right I-cosets and so in particular smooth. Thus $f \in \mathrm{I}(\theta)$ and $\operatorname{supp} f = \bar{\mathrm{P}}\mathrm{I}_{\bar{\mathrm{P}}}\mathrm{I}_{\mathrm{N}}n = \bar{\mathrm{P}}\mathrm{I}_{\mathrm{N}}n \subseteq \bar{\mathrm{P}}\mathrm{N}$, that is, $f \in \mathrm{I}(\theta)(\mathrm{N})$. Finally $f_{|\mathrm{N}} = \mathbf{1}_{\mathrm{I}_{\mathrm{N}}n}$, where the compact open subgroup $\mathrm{I}_{\mathrm{N}} \subseteq \mathrm{N}$ can be made arbitrarily small by choosing sufficiently small compact open $\mathrm{I} \subseteq \mathrm{G}$ in the neighborhood basis of 1 consisting of all compact open subgroups with Iwahori factorization. ◻

Let us denote the left-conjugation action of $g \in \mathrm{G}$ on G by ${}^{g}\cdot$ (and accordingly its right-conjugation action by $\cdot^{g}$).

Lemma 3.2.

(i) The group P *stabilizes* $\mathrm{I}(\theta)(\mathrm{N})$ *and operates on* $\mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ *by*

$$f^{p} = \theta(m) f(\cdot^{m} n) \quad \text{for } p = mn \in \mathrm{P} \quad \text{with } m \in \mathrm{M}, n \in \mathrm{N}.$$

In particular, $n^{-1}m\mathbf{1}_{\mathrm{U}} = \theta(m)\mathbf{1}_{{}^{m}\mathrm{U}n}$ *for a compact open subset* U *of* N.

(ii) The $\mathbf{K}[\mathrm{P}]$*-module* $\mathrm{I}(\theta)(\mathrm{N}) \cong \mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ *is generated by any* $f = \mathbf{1}_{\mathrm{U}}$ *with* $\mathrm{U} \subseteq \mathrm{N}$ *compact open.*

PROOF:

Ad (i): Let $p = mn \in \mathrm{P}$ with $m \in \mathrm{M}, n \in \mathrm{N}$ and f in $\mathrm{I}(\theta)(\mathrm{N})$. Then

$$pf = f(\cdot p) = f(mm^{-1} \cdot mn) = \theta(m) f(\cdot {}^{m}n).$$

If U is a compact open subset of N then $p\mathbf{1}_{\mathrm{U}} = \mathbf{1}_{\mathrm{U}}(\cdot p) = \mathbf{1}_{\mathrm{U}p^{-1}}$.

Ad (ii): By right translation with suitable $n \in \mathrm{N}$, we obtain $\mathrm{U}n = \mathrm{N}_c$ for a compact open neighborhood $\mathrm{N}_c \ni 1$. Let $f = \mathbf{1}_{\mathrm{N}_c}$. By [**Cas95**, Proposition 1.4.3] there is an element $m \in \mathrm{M}$ with $|\alpha(m)|_{\mathbf{F}}$ sufficiently small for all $\alpha \in \Delta$, such that $\{{}^{m^i}\mathrm{N}_c : i \in \mathbb{N}\}$ constitutes a system of neighborhoods of $1 \in \mathrm{N}$.

We just saw $f^m = \theta(m)\mathbf{1}_{{}^m\mathrm{N}_c}$ for $m \in \mathrm{M}$. The group N acts by right translation on these (scaled) indicator functions, therefore $\mathbf{K} \cdot \{f^p : p \in \mathrm{P}\} \supseteq \mathbf{K} \cdot \{\mathbf{1}_{\mathrm{U}}\}$ for a topological basis of compact open subsets $\{\mathrm{U}\}$ of N.

Every $f \in \mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$ is by definition a linear combination of such indicator functions $\mathbf{1}_{\mathrm{U}}$ of compact open subsets, so $\mathbf{K}[\mathrm{P}] \cdot f \supseteq \mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathrm{N}, \mathbf{K})$. □

Recall from Part 0 that K is the maximal compact open subgroup of G and $\bar{\mathrm{B}}$ the Iwahori subgroup in K.

Definition 3.3. Let ϕ_1 be the function in $\mathrm{I}(\theta)$ that has support $\bar{\mathrm{P}}\mathrm{B}$ and is equal to 1 on B (cf. [**Cas80**, Section 2]).

Let $\psi \colon \mathrm{M} \to \mathbf{K}^*$ be an algebraic dominant character and put $\chi = \theta\psi$.

Lemma 3.4. *Let* U *be an irreducible split algebraic* G*-representation. Then there is a unique (up to scalar multiplication) vector* $\bar{u}$ *fixed by* $\bar{\mathrm{N}}$ *and the* $\mathbf{K}[\mathrm{P}_0]$*-module* U *is generated by* $\bar{u}$ *for any compact open subgroup* $\mathrm{P}_0 \subseteq \mathrm{P}$.

PROOF: Let $\mathrm{N}_0 \subseteq \mathrm{N}$ be any open subgroup. By [**Bor91**, Theorem 21.20(i)], the rational points N are Zariski dense inside $\mathbf{N}$ and likewise, for N_0 inside N.

The proof of [**Hum81**, Proposition 31.2] shows that the $\mathrm{K}[\mathrm{P}_0]$-module U is generated by $\bar{u}$ for every open subgroup P_0 of P. □

Corollary 3.5. *Let* $\mathrm{U} = \mathrm{U}_\psi$ *be an irreducible algebraic* G*-representation and denote by* $\bar{u}$ *its unique (up to scalar multiplication) vector fixed by* $\bar{\mathrm{N}}$. *Then*

$$\mathrm{I}(\chi)(\mathrm{N}) = \mathbf{K}[\mathrm{P}] \cdot \phi_1 \otimes \bar{u}.$$

PROOF: Let $\mathrm{P}_{\bar{\mathrm{B}}} = \mathrm{P} \cap \bar{\mathrm{B}}$. Since $\mathrm{P}_{\bar{\mathrm{B}}} \subseteq \mathrm{P}$ is open, we find by the preceding Lemma 3.4 that

$$\mathbf{K}[\mathrm{P}_{\bar{\mathrm{B}}}] \cdot \phi_1 \otimes \bar{u} = \mathbf{K} \cdot \phi_1 \otimes \mathbf{K}[\mathrm{P}_{\bar{\mathrm{B}}}] \cdot \bar{u} = \mathbf{K} \cdot \phi_1 \otimes \mathrm{U}.$$

Therefore

$$\begin{aligned}\mathbf{K}[\mathrm{P}] \cdot \phi_1 \otimes \bar{u} &= \mathbf{K}[\mathrm{P}] \cdot (\mathbf{K}[\mathrm{P}_{\bar{\mathrm{B}}}] \cdot \phi_1 \otimes \bar{u}) \\ &= \mathbf{K}[\mathrm{P}] \cdot (\mathbf{K} \cdot \phi_1 \otimes \mathrm{U}) \\ &= \mathrm{I}(\theta)(\mathrm{N}) \otimes \mathrm{U} = \mathrm{I}(\chi)(\mathrm{N});\end{aligned}$$

the last equality by Lemma 3.2.(ii). □

4. The greatest unitary norm on the open cell

Let G be a group, let V be a **K**-vector space and let G act on V. Let **o** be the ring of integers of **K**.

Definition. A *non-Archimedean seminorm* on a **K**-vector space V is a function $\|\cdot\| \colon \mathrm{V} \to \mathbb{R}_{\geq 0}$ such that

- $\|v + w\| \leq \max\{\|v\|, \|w\|\}$ for all v, w in V, and
- $\|\lambda v\| = |\lambda| \|v\|$ for all v in V and λ in **K**.

A *non-Archimedean norm* $\|\cdot\|$ is a non-Archimedean seminorm such that

- $\|v\| = 0$ only if $v = 0$.

In what follows, every seminorm on a vector space over a non-Archimedean field is understood non-Archimedean.

- A seminorm $\|\cdot\|$ on V is *unitary* if $\|g\cdot\| = \|\cdot\|$ for every g in G. An equivalence class of seminorms is *unitary* if it contains a unitary norm.
- Let N′ and N″ be equivalence classes of seminorms. Then N′ is *greater* than N″ if for every seminorm $\|\cdot\|'$ in N′ and every seminorm $\|\cdot\|''$ in N″ there is a constant $\mathrm{C} > 0$ such that $\mathrm{C}\,\|\cdot\|' \geq \|\cdot\|''$.
- If there is a greatest among all equivalence classes of unitary seminorms, then it is unique and called the *universal* unitary seminorm. For example, we see below that it exists when V is finitely generated as module over **K**[G].

Definition. A *lattice* $\mathfrak{L}$ of a **K**-vector space V is an **o**-submodule such that for every $v \in \mathrm{V}$ there is λ in $\mathbf{K}^*$ such that λv in $\mathfrak{L}$.

Let $\|\cdot\|'$ and $\|\cdot\|''$ be two norms on a common **K**-vector space with corresponding lattices $\mathfrak{L}'$ and $\mathfrak{L}''$. Then $\|\cdot\|'$ and $\|\cdot\|''$ are equivalent if and only if $\mathfrak{L}'$ and $\mathfrak{L}''$ are *commensurable*, that is, there are λ' and λ'' in **K** such that

$$\mathfrak{L}' \subseteq \lambda'' \cdot \mathfrak{L}'' \quad \text{and} \quad \mathfrak{L}'' \subseteq \lambda' \cdot \mathfrak{L}'.$$

The notions of a commensurability class of lattices and an equivalence class of seminorms on a non-Archimedean vector space are equivalent; that is, the following assignments induce mappings between all equivalence classes of seminorms and all commensurability classes of lattices that are inverse to each other:

- Every lattice $\mathfrak{L}$ of a vector space V gives rise to a seminorm $\|\cdot\|_{\mathfrak{L}}$ on V given by

$$\|v\|_{\mathfrak{L}} := \sup |\{ \text{ all } \lambda \in \mathbf{K}^* \text{ such that } \lambda v \in \mathfrak{L}\}|, \quad \text{ and}$$

- every seminorm gives rise to a lattice given by its closed unit ball.

By definition, a lattice $\mathfrak{L}$ need not be free and can even coincide with its surrounding vector space V, just as its corresponding seminorm $\|\cdot\|$ need not be 0 solely on the 0 vector and can even be 0 everywhere. In fact, (if V is countably infinite-dimensional) then $\mathfrak{L}$ is free if and only if $\|\cdot\|$ is a norm ([**Sch02**, Proposition 10.4]).

The above notions for seminorms (unitarity, one equivalence class of seminorms being greater than another, and universality) correspond to the following notions for lattices:

- An equivalence class of norms is unitary if and only if its corresponding commensurability class contains a lattice stable under G.
- Let N′ and N″ be equivalence classes of norms with corresponding commensurability classes of lattices L′ and L″. Then N′ is greater than N″ if and only if L′ is *smaller* than L″, that is, for every lattice $\mathfrak{L}'$ in L′ and every lattice $\mathfrak{L}''$ in L″, there is λ in $\mathbf{o}$ such that $\lambda \mathfrak{L}' \subseteq \mathfrak{L}''$.
- An equivalence class of norms is the universal unitary norm if and only if its corresponding lattice is the smallest commensurability class of all lattices that are stable under G.

If V is a finitely generated $\mathbf{K}[G]$-module, that is, there are $v_1, \dots, v_n$ in V such that $V = \mathbf{K}[G]v_1 + \cdots + \mathbf{K}[G]v_n$, then its universal unitary lattice $\mathfrak{L}$ is given by the smallest $\mathbf{o}[G]$-module of V that contains $v_1, \dots, v_n$, that is,

$$\mathfrak{L} = \mathbf{o}[G]v_1 + \cdots + \mathbf{o}[G]v_n.$$

Lemma 4.1. *The universal unitary $\mathfrak{L}$ of the $\mathbf{K}[P]$-module $I(\chi)(N)$ is given by*

$$\mathfrak{L} = \mathbf{o}[P] \cdot \phi_1 \otimes \bar{u}.$$

Proof: By Corollary 3.5 the $\mathbf{K}[P]$-module $I(\chi)(N)$ is generated by $\phi_1 \otimes \bar{u}$. Thus, by our above observation, its universal unitary lattice is given by $\mathbf{o}[P] \cdot \phi_1 \otimes \bar{u}$. □

Put $f_0 := \phi_1 \otimes \bar{u}$. A lattice $\mathfrak{L}$ is in the smallest commensurability class of all lattices that contain all $p \cdot f_0$ for p in P if and only if its corresponding seminorm $\|\cdot\|$ is in the greatest equivalence class of all seminorms that satisfy $\|pf_0\| \leq \|f_0\|$ for all p in P. We conclude:

Corollary 4.2. *There is a unitary norm on the P-representation $I(\chi)(N)$ if and only if there is a norm $\|\cdot\|$ on $I(\chi)(N)$ such that there is a constant $C > 0$ for which $\|pf_0\| \leq C$ for all p in P.*

Part 2. Locally polynomial differentiable functions

We introduce a norm on the space of locally polynomial functions in many variables resembling that on the space of differentiable functions. We have in mind to endow next, in Part 3, the space of locally algebraic functions $\mathrm{I}(\chi)(\mathrm{N})$ with this norm, constructed so as to satisfy Corollary 4.2.

5. Basic estimates

Definition. A function $f\colon \mathbf{F}^d \to \mathbf{K}$ is *locally polynomial* if for every $x \in \mathbf{F}^d$ there is an open neighborhood U around x inside $\mathbf{F}^d$ such that $f_{|\mathrm{U}}$ is a polynomial function. Let $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d, \mathbf{K})$ be the $\mathbf{K}$-vector space of all locally polynomial functions $f\colon \mathbf{F}^d \to \mathbf{K}$ of compact support. For $a \in \mathbf{F}^d$ and $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}^d_{>0}$, let

$$\mathrm{B}_{\leq\delta}(a) := \{x \in \mathbf{F}^d : |x_1 - x_1| \leq \delta_1, \ldots, |x_d - a_d| \leq \delta_d\}$$

be the polydisc around a of radius δ. A function $f\colon \mathbf{F}^d \to \mathbf{K}$ is called δ-*polynomial* if $f_{|\mathrm{U}}$ is a polynomial function for every polydisc U of radius at most δ. The locally polynomial function $f\colon \mathbf{F}^d \to \mathbf{K}$ has *degree at most* $\boldsymbol{n} \in \mathbb{N}^d$ if there is an open covering $\{\mathrm{U}\}$ of its domain and matching polynomials $u(\mathrm{X}) = \sum_{i_1 \leq n_1, \ldots, i_d \leq n_d} a_{\boldsymbol{i}} \mathrm{X}^{\boldsymbol{i}}$ such that $f(x) = u(x)$ on all U in the covering.

For a locally polynomial function $f\colon \mathbf{F}^d \to \mathbf{K}$ put

$$\|f\|_{\sup} := \sup\{|f(x)| : x \in \mathrm{X}\}$$

and for a subset U of $\mathbf{F}^d$ put $\|f\|_{\mathrm{U}} := \|f_{|\mathrm{U}}\|_{\sup}$.

Lemma 5.1. *Fix $n \in \mathbb{N}$. There is a positive constant $c \leq 1$ such that for every compact open subset of the form $\mathrm{U} = \pi^k \cdot \mathbf{o}_\mathrm{F} \subseteq \mathbf{F}$ and every polynomial function $f = \sum_{i=0,\ldots,n} a_i *^i$ of degree at most n,*

$$c \cdot \max_{i=0,\ldots,n} |a_i| \|*^i\|_{\mathrm{U}} \leq \|\sum_{i=0,\ldots,n} a_i *^i\|_{\mathrm{U}} \leq \max_{i=0,\ldots,n} |a_i| \|*^i\|_{\mathrm{U}}.$$

Proof: Because the $\mathbf{K}$-vector space of polynomial functions $f\colon \mathbf{o}_\mathbf{K} \to \mathbf{K}$ of degree at most n is finite dimensional and $\mathbf{K}$ is complete, $\|\cdot\|_{\mathbf{o}_\mathbf{F}}$ is equivalent to the norm $\|\cdot\|$ given by the orthonormal basis $*^i$, that is, the norm $\|\cdot\|$ defined by $\|f\| = \max_{i=0,\ldots,n}|a_i|$ for $f = \sum_{i=0,\ldots,n} a_i *^i$. (See [Scho2, Proposition 4.13].) In particular, $c \cdot \max_{i=0,\ldots,n}|a_i| \leq \|f\|_{\mathbf{o}_\mathbf{F}}$ for some positive constant $c \leq 1$.

If $\mathrm{U} = \pi^k \cdot \mathbf{o}_\mathbf{F}$, then

$$\begin{aligned}\|f\|_{\mathrm{U}} &= \|\sum_{i=0,\ldots,n} a_i \pi^{ki} *^i\|_{\mathbf{o}_\mathbf{F}} \\ &\geq c \cdot \max_{i=0,\ldots,n} |a_i||\pi|^{ki} \\ &= c \cdot \max_{i=0,\ldots,n} |a_i| \|*^i\|_{\pi^k \cdot \mathbf{o}_\mathbf{F}} = c \cdot \max_{i=0,\ldots,n} |a_i| \|*^i\|_{\mathrm{U}}.\end{aligned}$$

□

REMARK. By [CCo6, Proposition 1.3], we find more exactly $c = |\pi|^{w(n)}$ with $w(n) := \sum_{i \geq 1} \lfloor n/q_{\mathbf{F}}^i \rfloor$.

For $a \in \mathbf{F}^d$ and $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}_{>0}^d$ let $\mathrm{B}_{\leq\delta}^{\bullet}(a) := \mathrm{B}_{\leq\delta}(a) - \{a\}$ be the *pointed polydisc* of radius δ around a.

Lemma 5.2. *There is a constant* $\mathrm{C} \geq 1$ *such that for every polynomial function* $p\colon \mathbf{F} \to \mathbf{K}$ *of degree at most n,*

$$\left\| \frac{|p(x+h) - p(x)|}{|h|^{\rho}} \right\|_{\mathrm{B}_{\leq\delta}(0) \times \mathrm{B}_{\leq\delta}^{\bullet}(0)} \leq \mathrm{C} \cdot 1/\delta^{\rho} \cdot \|p\|_{\mathrm{B}_{\leq\delta}(0)}.$$

PROOF: Write $p = \sum_{i=0,\ldots,n} a_i *^i$. We have

$$(x+h)^i - x^i = \sum_{j=0,\ldots,i-1} \binom{i}{j} x^j h^{i-j}$$

and thus

$$|p(x+h) - p(x)| \leq \max_{i=1,\ldots,n} |a_i| \cdot (\max_{j=0,\ldots,i-1} |x^j||h^{i-j}|),$$

yielding

$$\begin{aligned} \left\| \frac{|p(x+h) - p(x)|}{|h|^{\rho}} \right\|_{\mathrm{B}_{\leq\delta}(0) \times \mathrm{B}_{\leq\delta}^{\bullet}(0)} &\leq \max_{i=1,\ldots,n} |a_i| \delta^i / \delta^{\rho} \\ &= 1/\delta^{\rho} \cdot \max_{i=1,\ldots,n} |a_i| \| *^i \|_{\mathrm{B}_{\leq\delta}(0)}. \end{aligned}$$

By the preceding Lemma 5.1, there is a constant $\mathrm{C} \geq 1$ such that

$$\max_{i=1,\ldots,n} |a_i| \| *^i \|_{\mathrm{B}_{\leq\delta}(0)} \leq \max_{i=0,\ldots,n} |a_i| \| *^i \|_{\mathrm{B}_{\leq\delta}(0)} \leq \mathrm{C} \cdot \| \sum_{i=0,\ldots,n} a_i *^i \|_{\mathrm{B}_{\leq\delta}(0)}.$$

We conclude

$$\begin{aligned} \left\| \frac{|p(x+h) - p(x)|}{|h|^{\rho}} \right\|_{\mathrm{B}_{\leq\delta}(0) \times \mathrm{B}_{\leq\delta}^{\bullet}(0)} &\leq \mathrm{C} \cdot 1/\delta^{\rho} \cdot \| \sum_{i=0,\ldots,n} a_i *^i \|_{\mathrm{B}_{\leq\delta}(0)} \\ &= \mathrm{C} \cdot 1/\delta^{\rho} \cdot \|p\|_{\mathrm{B}_{\leq\delta}(0)}. \end{aligned}$$

□

Corollary 5.3. *There is a constant* $\mathrm{C} \geq 1$ *such that for every* δ*-polynomial function* $f\colon \mathbf{F} \to \mathbf{K}$ *of degree at most n of compact support,*

$$\left\| \frac{|f(x+h) - f(x)|}{|h|^{\rho}} \right\|_{\mathbf{F} \times \mathbf{F}^*} \leq \mathrm{C} \cdot 1/\delta^{\rho} \cdot \|f\|_{\sup}.$$

PROOF: We distinguish two cases: Firstly fix $|h| > \delta$. Then

$$\left\| \frac{|f(\cdot + h) - f|}{|h|^{\rho}} \right\|_{\mathbf{F}} < 1/\delta^{\rho} \cdot \|f\|_{\sup}.$$

Now let $h \in \mathbf{F}^*$ with $|h| \leq \delta$. Write $f = \sum_i \mathbf{1}_{\mathrm{B}_{\leq\delta}(x_i)} p_i$ with polynomial functions p_i. Since $|h| \leq \delta$,

$$\Big\|\frac{|f(x+h)-f(x)|}{|h|^\rho}\Big\|_{\sup} = \Big\|\frac{|\sum_i \mathbf{1}_{\mathrm{B}_{\leq\delta}(x_i)}[p_i(x+h)-p_i(x)]|}{|h|^\rho}\Big\|_{\sup}$$
$$\leq \max_i \Big\|\frac{|\mathbf{1}_{\mathrm{B}_{\leq\delta}(x_i)}[p_i(x+h)-p_i(x)]|}{|h^\rho|}\Big\|_{\sup}.$$

Because

$$\|f\|_{\sup} = \|\sum_i \mathbf{1}_{\mathrm{B}_{\leq\delta}(x_i)} p_i\|_{\sup} = \max_i \|\mathbf{1}_{\mathrm{B}_{\leq\delta}(x_i)} p_i\|_{\sup},$$

it suffices to prove

$$\Big\|\frac{|p_i(x+h)-p_i(x)|}{|h^\rho|}\Big\|_{\mathrm{B}_{\leq\delta}(x_i)\times\mathrm{B}^\bullet_{\leq\delta}(0)} \leq \mathrm{C}\cdot 1/\delta^\rho \cdot \|p_i\|_{\mathrm{B}_{\leq\delta}(x_i)}.$$

Let $q = p_i(\cdot + x_i)$. By the preceding Lemma 5.2, we obtain

$$\Big\|\frac{|p_i(x+h)-p_i(x)|}{|h^\rho|}\Big\|_{\mathrm{B}_{\leq\delta}(x_i)\times\mathrm{B}^\bullet_{\leq\delta}(0)} = \Big\|\frac{|q(x+h)-q(x)|}{|h|^\rho}\Big\|_{\mathrm{B}_{\leq\delta}(0)\times\mathrm{B}^\bullet_{\leq\delta}(0)}$$
$$\leq \mathrm{C}\cdot 1/\delta^\rho \cdot \|q\|_{\mathrm{B}_{\leq\delta}(0)}$$
$$= \mathrm{C}\cdot 1/\delta^\rho \cdot \|p_i\|_{\mathrm{B}_{\leq\delta}(x_i)}. \qquad \square$$

6. A norm of locally polynomial functions

Definition. Let $i \geq 0$ and $(h_1, \ldots, h_i) \in \mathbf{F}^i$. Then we define the $\mathbf{K}$-linear *iterated difference operator* $\Delta^i\_(\cdot; h_1, \ldots, h_i) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K})$ iteratively by $\Delta^0 f = f$ and

$$\Delta^{i+1} f(\cdot; h_1, \ldots, h_i, h_{i+1}) = \Delta^i f(\cdot + h_{i+1}; h_1, \ldots, h_i) - \Delta^i f(\cdot; h_1, \ldots, h_i).$$

Given a real number $r \geq 0$, we split it into

$$r = \nu + \rho$$

with

- an integral part $\nu := \lfloor r \rfloor$ in $\mathbb{N}$, and
- a fractional part $\rho := r - \nu$ in $[0, 1[$.

Definition. Let $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$.

- Let $\boldsymbol{h} = ({}^1\boldsymbol{h}; \ldots; {}^d\boldsymbol{h}) \in \mathbf{F}^{\nu_1} \times \cdots \times \mathbf{F}^{\nu_d}$. We define a $\mathbf{K}$-linear *iterated partial difference operator* $\Delta^\nu\_(\cdot; \boldsymbol{h}) \curvearrowright \mathscr{C}^{\mathrm{lc}}_{\mathrm{cp}}(\mathbf{F}^d, \mathbf{K})$ as follows: We have an isomorphism between $\mathbf{K}$-vector spaces

  $$\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K}) \otimes \cdots \otimes \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d, \mathbf{K}).$$

  Then we define

  $$\Delta^\nu\_(\cdot; \boldsymbol{h}) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K}) \otimes \cdots \otimes \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K})$$

by

$$\Delta^{\nu}\_(\cdot;\boldsymbol{h}) = \Delta^{\nu_1}\_(\cdot;{}^1\boldsymbol{h}) \otimes \cdots \otimes \Delta^{\nu_d}\_(\cdot;{}^d\boldsymbol{h}).$$

- Let henceforth $\mathbf{1} = (1,\dots,1) \in \mathbb{N}^d$. We put

$$\|f\|_{\boldsymbol{r}} = \sup_{\substack{x\in\mathbf{F}^d,\\ \boldsymbol{h}\in\mathbf{F}^{*\nu_1+1}\times\cdots\times\mathbf{F}^{*\nu_d+1}}} \frac{|\Delta^{\nu+\mathbf{1}}f(x;\boldsymbol{h})|}{\prod_{k=1,\dots,d}(|{}^k h_1|\cdots|{}^k h_{\nu_k}|\cdot|{}^k h_{\nu_k+1}|^{\rho_k})}.$$

For a tuple $\boldsymbol{n} \in \mathbb{N}^d$, let $|\boldsymbol{n}| := n_1 + \cdots + n_d \in \mathbb{N}$.

Remark 6.1. We define the operator $\Delta^{\nu}\_(\cdot;\boldsymbol{h}) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d,\mathbf{K})$ directly by recursion over $|\nu|$, suited for the proofs by induction on $|\nu|$ to come.

- We set $\Delta^{\mathbf{0}} = \mathrm{id}_{\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d,\mathbf{K})}$. Let $\nu^+ \in \mathbb{N}^d$ with $|\nu^+| \geq 1$, say $\nu^+ = \nu + \boldsymbol{e}_k$ and let $\boldsymbol{h}^+ \in \prod_{k=1,\dots,d}\mathbf{F}^{*\nu_k^+}$. Put

$$\boldsymbol{h} = ({}^l\boldsymbol{h})_{l=1,\dots,d} \quad \text{with} \quad {}^l\boldsymbol{h} = \begin{cases} ({}^l h_1^+,\dots,{}^l h_{\nu_l}^+), & \text{if } l = k,\\ {}^l\boldsymbol{h}^+, & \text{otherwise.}\end{cases}$$

Then

$$\Delta^{\nu^+}f(\cdot,\boldsymbol{h}^+) = \Delta^{\nu}f(\cdot + {}^k h_{\nu_k+1}\cdot\boldsymbol{e}_k,\boldsymbol{h}) - \Delta^{\nu}f(\cdot,\boldsymbol{h}).$$

We notice that $\Delta^{\nu}f(\cdot,\boldsymbol{h})/\prod_{k=1,\dots,d}{}^k h_1\cdots{}^k h_{\nu_k} \in \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d,\mathbf{K})$ can likewise be given by the iterated difference operator $\_^{]\nu[}(\cdot,\boldsymbol{h}) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d,\mathbf{K})$ defined recursively over $|\nu|$ by $f^{]\mathbf{0}[} = f$, and for $\nu^+ \in \mathbb{N}^d$ with $|\nu^+| = |\nu| + 1$, say $\nu^+ = \nu + \boldsymbol{e}_k$, we put

$$f^{]\nu^+[}(\cdot,\boldsymbol{h}^+) = \frac{f^{]\nu[}(\cdot + {}^k h_{\nu_k+1}\cdot\boldsymbol{e}_k,\boldsymbol{h}) - f^{]\nu[}(\cdot,\boldsymbol{h})}{{}^k h_{\nu_k+1}}.$$

- Let $\boldsymbol{h} \in \mathbf{F}^{*\nu_1+1}\times\cdots\times\mathbf{F}^{*\nu_d+1}$. Then

$$\frac{|\Delta^{\nu+\mathbf{1}}f(x;\boldsymbol{h})|}{\prod_{k=1,\dots,d}(|{}^k h_1|\cdots|{}^k h_{\nu_k}|\cdot|{}^k h_{\nu_k+1}|^{\rho_k})} = \frac{|\Delta^{\mathbf{1}}\mathrm{F}(x,({}^1 h_{\nu_1+1},\dots,{}^d h_{\nu_d+1}))|}{|{}^1 h_{\nu_1+1}|^{\rho_1}\cdots|{}^d h_{\nu_d+1}|^{\rho_d}}$$

with $\mathrm{F} = f^{]\nu[}(\cdot,\check{\boldsymbol{h}})$ and $\check{\boldsymbol{h}} = ({}^1 h_1,\dots,{}^1 h_{\nu_1};\dots;{}^d h_d,\dots,{}^d h_{\nu_d})$.

For the remainder of this interlude, we fix $n \in \mathbb{N}$ and denote by $\mathrm{C} \geq 1$ the corresponding constant appearing in the formulation of Corollary 5.3.

Lemma 6.2. *For every δ-polynomial function $f\colon \mathbf{F}^d \to \mathbf{K}$ of compact support of degree at most $\boldsymbol{n} = (n,\dots,n)$,*

$$\|f^{]\nu[}\|_{\sup} \leq \mathrm{C}^{|\nu|}/\delta_1^{\nu_1}\cdots\delta_d^{\nu_d}\cdot\|f\|_{\sup}.$$

Proof: This is proved by induction on $|\nu|$. In case $|\nu| = 0$, there is nothing to prove. Let $|\nu^+| \geq 1$, so that we can write $\nu^+ = \nu + \boldsymbol{e}_k$ for some coordinate

$k \in \{1, \ldots, d\}$. For notational convenience, assume $k = 1$. Let $x \in \mathbf{F}^d$ and $\boldsymbol{h}^+ \in \prod_{k=1,\ldots,d} \mathbf{F}^{*\nu_k}$. Put

$$\boldsymbol{h} = ({}^l\boldsymbol{h})_{l=1,\ldots,d} \quad \text{with} \quad {}^l\boldsymbol{h} = \begin{cases} ({}^l h_1^+, \ldots, {}^l h_{\nu_l}^+), & \text{if } l = 1, \\ {}^l\boldsymbol{h}^+, & \text{otherwise.} \end{cases}$$

Then

$$f^{]\nu^+[}(x, \boldsymbol{h}^+) = \frac{\mathrm{F}(x_1) - \mathrm{F}(x_1 + {}^k h_{\nu_1+1})}{{}^k h_{\nu_1+1}} \quad \text{with} \quad \mathrm{F}(x_1) := f^{]\nu[}(x, \boldsymbol{h}).$$

We fix any $(\cdot, x_2, \ldots, x_d) \in \mathbf{F}^{d-1}$ and $\boldsymbol{h} \in \prod_{k=1,\ldots,d} \mathbf{F}^{*\nu_k}$. Then the above defined function $\mathrm{F}\colon \mathbf{F} \to \mathbf{K}$ is given by $\mathrm{F} := f^{]\nu[}((\cdot, x_2, \ldots, x_d), \boldsymbol{h})$. It is a δ_1-polynomial function. By Corollary 5.3 therefore

$$(*) \qquad \left|\left|\left| \frac{\mathrm{F}(x) - \mathrm{F}(x+h)}{h} \right|\right|\right|_{\mathbf{F}\times\mathbf{F}^*} = \left\| \frac{|\mathrm{F}(x) - \mathrm{F}(x+h)|}{|h|} \right\|_{\mathbf{F}\times\mathbf{F}^*} \leq \mathrm{C}/\delta_k \cdot \|\mathrm{F}\|_{\sup}.$$

Since the above inequality $(*)$ holds for any choice of $(\cdot, x_2, \ldots, x_d)$ and $\boldsymbol{h} \in \prod_{k=1,\ldots,d} \mathbf{F}^{*\nu_k}$, we find

$$\|f^{]\nu^+[}\| \leq \mathrm{C}/\delta_1 \cdot \|f^{]\nu[}\| \leq \mathrm{C}^{|\nu^+|}/\delta_1^{\nu_1^+} \cdots \delta_d^{\nu_d^+},$$

the last inequality by the induction hypothesis. □

Lemma 6.3. *Let $f\colon \mathbf{F}^d \to \mathbf{K}$ be δ-polynomial of compact support of degree at most $\boldsymbol{n} = (n, \ldots, n)$. Then*

$$\|f\|_\rho = \sup_{\substack{x\in\mathbf{F}^d,\\ \boldsymbol{h}\in\mathbf{F}^*\times\cdots\times\mathbf{F}^*}} \frac{|\Delta^{\mathbf{1}} f(x; \boldsymbol{h})|}{|h_1|^{\rho_1} \cdots |h_d|^{\rho_d}} \leq \mathrm{C}^d/\delta_1^{\rho_1} \cdots \delta_d^{\rho_d} \cdot \|f\|_{\sup}.$$

Proof: For $\rho \in v(\mathbf{K}^*)$, let $*^\rho\colon \mathbf{F} \to \mathbf{K}$ be given by $x^\rho = a^{v_{\mathbf{F}}(x)}$ for any $a \in \mathbf{K}$ with $v(a) = \rho$. Then we define the operator $\_^{]\rho[}(\cdot, \boldsymbol{h}) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d, \mathbf{K})$ for $\boldsymbol{h} \in \mathbf{F}^d$ by

$$f^{]\rho[}(\cdot, \boldsymbol{h}) = \frac{\Delta^{\mathbf{1}} f(\cdot, \boldsymbol{h})}{h_1^{\rho_1} \cdots h_d^{\rho_d}},$$

so that

$$\frac{|\Delta^{\mathbf{1}} f(x; \boldsymbol{h})|}{|h_1|^{\rho_1} \cdots |h_d|^{\rho_d}} = |f^{]\rho[}(x, \boldsymbol{h})|.$$

For $\mathrm{I} \subseteq \{1, \ldots, d\}$ and $\boldsymbol{h} \in \mathbf{F}^{*\mathrm{I}}$, let us define $\__{\mathrm{I}}^{]\rho[}(\cdot; \boldsymbol{h}) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d, \mathbf{K})$ over $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}^d, \mathbf{K}) = \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K}) \otimes \cdots \otimes \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K})$ by

$$\__{\mathrm{I}}^{]\rho[}(\cdot, \boldsymbol{h}) = \bigotimes_{k\in\mathrm{I}} \_^{]\rho_k[}(\cdot, h_k) \otimes \bigotimes_{k\in\{1,\ldots,d\}-\mathrm{I}} \mathrm{id}_{\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F},\mathbf{K})}$$

with $\_^{]\rho[}(\cdot, h) \curvearrowright \mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K})$ defined by

$$f^{]\rho[}(\cdot, h) = \frac{f(\cdot + h) - f}{h^\rho}.$$

Then in particular $f^{]\rho[}(x,\boldsymbol{h}) = \mathrm{F}^{]\rho_1[}(x_1,h_1)$ with

$$\mathrm{F} := f^{]\rho[}_{\{2,\dots,d\}}((\cdot,x_2,\dots,x_d),(h_2,\dots,h_d)).$$

By induction on #I — the starting case #I = 0 holding true by definition — we may assume $\|\mathrm{F}\|_{\mathbf{F}} \le \mathrm{C}^{d-1}/\delta_2^{\rho_2}\cdots\delta_d^{\rho_d}\|f\|_{\sup}$. Then F is a δ_1-polynomial function in x_1 and so together with Corollary 5.3,

$$\|\mathrm{F}^{]\rho_1[}\|_{\mathbf{F}\times\mathbf{F}^*} \le \mathrm{C}/\delta_1^{\rho_1}\cdot\|\mathrm{F}\|_{\mathbf{F}} \le \mathrm{C}^d/\delta_1^{\rho_1}\cdots\delta_d^{\rho_d}\cdot\|f\|_{\sup}.$$

Because this holds for every $(\cdot,x_2,\dots,x_d)$ and $(h_2,\dots,h_d)$ in the definition of F, we conclude

$$\|f^{]\rho[}(x,\boldsymbol{h})\|_{\mathbf{F}^d\times\mathbf{F}^{*d}} \le \mathrm{C}^d/\delta_1^{\rho_1}\cdots\delta_d^{\rho_d}\cdot\|f\|_{\sup}.$$

Proposition 6.4. *Let $f\colon \mathbf{F}^d \to \mathbf{K}$ be a δ-polynomial function of compact support of degree at most $\boldsymbol{n} = (n,\dots,n)$. Then*

$$\|f\|_{\boldsymbol{r}} \le \mathrm{C}^{|\nu+\mathbf{1}|}/\delta_1^{r_1}\cdots\delta_d^{r_d}\cdot\|f\|_{\sup}.$$

Proof: By definition, we have

$$\|f\|_{\boldsymbol{r}} = \sup_{\substack{x\in\mathbf{F}^d,\\ \boldsymbol{h}\in\mathbf{F}^{*\nu_1+1}\times\cdots\times\mathbf{F}^{*\nu_d+1}}} \frac{|\Delta^{\nu+\mathbf{1}}f(x;\boldsymbol{h})|}{\prod_{k=1,\dots,d}(|^kh_1|\cdots|^kh_{\nu_k}|\cdot|^kh_{\nu_k+1}|^{\rho_k})}.$$

Let $\boldsymbol{h}\in\mathbf{F}^{*\nu_1+1}\times\cdots\times\mathbf{F}^{*\nu_d+1}$. By Remark 6.1, we have

$$\frac{|\Delta^{\nu+\mathbf{1}}f(x;\boldsymbol{h})|}{\prod_{k=1,\dots,d}(|^kh_1|\cdots|^kh_{\nu_k}|\cdot|^kh_{\nu_k+1}|^{\rho_k})} = \frac{|\Delta^{\mathbf{1}}\mathrm{F}_{\check{\boldsymbol{h}}}(x,(^1h_{\nu_1+1},\dots,{}^dh_{\nu_d+1}))|}{|^1h_{\nu_1+1}|^{\rho_1}\cdots|^dh_{\nu_d+1}|^{\rho_d}}$$

with $\mathrm{F}_{\check{\boldsymbol{h}}} = f^{]\nu[}(\cdot,\check{\boldsymbol{h}})$ and $\check{\boldsymbol{h}} = (^1h_1,\dots,{}^1h_{\nu_1};\dots;{}^dh_d,\dots,{}^dh_{\nu_d})\in\mathbf{F}^{*\nu_1}\times\cdots\times\mathbf{F}^{*\nu_d}$. By Lemma 6.3,

$$\frac{|\Delta^{\mathbf{1}}\mathrm{F}_{\check{\boldsymbol{h}}}(x,(^1h_{\nu_1+1},\dots,{}^dh_{\nu_d+1}))|}{|^1h_{\nu_1+1}|^{\rho_1}\cdots|^dh_{\nu_d+1}|^{\rho_d}} \le \mathrm{C}^d/\delta_1^{\rho_1}\cdots\delta_d^{\rho_d}\cdot\|\mathrm{F}_{\check{\boldsymbol{h}}}\|_{\sup}.$$

By Lemma 6.2, we have

$$\|\mathrm{F}_{\check{\boldsymbol{h}}}\|_{\sup} \le \|f^{]\nu[}(x,\check{\boldsymbol{h}})\|_{\mathbf{F}^d\times(\mathbf{F}^{*\nu_1}\times\cdots\times\mathbf{F}^{*\nu_d})} \le \mathrm{C}^{|\nu|}/\delta_1^{\nu_1}\cdots\delta_d^{\nu_d}\|f\|_{\sup}.$$

Because $\check{\boldsymbol{h}}\in\mathbf{F}^{*\nu_1}\times\cdots\times\mathbf{F}^{*\nu_d}$ was arbitrary, we can conclude

$$\|f\|_{\boldsymbol{r}} \le \mathrm{C}^{|\nu+\mathbf{1}|}/\delta_1^{r_1}\cdots\delta_d^{r_d}\cdot\|f\|_{\sup}.$$

Part 3. Construction of the greatest unitary norm on the open cell

Let $\mathbf{G}$ be a split connected reductive group over a p-adic number field $\mathbf{F}$. Every root group N_α for α in Φ is as valued group canonically isomorphic to $\mathbf{F}$; let $\chi_\alpha\colon \mathbf{F}\xrightarrow{\sim}\mathrm{N}_\alpha$ denote this canonical isomorphism. Let $\mathrm{N}(a)$ be the compact open subgroup of N_α defined by $\chi_\alpha(\mathbf{o_F})$ and, for i in $\mathbb{Z}$, let $\mathrm{N}(a+i)$ be the compact open subgroup of N_α given by $\chi_\alpha(v_{\mathbf{F}}^{-1}([i,\infty]))$. The conjugation action of M on N_α is under this isomorphism given by multiplication with $\alpha(m)$, and in particular ${}^m N(a) = N(a + v_{\mathbf{F}}(\alpha(m)))$.

Let N_0 be the compact open subgroup of N given by $N_0 = \prod_{\alpha\in\Phi^+} N(a)$.

Lemma 6.5. *Let $m \in M$. Then ${}^m N_0 \subseteq N_0$ if and only if $v(m)(\alpha) \geq 0$ for all $\alpha \in \Delta$.*

Proof: Let m in M and $\alpha \in \Phi$. Because ${}^m N(a) = N(a+\alpha(m))$, conjugation by m stabilizes $N(a)$ if and only if $v_{\mathbf{F}}(\alpha(m)) \geq 0$. The proposition follows, because

- m stabilizes N_0 if and only if it stabilizes each factor $N(a)$, and
- $\alpha(v(m)) \geq 0$ for all $\alpha \in \Delta$ if and only if $\alpha(m) \geq 0$ for all $\alpha \in \Phi^+$. □

Definition. Let $M^+ := \{m \in M : {}^m N_0 \subseteq N_0\}$.

Let $\chi = \theta\psi \colon M \to \mathbf{K}^*$ be a locally constant dominant character and $I(\chi) = \mathrm{Ind}_{\bar{P}}^{G} \theta^{lc} \otimes_{\mathbf{K}} U_\psi$. There is a monomorphism of $\mathbf{K}[P]$-modules

$$I(\theta)(N) \otimes U_\psi \xrightarrow{\sim} \mathscr{C}^{lc}_{cp}(N, \mathbf{K}) \otimes U_\psi \hookrightarrow \mathscr{C}^{lc}_{cp}(N, \mathbf{K}) \otimes \mathscr{C}^{alg}(N, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}^{la}_{cp}(N, \mathbf{K})$$

where :

1. The first isomorphism is that of Lemma 3.1 that becomes a morphism of $\mathbf{K}[P]$-modules by defining P to act on the right-hand side by $f^p = \psi(m) f(\cdot^m n)$ for every $p = nm$ with n in N and m in M.
2. The middle monomorphism is the restriction morphism between $\mathbf{K}$-vector spaces
$$\begin{aligned} \mathrm{Ind}_{\bar{P}}^{G}(\psi)^{alg} &\hookrightarrow \mathscr{C}^{alg}(N, \mathbf{K}) \\ f &\mapsto f_{\restriction N}. \end{aligned}$$
It is injective because $\bar{P}N$ is Zariski-dense inside G by [Bor91, Corollary 14.14 and Theorem 21.20].
3. The last morphism is given by $f \otimes g \mapsto [(x, y) \mapsto f(x)g(y)]$. It is bijective because the variety $\mathbf{N}$ is an affine space and by the Taylor polynomial expansion, every polynomial function on N is uniquely determined on an open subset.

Let $\mathscr{C}^{\psi-la}_{cp}(N, \mathbf{K})$ denote the image of this monomorphism, explicitly given by

$$\begin{aligned} \mathscr{C}^{\psi-la}_{cp}(N, \mathbf{K}) := \{ f \colon N \to \mathbf{K} \text{ of compact support } : \text{ For all } n \in N \text{ exists open} \\ U \ni n \text{ inside N such that } f_{|U} = p_{|U} \text{ for some } p \in \mathrm{Ind}_{\bar{P}}^{G}(\psi)^{alg} \}. \end{aligned}$$

We conclude that there is an isomorphism between $\mathbf{K}[P]$-modules

$$I(\chi)(N) \xrightarrow{\sim} \mathscr{C}^{\psi-la}_{cp}(N, \mathbf{K}),$$

where the right-hand side is endowed with the P-action by $f^n = (\cdot n)$ for $n \in N$ and $f^m = \chi(m) \cdot f(\cdot^m)$ for $m \in M$. This isomorphism maps $\phi_1 \otimes u$ to $\mathbf{1}_{N_0} \otimes u_{|N}$ for every u in U_ψ.

7. Necessity

Proposition 7.1. *There is unitary norm on* $\mathscr{C}_{\mathrm{cp}}^{\psi-\mathrm{la}}(\mathrm{N}, \mathbf{K})$ *only if* $|\chi(m)| \leq 1$ *for all* m *in* M^+.

PROOF BY CONTRAPOSITION: If u is a highest weight vector of U_ψ then $u_{\restriction \mathrm{N}} = 1$. Thus $f_0 = \mathbf{1}_{\mathrm{N}_0}$ in $\mathscr{C}_{\mathrm{cp}}^{\psi-\mathrm{la}}(\mathrm{N}, \mathbf{K})$. Let m in M and n in N. Because $mf = \chi(m) 1_{{}^m\mathrm{N}_0}$ and $nf_0 = f(\cdot n)$, we compute

$$f_0 = \mathbf{1}_{\mathrm{N}_0} = \sum_{n \in \mathrm{N}_0 / {}^m\mathrm{N}_0} \mathbf{1}_{{}^m\mathrm{N}_0 n} = 1/\chi(t) \sum_{n \in \mathrm{N}_0 / {}^m\mathrm{N}_0} mnf_0;$$

hence if $\|\cdot\|$ is a unitary seminorm on $\mathscr{C}_{\mathrm{cp}}^{\psi-\mathrm{la}}(\mathrm{N}, \mathbf{K})$, then

$$\|f_0\| \leq |1/\chi(m)| \max\{\|mnf_0\|\} \leq |1/\chi(m)| \|f_0\|.$$

We conclude that if there is m in M^+ such that $|\chi(m)| > 1$, then $\|f_0\| = 0$. □

8. Sufficiency

Let $\mathrm{X}^*(\mathbf{M}/\mathbf{Z}) = \{$ all algebraic group homomorphisms $\chi \colon \mathbf{M}/\mathbf{Z} \to \mathbb{G}_{\mathrm{m}}\}$. Let M_0 be the maximal compact open subgroup of M and put $\Lambda := \mathrm{M}/\mathrm{M}_0\mathrm{Z}$. By the group isomorphism (cf. [**Car79**, Section 3.2])

$$\Lambda \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}^*(\mathbf{M}/\mathbf{Z}), \mathbb{Z})$$
$$\lambda \mapsto [\gamma \mapsto v(\gamma(\lambda))]$$

we identify Λ with all additive mappings $\lambda \colon \mathrm{X}^*(\mathbf{M}/\mathbf{Z}) \to \mathbb{Z}$.

Proposition 8.1. *There is a unique basis* $\{\lambda_\alpha : \alpha$ *in* $\Delta\}$ *in* Λ *that is orthonormal to* Δ *in* $\mathrm{X}^*(\mathbf{M}/\mathbf{Z})$ *with respect to the natural pairing between* $\mathrm{X}^*(\mathbf{M}/\mathbf{Z})$ *and* $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}^*(\mathbf{M}/\mathbf{Z}), \mathbb{Z})$.

PROOF: We may assume $\mathbf{G} = \mathbf{G}/\mathbf{Z}$. Thus, because $\mathbf{G}$ is reductive, it is semi-simple, and because $\ker \mathrm{Ad} = \mathbf{Z}$, moreover $\mathbf{G}$ adjoint ([**Bor91**, Section 3.15]).

Thus by semi-simplicity of $\mathbf{G}$, the root basis Δ spans a finite free $\mathbb{Z}$-lattice Q inside $\mathrm{X}^*(\mathbf{M})$ ([**Bor66**, Section 6.5(2)]), and because $\mathbf{G}$ is adjoint, $\mathrm{Q} = \mathrm{X}^*(\mathbf{M})$. We conclude that Δ is a basis of the free $\mathbb{Z}$-module $\mathrm{X}^*(\mathbf{M})$.

The orthonormal basis $\{\lambda_\alpha : \alpha$ in $\Delta\}$ in Λ is the basis which identifies with that in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}^*(\mathbf{M}/\mathbf{Z}), \mathbb{Z})$ dual to Δ. □

Corollary 8.2. *Let* $\Lambda^+ := \mathrm{M}^+/\mathrm{M}_0\mathrm{Z}$. *Then* $\Lambda^+ = \bigoplus_{\alpha \in \Delta} \mathbb{N}\lambda_\alpha$.

PROOF: By orthogonality of $\{\lambda_\alpha\}$ with respect to Δ and Lemma 6.5. □

We assume that the locally constant dominant character $\chi \colon \mathrm{M} \to \mathbf{K}^*$ fulfills the condition of Proposition 7.1, that is,

$$|\chi(m)| \leq 1 \quad \text{for all } m \in \mathrm{M}^+. \tag{8.1}$$

In particular $|\chi(z)| = 1$ for all z in the center Z.

Let us assume that $\mathrm{N}(a)$ identifies with $\mathbf{o_F}$ under the canonical isomorphism between topological groups $\mathbf{F} \xrightarrow{\sim} \mathrm{N}_\alpha$. The conjugation action of M on $\mathrm{N}(a)$ is under this isomorphism given by

$$^{m}\mathbf{o_F} = \pi^{\alpha(m)} \cdot \mathbf{o_F}.$$

Definition. Define $\boldsymbol{r} \in \mathbb{R}_{\geq 0}^{\Phi^+}$ by

$$r_\alpha := \begin{cases} v_{\mathbf{K}}(\chi(\lambda_\alpha)) & \text{if } \alpha \in \Delta, \\ 0 & \text{otherwise,} \end{cases}$$

where λ_α for every α in Δ as in Proposition 8.1. This is well-defined because $|\chi|$ is by Equation (8.1) $|\chi|$ trivial on Z and $|\chi| = |\psi||\theta|$ is trivial on M_0 as

- $|\psi|$ is trivial on M_0, and
- θ is as locally constant character trivial on a compact open subgroup of finite index in M_0, thence $|\theta|$ is trivial on M_0.

The canonical isomorphism between affine algebraic varieties

$$\mathbf{N} \xrightarrow{\sim} \prod_{\alpha\in\Phi^+} \mathbf{N}_\alpha \xrightarrow{\sim} \mathbf{F}^{\Phi^+}$$

gives rise to an injection of $\mathbf{K}$-vector spaces

$$\iota\colon\ \mathscr{C}_{\mathrm{cp}}^{\psi-\mathrm{la}}(\mathrm{N}, \mathbf{K}) \hookrightarrow \mathscr{C}_{\mathrm{cp}}^{\mathrm{la}}(\mathrm{N}, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}_{\mathrm{cp}}^{\mathrm{la}}(\mathbf{F}^{\Phi^+}, \mathbf{K})$$

where $\mathscr{C}_{\mathrm{cp}}^{\mathrm{la}}(\mathrm{N}, \mathbf{K})$ are all locally algebraic functions $f\colon \mathrm{N} \to \mathbf{K}$ of compact support. We define a norm $\|\cdot\|$ on $\mathscr{C}_{\mathrm{cp}}^{\psi-\mathrm{la}}(\mathrm{N}, \mathbf{K})$ by

$$\|\cdot\| := \|\iota(\cdot)\|_{\boldsymbol{r}}.$$

Let us make $\|\cdot\|$ explicit. Given a real number $r \geq 0$, split it into $r = \nu + \rho$ with an integral part $\nu := \lfloor r \rfloor$ in $\mathbb{N}$ and a fractional part $\rho := r - \nu$ in $[0, 1[$. For every α in Φ^+ put $\nu_\alpha = \lfloor r_\alpha \rfloor$, and for every $\boldsymbol{n}_\alpha^*$ in $\mathbf{F}^{*\nu_\alpha}$ define the operator

$$\varDelta^{\nu}\_(\cdot; (\boldsymbol{n}_\alpha^*)) \curvearrowright \bigotimes_{\alpha\in\Phi^+} \mathscr{C}^{\mathrm{la}}(\mathbf{F}, \mathbf{K})$$

by

$$\varDelta^{\nu}\_(\cdot; (\boldsymbol{n}_\alpha^*)_{\alpha\in\Delta}) = \bigotimes_{\alpha\in\Phi^+} \varDelta^{\nu_\alpha}\_(\cdot; \boldsymbol{n}_\alpha^*).$$

Then, with $\mathbf{1} := (1, \ldots, 1)$ in $\mathbb{N}^{\Phi^+}$,

$$\|f\| := \sup_{n\in\mathrm{N}, (\boldsymbol{n}_\alpha^*)\in\prod_{\alpha\in\Phi^+} \mathbf{F}^{*\nu_\alpha+1}} \frac{|\varDelta^{\nu+\mathbf{1}} f(n; (\boldsymbol{n}_\alpha^*))|}{\prod_{\alpha\in\Delta}(|n_{\alpha,1}^*| \cdots |n_{\alpha,\nu_\alpha}^*| \cdot |n_{\alpha,\nu_\alpha+1}^*|^{\rho_\alpha})}.$$

Lemma 8.3. *The norm $\|\cdot\|$ satisfies the following two conditions:*

(i) It is invariant under translation by N.
(ii) There is a constant $\mathrm{C} \geq 1$ *such that* $\|\mathbf{1}_{^m\mathrm{N}_0} \otimes \bar{u}\| \leq \mathrm{C} \cdot 1/|\theta\bar{\psi}(m)|$ *for all* $m \in \mathrm{M}$.

PROOF:

Ad (i): Because $\|\cdot\|$ is a supremum over all of N.

Ad (ii): For every $\alpha \in \Phi^+$, we may assume the algebraic isomorphism between groups $N_\alpha \xrightarrow{\sim} \mathbf{F}$ to be chosen such that $N(a) \xrightarrow{\sim} \mathbf{o_F}$, where we let $a \in \Phi_0^+$ correspond to $\alpha \in \Phi^+$. We have ${}^m N_0 = \prod_{\alpha \in \Phi^+} N(a+ < \alpha, v(m) >)$ and $N(a+ < \alpha, v(m) >) \xrightarrow{\sim} \pi^{<\alpha, v(m)>} \cdot o_{\mathbf{F}}$. Therefore $\iota(\mathbf{1}_{N_0} \otimes \bar{u}) : \mathbf{F}^{\Phi^+} \xrightarrow{\sim} N \to \mathbf{K}$ is a δ-polynomial function with $\delta = (\delta_\alpha) \in \mathbb{R}_{>0}^{\Phi^+}$ given by $\delta_\alpha = |\pi|_{\mathbf{F}}^{<\alpha, v(m)>}$. Because U_ψ is a finite-dimensional $\mathbf{K}$-vector space, there is by Proposition 6.4 a constant $\tilde{C}$ such that

$$\|\mathbf{1}_{{}^m N_0} \otimes \bar{u}\| = \|\iota(\mathbf{1}_{{}^m N_0} \otimes \bar{u})\|_{\boldsymbol{r}} \leq \tilde{C} / \prod_{\alpha \in \Delta} \delta_\alpha^{r_\alpha} \cdot \|\iota(\mathbf{1}_{{}^m N_0} \otimes \bar{u})\|_{\sup}.$$

First write $m = \sum_{\alpha \in \Delta} i_\alpha \cdot m_\alpha$ with $i_\alpha \in \mathbb{Z}_{\geq 0}$. Then for every $\alpha \in \Delta$, we have $< \alpha, v(m) >= i_\alpha$. Therefore

$$\delta_\alpha^{r_\alpha} = |\pi|_{\mathbf{F}}^{i_\alpha \cdot r_\alpha} = |\pi|_{\mathbf{F}}^{i_\alpha \cdot v_{\mathbf{K}}(\chi(m_\alpha))}$$

where the last equality holds by definition of r_α. This gives

$$\begin{aligned} \prod_{\alpha \in \Delta} \delta_\alpha^{r_\alpha} &= |\pi|_{\mathbf{F}}^{\sum_{\alpha \in \Delta} i_\alpha \cdot v_{\mathbf{K}}(\chi(m_\alpha))} \\ &= |\pi|_{\mathbf{F}}^{v_{\mathbf{K}}(\chi(\sum_{\alpha \in \Delta} i_\alpha \cdot m_\alpha))} \\ &= |\pi|_{\mathbf{F}}^{v_{\mathbf{K}}(\chi(m))} \\ &= c_{\mathbf{K}}^{v_{\mathbf{K}}(\chi(m))} = |\chi(m)|_{\mathbf{K}}. \end{aligned}$$

Second,

$$\|\mathbf{1}_{{}^m N_0} \otimes \bar{u}\|_{\sup} = \|\bar{u}\|_{{}^m N_0} = \|\bar{u}({}^m \cdot)\|_{N_0} = |[\psi - \bar{\psi}](m)| \cdot \|\bar{u}\|_{N_0}.$$

Together,

$$\begin{aligned} \|\mathbf{1}_{{}^m N_0} \otimes \bar{u}\| &\leq \tilde{C} \cdot |\chi(m)|_{\mathbf{K}}^{-1} |[\psi - \bar{\psi}](m)| \cdot \|\bar{u}\|_{N_0} \\ &= \tilde{C} \cdot |\chi(m)|_{\mathbf{K}}^{-1} |[\psi - \bar{\psi}](m)| \cdot \|\mathbf{1}_{N_0} \otimes \bar{u}\|_{\sup}. \end{aligned}$$

We have $\chi = \theta\psi$ and therefore $\chi^{-1}[\psi - \bar{\psi}] = 1/\theta\bar{\psi}$. This gives

$$\|\mathbf{1}_{{}^m N_0} \otimes \bar{u}\| \leq \tilde{C} \cdot 1/|\theta\bar{\psi}(m)| \cdot \|\mathbf{1}_{N_0} \otimes \bar{u}\|_{\sup}.$$

We conclude $\|\mathbf{1}_{{}^m N_0} \otimes \bar{u}\| \leq C \cdot 1/|\theta\bar{\psi}(m)|$ with $C = \tilde{C} \cdot \|\mathbf{1}_{N_0} \otimes \bar{u}\|_{\sup}$. □

Corollary 8.4. *There is a unitary norm on the* P*-representation* I(χ)(N).

PROOF: Let $f_0 = \mathbf{1}_{N_0} \otimes \bar{u}$. By Corollary 4.2 there is a unitary norm on the P-representation I(χ)(N) if and only if there is a norm $\|\cdot\|$ on I(χ)(N) such that there is a constant $C > 0$ for which $\|p f_0\| \leq C$ for all p in P.

By the explicit description of the action of $P = MN$ on I(χ)(N) in Lemma 3.2 (i), this is asserted by Lemma 8.3. □

9. The example GL_2

Let $\mathbf{F} = \mathbb{Q}_p$ and $n = 2$, that is, $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $\psi\colon T \to \mathbf{K}^*$ be the dominant algebraic character given as the tensor product $\psi_1 \otimes \psi_2$ of the characters $\psi_1 = \cdot^{l+k}$ and $\psi_2 = \cdot^{l}$ on $\mathbb{Q}_p^*$ for l, k in $\mathbb{Z}$ with $k \geq 0$; and let $\theta = \theta_1 \otimes \theta_2 \colon T \to \mathbf{K}^*$ be an unramified algebraic character given as the tensor product $\theta_1 \otimes \theta_2$ of the characters θ_1 and θ_2 on $\mathbb{Q}_p^*$. Put $\chi = \theta\psi$. Let

$$\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Q}_p, \mathbf{K}) :=\{ \text{ all locally polynomial functions } f\colon \mathbb{Q}_p \to \mathbf{K} \text{ of degree} \leq k \text{ with compact support } \}.$$

Proposition 9.1. *If* $N = \mathbb{Q}_p$ *then as* $\mathbf{K}[P]$*-modules*

$$I(\chi)(N) = \mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Q}_p, \mathbf{K})$$

where the P*-action on the right-hand side is given by* $f^t = \chi(t) f(d/a \cdot \_)$ *for all* $t = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) \in T$ *and* $f^n = f(\cdot + n)$ *for* $n \in N$.

Proof: The unique irreducible algebraic representation $I(\psi)^{\mathrm{alg}}$ of highest weight ψ has a basis of functions $f\colon \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbf{K}$ given by k-fold products of the coordinate functions in the upper row and the determinant function. These functions are given on N by the monomial functions $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & X^k \\ & 1 \end{smallmatrix}\right)$. □

Let $r := v(\chi(\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right))) = v(\chi_1(p))$. By Proposition 7.1, there is a P-invariant norm on $I(\chi)(N)$ only if $r \geq 0$ and $|\chi(Z)| = 1$. By Lemma 4.1, the universal unitary lattice inside $I(\chi)(N)$ is generated by $1_{\mathbb{Z}_p} x^k$. Thus, by Corollary 4.2, the universal unitary norm is the greatest (up to equivalence) norm $\|\cdot\|$ on $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Q}_p, \mathbf{K})$ that

- is invariant under translation, and such that
- there is a constant $C > 0$ such that $\|1_{p^n \mathbb{Z}_p} x^k\| \leq C \cdot p^{(r-k)n}$ for all $n \in \mathbb{Z}$.

The norm $\|\cdot\|_r$ of Part 3 is given as follows:

Definition. Let f in $\mathscr{C}^{\mathrm{la}}_{\mathrm{cp}}(\mathbf{F}, \mathbf{K})$ and $h_1, \ldots, h_\nu \in \mathbf{F}$. The ν*-th iterated difference quotient* $\Delta^\nu f(\cdot; h_1, \ldots, h_\nu)\colon \mathbf{F} \to \mathbf{K}$ of f is given iteratively by $\Delta^0 f = f$ and

$$\Delta^{\nu+1} f(\cdot; h_1, \ldots, h_\nu, h_{\nu+1}) = \Delta^\nu f(\cdot + h_{\nu+1}; h_1, \ldots, h_\nu) - \Delta^\nu f(\cdot; h_1, \ldots, h_\nu).$$

Definition. We put

$$\|f\|_r = \sup_{x \in \mathbf{F},\ h_1, \ldots, h_{\nu+1} \in \mathbf{F}^*} \frac{|\Delta^{\nu+1} f(x; h)|}{|h_1| \cdots |h_\nu| \cdot |h_{\nu+1}|^\rho}.$$

If the functions in $\mathscr{C}^{\mathrm{la}\leq k}_{\mathrm{cp}}(\mathbb{Q}_p, \mathbf{K})$ whose support is included in $\mathbb{Z}_p$ are by restriction identified with $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p, \mathbf{K})$, then under this restriction $\|\cdot\|_r$ becomes

$$\|f\|_{\mathscr{C}^r} := \max\{\|f\|^0_{\sup}, \|f\|^0_1, \ldots, \|f\|^0_\nu, \|f\|^0_r\}$$

where

$$\|f\|_r = \sup_{x\in\mathbb{Z}_p,\ h_1,\dots,h_\nu,h_{\nu+1}\in\mathbb{Z}_p-\{0\}} \frac{|\Delta^{\nu+1} f(x;h_1,\dots,h_\nu,h_{\nu+1})|}{|h_1|\cdots|h_\nu|\cdot|h_{\nu+1}|^\rho}.$$

If $r=\nu$ in $\mathbb{N}$ then $\|\cdot\|_{\mathscr{C}^\nu}$ is the norm on $\mathscr{C}^\nu(\mathbb{Z}_p,\mathbf{K})$ as defined in [**Bar73**]. Let $\mathrm{P}^+ = \mathrm{T}^+\mathrm{N}_0$. Then $\mathscr{C}^r(\mathbb{Z}_p,\mathbf{K})$ is stable under the action of P^+. We show that $\|\cdot\|_{\mathscr{C}^r}$ is the universal unitary norm for the $\mathbf{K}[\mathrm{P}^+]$-module $\mathscr{C}^r(\mathbb{Z}_p,\mathbf{K})$.

Definition. A function $f\colon \mathbb{Z}_p \to \mathbf{K}$ is r**-times continuously differentiable** if the function

$$(x;h_1,\dots,h_\nu,h_{\nu+1}) \mapsto \frac{|\Delta^{\nu+1} f(x;h_1,\dots,h_\nu,h_{\nu+1})|}{|h_1|\cdots|h_\nu|\cdot|h_{\nu+1}|^\rho}$$

that is defined on all x and nonzero $h_1,\dots,h_\nu,h_{\nu+1}$ in $\mathbb{Z}_p$ and takes nonnegative values in $\mathbb{R}$ extends to a continuous function on all of $\mathbb{Z}_p\times\mathbb{Z}_p\times\cdots\times\mathbb{Z}_p$. Let $\mathscr{C}^r(\mathbb{Z}_p,\mathbf{K})$ be the $\mathbf{K}$-Banach space of all r-times continuously differentiable functions with norm $\|\cdot\|_{\mathscr{C}^r}$.

Proposition 9.2. *Let $r\geq 0$ and $k\in\mathbb{N}$. The universal unitary norm of $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p,\mathbf{K})$ is given by $\|\cdot\|_{\mathscr{C}^r}$ (and its completion is $\mathscr{C}^r(\mathbb{Z}_p,\mathbf{K})$).*

Proof: By Lemma 8.3 the norm $\|\cdot\|_{\mathscr{C}^r}$ is

(i) invariant under translation, and
(ii) there is a constant $\mathrm{C}>0$ such that $\|1_{p^n\mathbb{Z}_p}x^k\|_1 \leq \mathrm{C}\cdot p^{(r-k)n}$ for all $n\in\mathbb{Z}$.

We have to prove that $\|\cdot\| := \|\cdot\|_{\mathscr{C}^r}$ is the greatest norm on $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p,\mathbf{K})$ that satisfies (i) and (ii). By [**Nag13**, Theorem 3.8] the van der Put-basis

$$\{e_n^i := 1_{p^{l(n)}\mathbb{Z}_p}x^i(\cdot - n) : (n,i)\in\mathbb{N}\times\{0,\dots,k-1\}\},$$

with $l(0)=0$ and $l(n)=\lfloor\log_p(n)\rfloor$ for $n>0$, is an orthogonal basis of $\mathscr{C}^r(\mathbb{Z}_p,\mathbf{K})$ with $\|e_n^i\| = p^{(r-i)l(n)}$. In particular,

(a) $\mathscr{C}^{\mathrm{la}\leq\nu}(\mathbb{Z}_p,\mathbf{K})$ is dense in $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p,\mathbf{K})$ for $\|\cdot\|_{\mathscr{C}^r}$, and
(b) $\|\cdot\|$ is the greatest of all norms on $\mathscr{C}^{\mathrm{la}\leq\nu}(\mathbb{Z}_p,\mathbf{K})$ that satisfy (i) and (ii).

Therefore $\mathscr{C}^{\mathrm{la}\leq\nu}(\mathbb{Z}_p,\mathbf{K})$ is dense in $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p,\mathbf{K})$ for every norm that satisfies (i) and (ii). Because every norm is a continuous function, $\|\cdot\|$ is thus the greatest among all norms on all of $\mathscr{C}^{\mathrm{la}\leq k}(\mathbb{Z}_p,\mathbf{K})$ by (a) and (b). □

Remark. The norm $\|\cdot\|_r$ generalizes (from the domain $\mathbb{Z}_p$ and $\boldsymbol{r}=\nu\in\mathbb{N}$) the norm given in [**Bar73**] instead of that given in [**Sch84**]. The norms by Barsky and Schikhof are equivalent on all locally polynomial functions on the p-adic integers because their completions given by ν-times differentiable functions coincide; this can be seen by comparing their Mahler expansions at [**Bar73**, Section II] and [**Sch84**, Section 54].

However, let f be an additive differentiable function on the valuation ring $\mathbf{o}$ of a local field $\mathbf{F}$ such that $f' = 0$. If $\operatorname{char}\mathbf{F} = 0$ then $\mathbb{Z}$ is dense inside $\mathbb{Z}_p$ and thus by continuity f is $\mathbb{Z}_p$-linear. Thus $f = 0$, since $f' = 0$; whereas if $\operatorname{char}\mathbf{F} = p > 0$, an additive map is only $\mathbf{F}_p$-linear. Since by additivity $\Delta^2 f = 0$, such functions will be twice differentiable in the sense of [**Bar73**]. But there are examples that do not have a Taylor polynomial of order greater than 1 (see [**Glö07**, Theorem 3.7] and [**Yan04**, pg. 372]) and hence are not twice differentiable in the sense of Schikhof ([**Sch84**, Proposition 28.4]).

We conclude that if $\nu > 1$ and $\operatorname{char}\mathbf{F} > 0$ then the condition for ν-fold differentiability by Schikhof on a function on an open subset of $\mathbf{F}$ is stricter than that by Barsky.

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