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Partial Fractional non-Archimedean Differentiability

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Abstract. Recently the notion of an r-times differentiable function for a real number $r \geq 0$ taking values in a non-Archimedean Banach space has proved a valuable tool in the representation theory of p-adic Lie groups. We will first off briefly introduce those $\mathscr{C}^r$-functions of one variable through a description by iterated difference quotients and recall their basic properties such as the density of the set of locally polynomial functions of degree $\leq r$ inside the $\mathbf{K}$-Banach space $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$.

We will then give a complete characterization of the canonical (van der Put-) basis of locally polynomial functions in $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$ on the ring of integers $\mathbf{o} \subseteq \mathbf{K}$ inside a locally compact non-Archimedeanly valued field $\mathbf{K}$. Finally we introduce a new notion of $\boldsymbol{r}$-fold differentiable function for a tuple $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$. An exponential law analogous to the one of [Glö13] will then allow us to extend our results to the multivariate case with a single or multiple parameters of differentiability.

Contents

Introduction

Let $\mathbb{Z}_p$ denote the p-adic integers and $\mathbf{K}$ a complete non-Archimedeanly nontrivially valued field of characteristic zero. In [**BB10**] the authors introduce the notion of an *r-times differentiable function* $f\colon \mathbb{Z}_p \to \mathbf{K}$ for a real number $r \geq 0$ in an ad hoc manner through a growth condition on its *Mahler coefficients* $(a_n)_{n\in\mathbb{N}}$ by demanding $|a_n|n^r \to 0$ for $n \to \infty$. These describe, for any continuous functions on $\mathbb{Z}_p$, the expansion with respect to the orthogonal basis of Mahler polynomials which arises naturally from the cyclic topological group structure of $\mathbb{Z}_p$. They use this notion of fractional differentiability on the p-adic integers to describe a certain completion of a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation whose underlying $\mathbf{K}$-vector space can be given by locally polynomial functions on two copies of $\mathbb{Z}_p$. To make the connection between these two descriptions of this space of fractionally differentiable functions $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$, the one given by all those functions on $\mathbb{Z}_p$ obeying a certain growth condition of their Mahler coefficients and the other as that completion of the space of locally polynomial functions on $\mathbb{Z}_p$, the theorem of Amice-Velu-Vishik (cf. [**AV75**] respectively [**Viš76**]) is employed, giving a description of the dual space of the first space by the latter one and so showing these two definitions to coincide.

We will firstly revisit in Part 1 the situation of fractional differentiability in one variable: In Part 1 we recall the general pointwise definition of r-fold differentiability for $r \in \mathbb{R}_{\geq 0}$ through iterated difference quotients for a function $f\colon X \to \mathbf{K}$ defined on a subset $X \subseteq \mathbf{K}$ without isolated points as established in [**Nag12**] and residing upon [**Sch84**]. (We note that on $\mathbb{Z}_p$ in [**Col10**] and more recently, building thereupon, on the ring of integers of a local field in [**DI13**] other notions of r-fold differentiability were introduced, shown in [**Nag12**, Section II.4] to coincide with the one adopted here.) In this way many natural properties of this notion become readily accessible whereas they can hardly be read off by their description through the orthogonal Mahler basis.

As an example of this, we will in Section 4 prove the density of locally polynomial functions inside $\mathscr{C}^r(X, \mathbf{K})$ by a natural downward induction on $\lfloor r \rfloor$. This was already established in the multivariate case via a similar strategy in [**Nag13**]; the possible restriction to the case of a single variable allowing for the reduction of a lot of technicalities. We can then infer by the key lemma [**AS94**, Corollary 1.3] the polynomial functions to be dense inside $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$.

In Section 5, we will then construct a canonical basis of locally polynomial functions inside $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$ for a locally compact field $\mathbf{K}$ with valuation ring $\mathbf{o}$, the so called *van der Put basis* of $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$. This notion is a natural extension of the classic one over the ring of p-adic integers (cf. [**Sch84**, Part 3]) in the vein of [**DI13**]. In doing so, we will give explicit formulas for the coefficients of a function with respect to its expansion in terms of the van der Put basis, generalizing the results of [**DS94b**]. At this occasion, we also show in Section 6 how one can briefly arrive at the characterization of a $\mathscr{C}^r$-function by its Mahler coefficients obeying $|a_n|n^r \to 0$ for $n \to \infty$, thus recovering the initial definition given in [**BB10**].

In the final Part 3 of this article we will first off introduce a new notion of fractional differentiability in many variables with several orders of differentiability. This generalizes the notion of partial iterated differentiability in several variables in [**Glö13**]. In Section 8 will then extend the exponential law established at loc.cit. to the notion of fractional differentiability considered at this place. In particular, we can infer by the density of locally polynomials functions the space of differentiable functions of order $\boldsymbol{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d_{\geq 0}$ to be the tensor product of those of r_1-,...,r_d-differentiable functions of one variable and as immediate consequences, recorded in Section 9, find suitable van der Put- respectively Mahler-basiss in many variables. This generalizes results in exemplary cases carried out by Stany de Smedt in [**DS94a**].

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Notations and Conventions

Throughout this paper $\mathbf{K}$ will denote a complete non-Archimedeanly valued field whose valuation v is nontrivial. If we fix a positive real constant $c_v < 1$, we obtain a norm $|x| := c_v^{v(x)}$.

Metric and normed spaces

We will throughout assume all norms to be non-Archimedean. All normed respectively metric spaces are implicitly assumed to be endowed with a norm $\|\cdot\|$ respectively metric d, through whose arguments it will be clear where it is defined. Every normed space gives rise to a metric $\mathrm{d}(x, y) := \|x - y\|$.

Given a subset A inside a metric space, we denote by dia A its *diameter*

$$\operatorname{dia} \mathrm{A} = \sup\{\mathrm{d}(a, b) : a, b \in \mathrm{A}\}.$$

Let $X = X_1 \times \cdots \times X_d$ be the product of normed spaces $(X_1, \|\cdot\|_1, \ldots, (X_d, \|\cdot\|_d)$. Then we endow X with the structure of a normed space through the norm

$$\|x\| = \max\{\|x_1\|_1, \ldots, \|x_d\|_d\}.$$

If X is a set and Y a normed space, we define a *quasinorm* $\|\cdot\|_{\sup}$ (a map with image in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying all axioms of a norm) on all functions $f\colon X \to Y$ by

$$\|f\|_{\sup} = \begin{cases} \sup\{\|f(x)\| : x \in X\}, & \text{if this supremum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

For a subset $A \subseteq X$, we define $\|f\|_A := \|f_{|A}\|_{\sup}$.

Notational conventions

Let X and I be sets. Then we view X^I as the set of all tuples $(x_i)_{i \in I}$. We will often abbreviate $\max\{a, b\}$ for two real numbers a and b by $a \vee b$.

Part 1. A review on fractional differentiability in one Variable

1. $\mathscr{C}^\rho$-functions

Definition. Let X be a metric space, Y a complete metric space, $f\colon A \to Y$ a mapping defined on a subset $A \subseteq X$ and a some point in X; we will say that f is $\mathscr{C}^\rho$ **at** a, if for every $\varepsilon > 0$ there is a neighborhood $U \ni a$ in X such that

$$d(f(x), f(y)) \leq \varepsilon \cdot d(x, y)^\rho \quad \text{for all } x, y \in U \cap A.$$

Then f will be a $\mathscr{C}^\rho$**-function** if f is $\mathscr{C}^\rho$ at all points $a \in A$, where we note that this notion is independent of the ambient space X. We will denote the set of all $\mathscr{C}^\rho$-functions $f\colon A \to Y$ by $\mathscr{C}^\rho(A, Y)$.

Remark 1.1. We emphasize that we also defined what it means for a point $a \in X$ not in the function's domain A to be $\mathscr{C}^\rho$. If there is a neighborhood of a disjoint to A, then this condition will be void. The interesting case occurs whenever a is a boundary point of A inside X as then by completeness of Y the function's domain can be extended to include all these $\mathscr{C}^\rho$-points in the boundary as made precise below.

Proposition 1.2. *Let* X *be a metric space,* Y *a complete metric space and* $f\colon A \to Y$ *a* $\mathscr{C}^\rho$*-function defined on subset* $A \subseteq X$*. Let* B *with* $A \subseteq B \subseteq \bar{A}$ *denote the set of* $\mathscr{C}^\rho$*-points of* f*. Then* f *extends uniquely to a* $\mathscr{C}^\rho$*-function* $F\colon B \to Y$*.*

Definition. Let X be a metric space, **E** a non-Archimedean Banach space and $f\colon X \to \mathbf{E}$. We define $|f^{]\rho[}|$ on all pairs $(x, y) \in X \times X$ with distinct entries by

$$|f^{]\rho[}|(x, y) = \frac{\|f(x) - f(y)\|}{d(x, y)^\rho} \in \mathbb{R}_{\geq 0}.$$

Then the mapping $f\colon \mathrm{X} \to \mathrm{E}$ is $\mathscr{C}^\rho$ if and only if the function $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}_{\geq 0}$ vanishing on all diagonal entries (x,x) for $x \in \mathrm{X}$. Therefore the following definition is meaningful.

Definition. Let X be a compact metric space. We endow $\mathscr{C}^\rho(\mathrm{X},\mathbf{E})$ with the norm $\|\cdot\|_{\mathscr{C}^\rho}$ on $\mathscr{C}^\rho(\mathrm{X},\mathbf{E})$ by

$$\|f\|_{\mathscr{C}^\rho} = \|f\|_{\sup} \vee \||f^{[\rho]}|\|_{\sup}.$$

In this way $\mathscr{C}^\rho(\mathrm{X},\mathbf{E})$ becomes a $\mathbf{K}$-Banach space.

The following lemma shows the locally constant functions to be dense inside the $\mathscr{C}^\rho$-functions. The precise nature of the locally constant function approximating our given $\mathscr{C}^\rho$-function will be indispensable further below, when we prove the locally polynomial functions to be dense inside the differentiable ones. To this end, a function f on a metric space X will be called δ**-constant** if $\mathrm{d}(x,y) \leq \delta$ implies $f(x) = f(y)$ for all $x, y \in \mathrm{X}$.

Lemma 1.3. *Let* X *be a compact metric space,* $\mathbf{E}$ *a non-Archimedean Banach space and* $f\colon \mathrm{X} \to \mathbf{E}$ *a mapping such that for fixed* $\varepsilon > 0$*, there is* $0 < \delta \leq 1$ *such that* $\mathrm{d}(x,y) \leq \delta$ *implies* $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x,y)^\rho$ *for all* $x, y \in \mathrm{X}$*. Then there is a* δ*-constant function* $g\colon \mathrm{X} \to \mathbf{E}$ *with* $\|f - g\|_{\mathscr{C}^\rho} \leq \varepsilon$.

Proof: Because $\mathbf{E}$ is non-Archimedean, we can partition X into finitely many equivalence classes U_i by declaring

$$x \sim y \quad \text{if} \quad \|f(x) - f(y)\| \leq \varepsilon\delta^\rho.$$

By assumption on f, two points x and y will be equivalent if $\mathrm{d}(x,y) \leq \delta$. In particular every U_i is open.

We now choose an element a_i from each U_i and define δ-constant $g\colon \mathrm{X} \to \mathbf{E}$ by

$$g(x) := f(a_i) \quad \text{if} \quad x \in \mathrm{U}_i.$$

Then $\|f - g\|_{\sup} \leq \varepsilon\delta^\rho \leq \varepsilon$ and

$$\begin{aligned}
&\||(f-g)^{[\rho]}|\|_{\sup} \\
=&\||(f-g)^{[\rho]}|\|_{\{(x,y)\in\mathrm{X}^2:\mathrm{d}(x,y)\leq\delta\}} \vee \||(f-g)^{[\rho]}|\|_{\{(x,y)\in\mathrm{X}^2:\mathrm{d}(x,y)>\delta\}} \\
\leq&\quad \||f^{[\rho]}|\|_{\{(x,y)\in\mathrm{X}^2:\mathrm{d}(x,y)\leq\delta\}} \vee \||g^{[\rho]}|\|_{\{(x,y)\in\mathrm{X}^2:\mathrm{d}(x,y)\leq\delta\}} \\
&\vee \max_{x,y\in\mathrm{X}:\mathrm{d}(x,y)>\delta}\Big(\frac{\|f(x)-g(x)\|}{\mathrm{d}(x,y)^\rho} \vee \frac{\|f(y)-g(y)\|}{\mathrm{d}(x,y)^\rho}\Big) \\
\leq&\varepsilon \vee 0 \vee \varepsilon\delta^\rho/\delta^\rho = \varepsilon.
\end{aligned}$$

□

2. Definition of $\mathscr{C}^r$-functions

Following [**Sch84**, Section 29 ff.], we recall the notion of the iterated difference quotient of a function on a non-Archimedeanly valued domain.

Definition. Let X be a subset of $\mathbf{K}$ and $f\colon \mathrm{X} \to \mathbf{E}$. For $\nu \in \mathbb{N}$ put

$$\mathrm{X}^{[\nu]} = \mathrm{X}^{\{0,\dots,\nu\}} \quad \text{and} \quad \mathrm{X}^{]\nu[} := \{(x_0,\dots,x_\nu) \in \mathrm{X}^{[\nu]} : x_i = x_j \text{ only if } i = j\}.$$

The ν**-th difference quotient** $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ of a function $f\colon \mathrm{X} \to \mathbf{E}$ is inductively given by $f^{]0[} := f$ and for $n \in \mathbb{N}$ and $(x_0,\dots,x_\nu) \in \mathrm{X}^{]\nu[}$ by

$$f^{]\nu[}(x_0,\dots,x_\nu) := \frac{f^{]\nu-1[}(x_0,x_2,\dots,x_\nu) - f^{]\nu-1[}(x_1,x_2,\dots,x_\nu)}{x_0 - x_1}.$$

Having already defined $\mathscr{C}^\rho$-functions for $\rho \in [0,1[$, we add up our definitions to obtain our notion of fractional differentiability over (non-Archmideanly valued) complete fields.

We will from now on fix a real number $r \geq 0$ together with its decomposition $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ into its *integral part* $\nu \in \mathbb{N}$ and *fractional part* $\rho \in [0,1[$.

Definition 2.1. Let X be a subset of $\mathbf{K}$ and $f\colon \mathrm{X} \to \mathbf{E}$.

(i) We will say that f is $\mathscr{C}^r$ (or r **times continuously differentiable**) at a point $a \in \mathrm{X}$ if $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a,\dots,a) \in \mathrm{X}^{[\nu]}$.

(ii) Then f will be a $\mathscr{C}^r$**-function** (or an r**-times continuously differentiable function**) if f is $\mathscr{C}^r$ at all points $a \in \mathrm{X}$. The set of all $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathbf{E}$ will be denoted by $\mathscr{C}^r(\mathrm{X},\mathbf{E})$.

Definition. Let $a \in \mathrm{X}$ and f be $\mathscr{C}^\nu$ at a. We define $\mathrm{D}_\nu f(a) := \lim_{x \to \vec{a}} f^{]\nu[}(x)$ for $x \in \mathrm{X}^{]\nu[}$.

Remark. Let $a \in \mathrm{X}$ and let f be $\mathscr{C}^\nu$ at a. By [Sch84, Theorem 29.5] we have $\nu!\,\mathrm{D}_\nu f(a) = f^{(\nu)}(a)$ with $f^{(\nu)}$ denoting the usual Archimedean ν-th derivative of f.

Remark.

(i) Let a be some accumulation point in X. Then $\vec{a}$ is an accumulation point of $\mathrm{X}^{]\nu[}$. As $\mathbf{E}$ is complete, we find by Remark 1.1 that $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ is $\mathscr{C}^0$ at $\vec{a} \in \mathrm{X}^{[\nu]}$ if and only if their exists a limit $\mathrm{D}_\nu f(a) \in \mathbf{E}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f^{]\nu[}(x) - \mathrm{D}_\nu f(a)| \leq \varepsilon \quad \text{for all } x \in \mathrm{X}^{]\nu[} \text{ with } |x_0 - a|,\dots,|x_\nu - a| \leq \delta.$$

(ii) The previous point shows that our notion coincides with the common notion of ν-fold differentiability of f at an accumulation point a in the domain of f, as considered for example, in [Sch84, Section 29] in case $r = \nu \in \mathbb{N}$.

We cite the following natural observation. (And refer the reader to [Nag11, Lemma 2.3] for a complete proof.)

Lemma 2.2. *Let* $\mathrm{X} \subseteq \mathbf{K}$ *be a subset,* a *some point in* X *and* $f\colon \mathrm{X} \to \mathbf{E}$*. If* f *is* $\mathscr{C}^r$ *at* a*, then* f *will be* $\mathscr{C}^s$ *at* a *for every* $s \leq r$*.*

Proposition 2.3. *Let* X *be a nonempty subset of* $\mathbf{K}$ *without isolated points and* $f\colon \mathrm{X} \to \mathbf{E}$*. Then* $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ *if and only if* $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ *extends to a* $\mathscr{C}^\rho$*-function* $f^{[\nu]}\colon \mathrm{X}^{[\nu]} \to \mathbf{E}$.

Proof: By an induction on $\nu \in \mathbb{N}$ and the previous Lemma 2.2, one obtains that $f^{]\nu[}$ is a $\mathscr{C}^\rho$-function. Because $\mathrm{X}^{]\nu[}$ is a dense subset of $\mathrm{X}^{[\nu]}$, one can then apply Proposition 1.2. (See [**Nag11**, Proposition 2.5] for an elaborate proof.) □

Definition. Let $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$. Then $f^{]0[}, \dots, f^{]\nu-1[}$ and $f^{]\nu[}$ extend by Lemma 2.2 and Proposition 2.3 to continuous functions $f^{[0]}, \dots, f^{[\nu-1]}$ and a $\mathscr{C}^\rho$-function $f^{[\nu]}$. If X is compact, we can thence define the norm $\|\cdot\|_{\mathscr{C}^r}$ on $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^r} := \|f^{[0]}\|_{\sup} \vee \dots \vee \|f^{[\nu-1]}\|_{\sup} \vee \|f^{[\nu]}\|_{\mathscr{C}^\rho}.$$

3. The r-fold difference quotient

Once one recognizes the symmetry of the difference quotient $f^{]\nu[}$ (See [**Sch84**, Lemma 29.2(ii)]) the following two observations are natural. We will only state them here and refer the reader to [**Nag11**, Lemma 2.1] respectively [**Nag11**, Corollary 2.11] for complete proofs.

Lemma 3.1. *Let* $\mathrm{X} \subseteq \mathbf{E}$*. Then a function* $f\colon \mathrm{X} \to \mathbf{E}$ *is* $\mathscr{C}^r$ *at a point* $a \in \mathrm{X}$ *if and only if for every* $\varepsilon > 0$*, there is a neighborhood* $\mathrm{U} \ni a$ *in* X *such that*

$$|f^{]\nu[}(x_0, x_1, \dots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \dots, x_\nu)| \le \varepsilon |x_0 - \tilde{x}_0|^\rho$$

for distinct $x_0, \tilde{x}_0, x_1, \dots, x_\nu \in \mathrm{U}$.

Proposition 3.2. *Let* $f\colon \mathrm{X} \to \mathbf{E}$ *be a mapping defined on a nonempty subset* $\mathrm{X} \subseteq \mathbf{K}$ *without isolated points. Define a function* $|f^{]r[}|\colon \mathrm{X}^{]\nu+1[} \to \mathbb{R}_{\ge 0}$ *by*

$$|f^{]r[}|(x_0, \tilde{x}_0, x_1, \dots, x_\nu) := \frac{|f^{]\nu[}(x_0, x_1, \dots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \dots, x_\nu)|}{|x_0 - \tilde{x}_0|^\rho}.$$

Then $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ *if and only if* $|f^{]r[}|\colon \mathrm{X}^{]\nu+1[} \to \mathbb{R}_{\ge 0}$ *extends to a continuous function* $|f^{[r]}|\colon \mathrm{X}^{[\nu+1]} \to \mathbb{R}_{\ge 0}$ *which will vanish if* $x_0 = \tilde{x}_0$.

In this case, we have $\|f^{[\nu]}\|_{\mathscr{C}^\rho} = \|f^{[\nu]}\|_{\sup} \vee \||f^{[r]}|\|_{\sup}$ *for compact* X.

Lemma 3.3. *Let* $x := (x_0, \dots, x_{\nu+1}) \in \mathrm{X}^{]\nu+1[}$ *and* $\sigma : \{x_0, x_1, \dots, x_{\nu+1}\} \circlearrowleft$ *be a permutation. Then*

$$|f^{]r[}|(x^\sigma) = \Big|\frac{x_0^\sigma - x_1^\sigma}{x_0 - x_1}\Big|^{1-\rho} |f^{]r[}|(x).$$

Proof: Let σ swap the indices $0, 1$ with $i, j \in \{0, \dots, \nu+1\}$. We notice

$$|f^{]r[}|(x) = |f^{]\nu+1[}(x)||x_0 - x_1|^{1-\rho}.$$

By symmetry of $f^{]\nu+1[}$ (cf. [**Sch84**, Lemma 29.2(ii)]) therefore holds

$$
\begin{aligned}
|f^{]r[}|(x^\sigma) =& |f^{]\nu+1[}(x^\sigma)||x_i - x_j|^{1-\rho} = |f^{]\nu+1[}(x)||x_0 - x_1|^{1-\rho} \frac{|x_i - x_j|^{1-\rho}}{|x_0 - x_1|^{1-\rho}} \\
=& |f^{]r[}|(x) \frac{|x_i - x_j|^{1-\rho}}{|x_0 - x_1|^{1-\rho}} = |f^{]r[}|(x) \Big| \frac{x_0^\sigma - x_1^\sigma}{x_0 - x_1} \Big|^{1-\rho}. \qquad \square
\end{aligned}
$$

Corollary 3.4. *Let* $(x_0, x_1, \ldots, x_{\nu+1}) \in X^{]\nu+1[}$ *and let us assume that* $|x_0 - x_1| = \operatorname{dia}\{x_0, x_1, \ldots, x_{\nu+1}\}$. *Then*

$$|f^{]r[}(x)| \geq |f^{]r[}|(x^\sigma)$$

for any permutation σ *of* $\{x_0, x_1, \ldots, x_{\nu+1}\}$.

Part 2. Orthogonal Bases

4. Density of locally polynomial functions

The total disconnectedness of a non-Archimedeanly valued domain expresses itself through the density of the locally constant functions inside all continuous functions with respect to the topology of uniform convergence and is at the heart of all the functional analytic properties of those function spaces (See for example, [?]). Conformly, in the case of a r-times differentiable function f, the function's i-fold derivatives $D_i f$ for $i \leq r$ intervene and we have to ensure by a downward induction that for each differentiability degree i we find a locally polynomial function g_i such that the locally constant function $D_i g_i$ and $D_i f$ are close to each other.

We let $X \subseteq \mathbf{K}$ denote a nonempty subset without isolated points and $\mathbf{E}$ a $\mathbf{K}$-Banach space. We will adopt the following natural notion of a locally polynomial function of maximal degree: We call $f \colon X \to \mathbf{E}$ a **locally polynomial function of degree at most** $g \in \mathbb{N}$, if for every point $a \in X$, there is a neighborhood $U \ni a$ such that $f_{|U} = p_{|U}$ is a polynomial function of degree at most g. In this context we will distinguish the identity function by $*$ so that for $i \in \mathbb{N}$ a natural number, $*^i \colon X \to \mathbf{K}$ denotes the monomial function $x \mapsto x^i$. Thence any polynomial function of highest degree g can be written $p = \sum_{i=0,\ldots,g} *^i \cdot a_i$ with $a_i \in \mathbf{E}$. By abuse of notation, we will also write $\lambda \cdot *^i$ for $*^i \cdot \lambda$ with $\lambda \in \mathbf{E}$.

Lemma 4.1. *Let* $f \in \mathscr{C}^n(X, \mathbf{E})$. *Fix* $\delta, \varepsilon > 0$. *If for all* $(x_0, \ldots, x_n) \in X^{n+1}$ *holds*

$$\operatorname{dia}\{x_0, \ldots, x_n\} \leq \delta \text{ implies } \|f^{[n]}(x_0, \ldots, x_n)\| \leq \varepsilon,$$

then for all $(x_0, \ldots, x_{n-1}), \vec{a} \in X^n$ *holds*

$$\|(x_0, \ldots, x_{n-1}) - \vec{a}\| \leq \delta \text{ implies } \|f^{[n-1]}(x_0, \ldots, x_{n-1}) - f^{[n-1]}(\vec{a})\| \leq \varepsilon \cdot \delta.$$

PROOF: Let $(x_0, \ldots, x_n), \vec{a} \in X^{n+1}$ with $\|(x_0, \ldots, x_n) - \vec{a}\| \leq \delta$. We calculate

$$\begin{aligned}
&\|f^{[n-1]}(x_0, \ldots, x_n) - f^{[n-1]}(\vec{a})\| \\
=&\|\sum_{j=0,\ldots,n-1} f^{[n-1]}(x_0, \ldots, x_{j-1}, x_j, a, \ldots, a) - f^{[n-1]}(x_0, \ldots, x_{j-1}, a, a, \ldots, a)\| \\
\leq& \max_{j=0,\ldots,n-1} \|f^{[n-1]}(x_0, \ldots, x_{j-1}, x_j, a, \ldots, a) - f^{[n-1]}(x_0, \ldots, x_{j-1}, a, a, \ldots, a)\| \\
=& \max_{j=0,\ldots,n-1} \|x_j - a\| \|f^{[n]}(x_j, a, x_0, \ldots, x_{j-1}, a, \ldots, a)\| \\
\leq&\delta \max_{j=0,\ldots,n-1} \|f^{[n]}(x_j, a, x_0, \ldots, x_{j-1}, a, \ldots, a)\| \\
\leq&\delta \cdot \varepsilon;
\end{aligned}$$

the last inequality as by the triangle inequality $\mathrm{dia}\{x_0, \ldots, x_j, a, \ldots, a\} \leq \delta$. □

Lemma 4.2. *Let $f \in \mathscr{C}^n(X, \mathbf{E})$. Fix $\delta, \varepsilon > 0$. If for all $(x_0, \ldots, x_n), \vec{a} \in X^{n+1}$ holds*

$$\|(x_0, \ldots, x_n) - \vec{a}\| \leq \delta \textit{ implies } \|f^{[n]}(x_0, \ldots, x_n) - f^{[n]}(\vec{a})\| \leq \varepsilon,$$

then there is a δ-constant $g\colon X \to \mathbf{E}$ such that $\tilde{f} := f - g*^n$ satisfies

$$\mathrm{dia}\{x_0, \ldots, x_n\} \leq \delta \textit{ implies } \|\tilde{f}^{[n]}(x_0, \ldots, x_n)\| \leq \varepsilon.$$

PROOF: For all $(x_0, \ldots, x_n), \vec{a} \in X^{n+1}$, we have

$$\|(x_0, \ldots, x_n) - \vec{a}\| \leq \delta \text{ implies } \|f^{[n]}(x_0, \ldots, x_n) - f^{[n]}(x'_0, \ldots, x'_n)\| \leq \varepsilon.$$

In particular for all $a, a' \in X$, we have

$$\|a - a'\| \leq \delta \text{ implies } \|D_n f(a) - D_n f(a')\| \leq \varepsilon.$$

By Lemma 1.3, there is a δ-constant $g\colon X \to \mathbf{E}$ such that $\|D_n f - g\|_{\sup} \leq \varepsilon$. Because g is locally constant, we find $D_n(g*^n) = g$. Hence $\tilde{f} := f - g*^n$ satisfies

$$\|D_n \tilde{f}\|_{\sup} = \|D_n f - D_n(g*^n)\|_{\sup} = \|D_n f - g\|_{\sup} \leq \varepsilon.$$

Then $\|(x_0, \ldots, x_n) - \vec{a}\| \leq \delta$ implies

$$\begin{aligned}
&\|\tilde{f}^{[n]}(x_0, \ldots, x_n) - \tilde{f}^{[n]}(\vec{a})\| \\
=&\|(f - g*^n)^{[n]}(x_0, \ldots, x_n) - (f - g*^n)^{[\vec{n}]}(\vec{a})\| \\
=&\|(f^{[n]}(x_0, \ldots, x_n) - f^{[n]}(\vec{a})) - ((g*^n)^{[n]}(x_0, \ldots, x_n) - (g*^n)^{[n]}(\vec{a}))\| \\
\leq&\|f^{[n]}(x_0, \ldots, x_n) - f^{[n]}(\vec{a})\| \vee \|(g*^n)^{[n]}(x_0, \ldots, x_n) - (g*^n)^{[n]}(\vec{a})\| \\
\leq&\varepsilon \vee \|g(a) - g(a)\| \\
=&\varepsilon;
\end{aligned}$$

the last inequality by g being constant on $B_{\leq\delta}(a)$ and $(*^n)^{[n]} = 1$. Since $\|D_n \tilde{f}\|_{\sup} \leq \varepsilon$, it follows

$$\|\tilde{f}^{[n]}(x_0, \ldots, x_n)\| \leq \|\tilde{f}^{[n]}(x_0, \ldots, x_n) - \tilde{f}^{[n]}(\vec{a})\| \vee \|\tilde{f}^{[n]}(\vec{a})\| \leq \varepsilon.$$

By the ultrametric triangle inequality, for $x \in X^{n+1}$, there is a $\vec{a} \in \triangle X^{n+1}$ such that $\|x - \vec{a}\| \leq \delta$ if and only if $\mathrm{dia}\{x_0, \ldots, x_n\} \leq \delta$. Hence

$$\mathrm{dia}\{x_0, \ldots, x_n\} \leq \delta \text{ implies } \|\tilde{f}^{[n]}(x_0, \ldots, x_n)\| \leq \varepsilon.$$

□

Lemma 4.3.

(i) We have $f \in \mathscr{C}^r(X, \mathbf{E})$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all pairwise distinct $x_0, \ldots, x_{\nu+1} \in X$, the following holds:

$$\mathrm{dia}\{x_0, \ldots, x_{\nu+1}\} \leq \delta \textit{ implies } |f^{]r[}|(x_0, \ldots, x_{\nu+1}) \leq \varepsilon.$$

(ii) In this case for all $x_0, \ldots, x_\nu \in X$ and $a \in X$, it holds:

$$\|(x_0, \ldots, x_\nu) - \vec{a}\| \leq \delta \textit{ implies } |f^{[\nu]}(x_0, \ldots, x_\nu) - f^{[\nu]}(\vec{a})| \leq \varepsilon\delta^\rho.$$

Proof: Ad (i): Firstly, by Lemma 3.1, we find $f \in \mathscr{C}^r(X, \mathbf{E})$ if and only if for every $\vec{a} \in X^{[r]}$, there is a neighborhood $U \subseteq X^{]r[}$ of $\vec{a}$ such that $|f^{]r[}|(x) \leq \varepsilon$ for all $x \in U$; here we recall $|f^{]r[}|: X^{]r[} \to \mathbb{R}_{\geq 0}$ to be defined by

$$|f^{]r[}|(x_0, x_1, x_2, \ldots, x_{\nu+1}) = \frac{|f^{]\nu[}(x_0, x_2, \ldots, x_{\nu+1}) - f^{]\nu[}(x_1, x_2, \ldots, x_{\nu+1})|}{|x_0 - x_1|^\rho}.$$

If there is such δ, then given $a \in X$, we put $U = B_{\leq\delta}(\vec{a})$. Then $\mathrm{dia}\{x_0, \ldots, x_{\nu+1}\} \leq \delta$ implies $x \in U$ and therefore $|f^{]r[}|(x) \leq \varepsilon$. Contrariwise, if $f \in \mathscr{C}^r(X, \mathbf{E})$, then by Proposition 3.2, this is the case if and only if $|f^{]r[}|: X^{]r[} \to \mathbb{R}_{\geq 0}$ extends to a continuous function $|f^{[r]}|: X^{[r]} \to \mathbb{R}_{\geq 0}$ vanishing on $\diamond X^{[r]} := \{x \in X^{[r]} : x_0 = x_1\} \subseteq X^{[r]}$. In particular for all $\vec{a} = (a, \ldots, a) \in \diamond X^{[r]}$, there exists a neighborhood $U \ni \vec{a}$ such that for all $x \in U$, we have $|f^{[r]}|(x) \leq \varepsilon$. Since X and so $X^{[r]}$ is compact and ultrametric, given $x \in X^{[r]}$, there is $\vec{a} \in \triangle X^{[r]}$ and a neighborhood $U \ni \vec{a}$ in $X^{[r]}$ with $x \in U$ if and only if there is $\delta > 0$ such that $\mathrm{dia}\{x_0, \ldots, x_{\nu+1}\} \leq \delta$. Therefore in particular for all $x \in X^{]r[}$, we find $\mathrm{dia}\{x_0, \ldots, x_{\nu+1}\} \leq \delta$ to imply $|f^{]r[}|(x) \leq \varepsilon$. Ad (ii): If $\|(x_0, \ldots, x_\nu) - \vec{a}\| \leq \delta$, then $\mathrm{dia}\{x_0, \ldots, x_\nu, a\} \leq \delta$. By (i) , we obtain

$$\begin{aligned}
&|f^{[\nu]}(x_0, \ldots, x_\nu) - f^{[\nu]}(\vec{a})| \\
=&|\sum_{i=0,\ldots,n} f^{[\nu]}(x_0, \ldots, x_{i-1}, x_i, a, \ldots, a) - f^{[\nu]}(x_0, \ldots, x_{i-1}, a, a, \ldots, a)| \\
=&|x_i - a|^\rho |f|^{[r]}(x_i, a, x_0, \ldots, x_{i-1}, a, \ldots, a) \leq \delta^\rho \varepsilon.
\end{aligned}$$

□

Proposition 4.4. *The locally polynomial functions of degree $g \leq \nu$ are dense in $\mathscr{C}^r(X, \mathbf{E})$.*

Proof: Fix $\varepsilon > 0$ and $f \in \mathscr{C}^r(X, \mathbf{E})$. By compactness of X, there is by Lemma 4.3(i) a $\delta > 0$ such that for all $(x_0, \ldots, x_{\nu+1})$,

$$|f^{[r]}|(x_0, \ldots, x_{\nu+1}) \leq \varepsilon \quad \text{if } \mathrm{dia}\{x_0, \ldots, x_{\nu+1}\} \leq \delta.$$

We will fix δ for the rest of the proof.

By downward induction on $n = \nu, \dots, 0$, we will inductively construct a sequence of δ-constant functions $g_\nu, \dots, g_0 \colon \mathrm{X} \to \mathbf{E}$ such that $f_n = f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \cdots - g_n *^n$ satisfies

$$|f_n^{[n]}(x_0, \dots, x_n)| \leq \varepsilon\delta^{r-n} \quad \text{if } \operatorname{dia}\{x_0, \dots, x_n\} \leq \delta.$$

Let $n = \nu$. Then by Lemma 4.3(ii), for all $x_0, \dots, x_n \in \mathrm{X}$ and $a \in \mathrm{X}$, we find that

$$|f^{[n]}(x_0, \dots, x_n) - f^{[n]}(\vec{a})| \leq \varepsilon\delta^\rho \quad \text{if } \|(x_0, \dots, x_n) - \vec{a}\| \leq \delta.$$

Therefore such g_n exists by Lemma 4.2.

Let $n < \nu$ and assume that assume that we have constructed a sequence of δ-constant functions $g_\nu, \dots, g_n \colon \mathrm{X} \to \mathbf{E}$ such that $f_n = f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \cdots - g_n *^n$ satisfies

$$|f_n^{[n]}(x_0, \dots, x_n)| \leq \varepsilon\delta^{r-n} \quad \text{for all } \boldsymbol{x} \text{ with } \operatorname{dia}\{x_0, \dots, x_n\} \leq \delta.$$

By Lemma 4.1, for all $(x_0, \dots, x_{n-1}), \vec{a} \in \mathrm{X}^n$ with $\|(x_0, \dots, x_{n-1}) - \vec{a}\| \leq \delta$ we have:

$$|f_n^{[n-1]}(x_0, \dots, x_{n-1}) - f_n^{[n-1]}(\vec{a})| \leq \varepsilon\delta^{r-n} \cdot \delta = \varepsilon \cdot \delta^{r-(n-1)}.$$

By Lemma 4.2, there is a δ-constant function $g_{n-1} \colon \mathrm{X} \to \mathbf{E}$ such that $f_{n-1} = f_n - g_{n-1} *^{n-1}$ satisfies

$$|f_{n-1}^{[n-1]}(x_0, \dots, x_{n-1})| \leq \varepsilon\delta^{r-(n-1)} \quad \text{for all } \boldsymbol{x} \text{ with } \operatorname{dia}\{x_0, \dots, x_\nu\} \leq \delta.$$

This finishes the construction of $g_0, \dots, g_\nu$.

We will prove by induction on $n = 0, \dots, \nu$ that $\|f_0^{[n]}\|_{\sup} \leq \varepsilon\delta^{r-n}$. Let $n = 0$. Then $\operatorname{dia}\{x_0\} = 0 \leq \delta$ for all $x_0 \in \mathrm{X}$. Hence $|f_0^{[0]}(x_0)| \leq \varepsilon\delta^\nu$ for all $x_0 \in \mathrm{X}$, that is, $\|f_0^{[0]}\|_{\sup} \leq \varepsilon\delta^\nu$.

Let $n + 1 > 0$. Then

$$\begin{aligned} \|f_0^{[n+1]}\|_{\sup} = \quad & \|f_0^{[n+1]}\|_{\{(x_0, \dots, x_{n+1}) \text{ with } \operatorname{dia}\{x_0, \dots, x_{n+1}\} \leq \delta\}} \\ & \vee \|f_0^{[n+1]}\|_{\{(x_0, \dots, x_{n+1}) \text{ with } \|x_k - x_l\| > \delta \text{ for some } k, l\}}. \end{aligned}$$

Ad $\|f_0^{[n+1]}\|_{\{(x_0, \dots, x_{n+1}) \text{ with } \operatorname{dia}\{x_0, \dots, x_{n+1}\} \leq \delta\}} \leq \varepsilon\delta^{r-(n+1)}$:

Let $i = 0, \dots, n$. Since g_i is δ-constant, the function $(g_i *^i)^{[i]}$ is constant on all $(x_0, \dots, x_i)$ with $\operatorname{dia}\{x_0, \dots, x_i\} \leq \delta$. As $i < n + 1$, the function $(g_i *^i)^{[n+1]}$ thus vanishes on all $(x_0, \dots, x_{n+1})$ with $\operatorname{dia}\{x_0, \dots, x_n + 1\} \leq \delta$. Therefore

$$\begin{aligned} f_0^{[n+1]} &= (f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \cdots - g_0)^{[n+1]} \\ &= (f - g_\nu *^\nu - \cdots - g_{n+1} *^{n+1})^{[n+1]} \\ &= f_{n+1}^{[n+1]} \end{aligned}$$

restricted to all $(x_0, \dots, x_{n+1})$ with $\operatorname{dia}\{x_0, \dots, x_{n+1}\} \leq \delta$. By construction of $g_\nu, \dots, g_0 \colon \mathrm{X} \to \mathbf{E}$, the function $f_{n+1} = f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \cdots - g_{n+1} *^{n+1}$ satisfies:

$$\operatorname{dia}\{x_0, \dots, x_{n+1}\} \leq \delta \text{ implies } |f_{n+1}^{[n+1]}(x_0, \dots, x_{n+1})| \leq \varepsilon\delta^{r-(n+1)}.$$

Therefore

$$\begin{aligned}&\|f_0^{[n+1]}\|_{\{(x_0,\ldots,x_{n+1}) \text{ with } \operatorname{dia}\{x_0,\ldots,x_{n+1}\}\leq\delta\}}\\ =&\|f_{n+1}^{[n+1]}\|_{\{(x_0,\ldots,x_{n+1}) \text{ with } \operatorname{dia}\{x_0,\ldots,x_{n+1}\}\leq\delta\}} \leq \varepsilon\delta^{r-(n+1)}.\end{aligned}$$

Ad $\|f_0^{[n+1]}\|_{\{(x_0,\ldots,x_{n+1}) \text{ with } \|x_k-x_l\|>\delta \text{ for some } k,l\}}$:

Let $(x_0,\ldots,x_{n+1})$ with $\|x_k - x_l\| > \delta$ for some coordinates k, l. Assume we have proved

$$|f_0^{[n+1]}(x_0,\ldots,x_{n+1})| \leq \varepsilon\delta^{r-(n+1)} \text{ for all } (x_0,\ldots,x_{n+1}) \text{ with } \|x_0 - x_1\| > \delta.$$

We renumber $x_0,\ldots,x_{n+1} = x'_0,\ldots,x'_{n+1}$ such that $x'_k = x_0, x'_l = x_1$. Then

$$\begin{aligned}\varepsilon\delta^{r-(n+1)} &\geq |f_0^{[n+1]}(x'_0,\ldots,x'_{n+1})|\\ &= |f_0^{[n+1]}(x_k, x_l,\ldots,x_{n+1})|\\ &= |f_0^{[n+1]}(x_0, x_1,\ldots,x_{n+1})|\end{aligned}$$

by symmetry of $f_0^{[n+1]}$. Hence we are reduced to the case $\|x_0 - x_1\| > \delta$. Then

$$\begin{aligned}&|f_0^{[n+1]}(x_0,x_1,\ldots,x_{n+1})|\\ =&|f_0^{[n]}(x_0,x_2,\ldots,x_{n+1}) - f_0^{[n]}(x_1,x_2,\ldots,x_{n+1})|/|x_0-x_1|\\ <&\|f_0^{[n]}\|/\delta \leq \varepsilon\delta^{r-n}/\delta = \varepsilon\delta^{r-(n+1)};\end{aligned}$$

the last inequality by the induction hypothesis. This finishes the proof of $\|f_0^{[n]}\|_{\sup} \leq \varepsilon\delta^{r-n}$ for $n = 0,\ldots,\nu$.

Finally we show $|||f_0^{[r]}|||_{\sup} \leq \varepsilon$. We have

$$\begin{aligned}|||f_0^{[r]}|||_{\sup} = &|||f_0^{[r]}|||_{\{(x_0,\ldots,x_{\nu+1}) \text{ with } \operatorname{dia}\{x_0,\ldots,x_{\nu+1}\}\leq\delta\}}\\ &\vee |||f_0^{[r]}|||_{\{(x_0,\ldots,x_{\nu+1}) \text{ with } \|x_k-x_l\|>\delta \text{ for some } k,l\}}.\end{aligned}$$

Ad $|||f_0^{[r]}|||_{\{(x_0,\ldots,x_{\nu+1}) \text{ with } \operatorname{dia}\{x_0,\ldots,x_{\nu+1}\}\leq\delta\}}$:

Let $i = 0,\ldots,n$. We just saw $f_0^{[\nu]} = f_\nu^{[\nu]}$ with $f_\nu = f - g_\nu *^\nu$ restricted to all $(x_0,\ldots,x_{\nu+1})$ with $\operatorname{dia}\{x_0,\ldots,x_{\nu+1}\} \leq \delta$. Since g_ν is δ-constant, the function $(g_\nu *^\nu)^{[\nu]}$ is constant on all $(x_0,\ldots,x_\nu)$ with $\operatorname{dia}\{x_0,\ldots,x_\nu\} \leq \delta$. Thus $|(g_\nu *^\nu)^{[r]}|$ vanishes on all $(x_0,\ldots,x_{\nu+1})$ with $\operatorname{dia}\{x_0,\ldots,x_{\nu+1}\} \leq \delta$. By the choice of δ, we find

$$|||f_0^{[r]}|||_{\{(x_0,\ldots,x_{\nu+1}) \text{ with } \operatorname{dia}\{x_0,\ldots,x_{\nu+1}\}\leq\delta\}} = |||f^{[r]}|||_{\{(x_0,\ldots,x_{\nu+1}) \text{ with } \operatorname{dia}\{x_0,\ldots,x_{\nu+1}\}\leq\delta\}} \leq \varepsilon.$$

Ad $|||f_0^{[r]}|||_{\{(x_0,\ldots,x_{\nu+1}) \text{ with } \|x_k-x_l\|>\delta \text{ for some } k,l\}}$:

If $\operatorname{dia}\{x_0, \ldots, x_{\nu+1}\} > \delta$, then by Corollary 3.4, to prove $|f^{[r]}|(x_0, \ldots, x_{\nu+1}) \leq \varepsilon$, we may assume $|x_0 - x_1| = \operatorname{dia}\{x_0, x_1, \ldots, x_{\nu+1}\} > \delta$. We compute

$$|f_0^{[r]}|(x_0, \ldots, x_{\nu+1}) = \frac{|f_0^{[\nu]}(x_0, x_2, \ldots, x_{\nu+1}) - f_0^{[\nu]}(x_1, x_2, \ldots, x_{\nu+1})|}{|x_0 - x_1|^{\rho}}$$
$$< \varepsilon\delta^{\rho}/\delta^{-\rho} = \varepsilon;$$

the last inequality by our estimate for $\|f_0^{[\nu]}\|_{\sup}$ just obtained.

Put $g := g_0 + g_1 *^1 + \cdots + g_\nu *^\nu$. Then g is a locally polynomial function of degree $g \leq \nu$ and $f_0 = f - g$. Then $\|f - g\|_{\mathscr{C}^r} = \max_{n=0,\ldots,\nu} \|f_0^{[n]}\|_{\sup} \vee \||f_0|^{[r]}\|_{\sup} \leq \varepsilon$. □

Corollary 4.5. *The polynomial functions are dense in $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$.*

Proof: We make firstly the following observation: Let $\mathbf{1}_{\mathrm{B}} \colon \mathrm{X} \to \mathbf{K}$ be the indicator function of a closed ball $\mathrm{B} \subseteq \mathrm{X}$ of positive radius and $\varepsilon > 0$. Then there is by [**AS94**, Corollary 1.3] a polynomial function $p \colon \mathrm{X} \to \mathbf{K}$ such that $\|\mathbf{1}_{\mathrm{B}} - p\|_{\mathscr{C}^{\nu+1}} \leq \varepsilon$. Therefore $\|\mathbf{1}_{\mathrm{B}} - p\|_{\mathscr{C}^r} \leq \|\mathbf{1}_{\mathrm{B}} - p\|_{\mathscr{C}^{\nu+1}} \leq \varepsilon$.

This, together with the already obtained density of locally polynomial functions inside $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ allows us to infer the result by general reasoning in two steps as follows:

(i) The closure of the polynomial functions inside $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ contains all locally constant functions.
(ii) The polynomial functions are dense in $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$.

Ad 1.: The closed balls $\mathrm{B} \subseteq \mathrm{X}$ constitute a basis of the topological space $\mathrm{X} \subseteq \mathbf{K}$. Hence by compactness of X, every locally constant function g is the finite sum $f = \sum_i \lambda_i \mathbf{1}_{\mathrm{B}_i}$ with $\lambda_i \in \mathbf{K}$ and indicator functions $\mathbf{1}_{\mathrm{B}_i}$ of closed balls $\mathrm{B}_i \subseteq \mathrm{X}$ for $i \in \mathrm{I}$. By the above observation, for every $\varepsilon > 0$, there are polynomial functions $p_i \colon \mathrm{X} \to \mathbf{K}$ such that $\|p_i - \mathbf{1}_{\mathrm{B}_i}\|_{\mathscr{C}^r} \mathrm{M}_i \leq \varepsilon$ with $\mathrm{M}_i := |\lambda_i| \geq 0$. Then $p := \sum_i \lambda_i p_i \colon \mathrm{X} \to \mathbf{K}$ satisfies $\|p - f\|_{\mathscr{C}^r} \leq \max_i |\lambda_i| \|p_i - \mathbf{1}_{\mathrm{B}_i}\|_{\mathscr{C}^r} \leq \varepsilon$.

Ad 2.: Let us fix $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K})$ and $\varepsilon > 0$. By the above Proposition 4.4, there is a locally polynomial function $g = \sum_{i=0,\ldots,\nu} g_i *^i \colon \mathrm{X} \to \mathbf{K}$ with locally constant g_i such that $\|f - g\|_{\mathscr{C}^r} \leq \varepsilon$. By Step 1., there are polynomial functions $p_i \colon \mathrm{X} \to \mathbf{K}$ with $\|p_i - g_i\|_{\mathscr{C}^r} \cdot \mathrm{M}_i \leq \varepsilon$ with $\mathrm{M}_i = \|*^i\|_{\mathscr{C}^r} > 0$ for all $i = 0, \ldots, \nu$. Then the polynomial function $p := \sum_{i=0,\ldots,\nu} p_i *^i \colon \mathrm{X} \to \mathbf{K}$ satisfies

$$\|p - g\|_{\mathscr{C}^r} \leq \max_{i=0,\ldots,\nu} \|p_i - g_i\|_{\mathscr{C}^r} \cdot \|*^i\|_{\mathscr{C}^r} \leq \varepsilon$$

and therefore $\|p - f\|_{\mathscr{C}^r} \leq \|p - g\|_{\mathscr{C}^r} \vee \|g - f\|_{\mathscr{C}^r} \leq \varepsilon$. □

5. The van der Put-basis

Let us assume $\mathbf{K}$ to be a non-Archimedeanly valued and locally compact field. Define $\mathbf{o}_{\leq\lambda} = \{x \in \mathbf{K} : |x| \leq \lambda\}$ for $\lambda \in \mathbb{R}_{\geq 0}$; let $\mathbf{o} = \mathbf{o}_{\leq 1}$ be the ring of integers. Because $\mathbf{K}$ is locally compact the image of the valuation map is discrete and we assume $v(\mathbf{K}^*) = \mathbb{Z}$. Let $\pi \in \mathbf{o}$ be a uniformizer.

Notation. We will denote the space of all locally polynomial functions $f\colon \mathbf{o} \to \mathbf{K}$ of highest degree g by $\mathscr{C}^{\mathrm{lp}\leq g}(\mathbf{o}, \mathbf{K})$.

Notation. Let $\mathrm{U} \subseteq \mathbf{o}$ be an open subset. Then we will define the **indicator function** $\mathbf{1}_{\mathrm{U}}\colon \mathbf{o} \to \mathbf{K}$ of U by

$$\mathbf{1}_{\mathrm{U}}(x) = \begin{cases} 1, & \text{if } x \in \mathrm{U}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.1. *Let $d \leq r$. Then we have $\|*^d\|_{\mathscr{C}^r} = 1$ and for $n \geq 1$ holds*

$$\|\mathbf{1}_{\pi^n\mathbf{o}} *^d\|_{\mathscr{C}^r} = |1/\pi|^{(n-1)(r-d)}.$$

Proof: We firstly make the following observation: Let $f\colon \mathbf{o} \to \mathbf{K}$ be a polynomial function of degree d. Let $\nu \leq d$. Then $f^{]\nu[}\colon \mathbf{o}^{[\nu]} \to \mathbf{K}$ is a polynomial function in $x_0, \ldots, x_\nu$ of total degree $d - \nu$.

It surely holds $\|\mathbf{1}_{\mathbf{o}} *^k\|_{\mathscr{C}^r} = 1$. So let $n \geq 1$ and let us abbreviate $\mathrm{U} := \pi^n\mathbf{o}$, $\delta := \mathrm{dia}(\mathrm{U}) = c^n$, $\breve{\delta} = \delta/c > \delta$ with $c = |\pi|$ and $f := \mathbf{1}_{\mathrm{U}} *^d$.

We firstly want to show that $\|f\|_{\mathscr{C}^r} \leq c^{n(r-d)}$. We first prove by induction $\|f^{[k]}\|_{\sup} \leq \breve{\delta}^{k-d}$ for $k \leq \nu$. If $k = 0$ the conclusion surely holds. Thus let $k \geq 1$ and $(x_0, \ldots, x_k) \in \mathrm{X}^{]k[}$. We distinguish two cases:

(i) It holds $\mathrm{dia}\{x_0, \ldots, x_k\} \leq \delta$. We distinguish two cases:
  (a) It holds $|x_i| \leq \delta$ for all $i = 0, \ldots, k$. Then by the preceding observation $|f^{[i]}(x_0, \ldots, x_k)| \leq \delta^{d-k}$.
  (b) It holds $|x_i| > \delta$ for all $i = 0, \ldots, k$. Then $x_0, \ldots, x_k \notin \mathrm{U}$ and thus $f^{[i]} = 0$.

(ii) It holds $|x_i - x_j| > \delta$ for some $i, j \in \{0, \ldots, k\}$. Then by symmetry of $f^{[k]}$ we may assume $(i, j) = (0, 1)$. Therefore

$$\begin{aligned} |f^{[k]}(x_0, \ldots, x_k)| &\leq \left|\frac{f^{[k-1]}(x_0, x_2, \ldots, x_k) - f^{[k-1]}(x_1, x_2, \ldots, x_k)}{x_0 - x_1}\right| \\ &\leq \breve{\delta}^{d-(k-1)}/\breve{\delta} \leq \breve{\delta}^{d-k}. \end{aligned}$$

We finally prove $\||f^{[r]}|\|_{\sup} \leq \breve{\delta}^{r-d}$. We distinguish two cases:

(i) It holds $\mathrm{dia}\{x_0, \ldots, x_{\nu+1}\} \leq \delta$. We distinguish two cases:
  (a) It holds $|x_i| \leq \delta$ for all $i = 0, \ldots, \nu + 1$. Then by the preceding observation it holds $|f^{[\nu+1]}(x_0, \ldots, x_{\nu+1})| \leq \delta^{d-(\nu+1)}$. In particular, we see that $|f^{[r]}(x_0, \ldots, x_{\nu+1})| \leq \delta^{d-r}$.
  (b) It holds $|x_i| > \delta$ for all $i = 0, \ldots, \nu + 1$. Then $x_0, \ldots, x_{\nu+1} \notin \mathrm{U}$ and thus $f^{[\nu]} = 0$.

(ii) It holds $|x_i - x_j| > \delta$ for some $i, j \in \{0, \ldots, \nu+1\}$. Then by Corollary 3.4 we may assume $(i, j) = (0, 1)$. Therefore

$$\begin{aligned} |f^{[r]}(x_0, \ldots, x_{\nu+1})| &\leq \frac{|f^{[\nu]}(x_0, x_2, \ldots, x_{\nu+1}) - f^{[\nu]}(x_1, x_2, \ldots, x_{\nu+1})|}{|x_0 - x_1|^{\rho}} \\ &\leq \breve{\delta}^{d-\nu}/\breve{\delta}^{\rho} \leq \breve{\delta}^{d-r}. \end{aligned}$$

Regarding the inverse equality, we have to find $x \in X^{[\nu+1]}$ such that $|f^{[r]}|(x) = \check{\delta}^{r-d}$. Choose elements $x_{d+1}, \dots, x_{\nu+1} \in \mathbf{o}$ such that $|x_i - x_j| = \check{\delta}$ for all $i, j \in \{d+1, \dots, \nu+1\}$ distinct and put $x = (0, \dots, 0, x_{d+1}, \dots, x_{\nu+1}) \in X^{[\nu+1]}$. We show that x fulfills the desired equality. Let $f = gh$ with $g = *^d$ and $h = \mathbf{1}_{\pi^n \mathbf{o}}$. By [**Sch84**, Lemma 29.2(v)] we have

$$\begin{aligned} f^{[\nu+1]}(x) &= \sum_{i=0,\dots,\nu+1} g(x_0, \dots, x_i) h(x_{j+1}, \dots, x_\nu) \\ &= \sum_{i=0,\dots,d} g^{[i]}(0, \dots, 0) h^{[\nu+1-i]}(0, \dots, 0, x_{d+1}, \dots, x_{\nu+1}) \\ &= h^{[\nu+1-d]}(0, x_d, \dots, x_{\nu+1}). \end{aligned}$$

By [**Sch84**, Exercise 29.A], we find

$$|h^{[\nu+1-d]}(0, x_d, \dots, x_{\nu+1})| = |h(0)/x_d \cdots x_{\nu+1}| = |1/x_d \cdots x_{\nu+1}| = \check{\delta}^{\nu+1-d}$$

and thus, where we use the symmetry of $f^{[\nu+1]}$ so as to assume $|x_0 - x_1| = \check{\delta}$, we can conclude

$$|f^{[r]}|(x) = |f^{[\nu+1]}|(x)|x_0 - x_1|^{1-\rho} = \check{\delta}^{r-d}.$$

□

Definition 5.2.

(i) We choose for $i \in \mathbb{N}$ an *increasing* family of systems of representatives $(S_{\leq i})$ of $\mathbf{o}/\mathbf{o}_{\leq i}$, that is, $S_{\leq i} \subseteq S_{\leq j}$ for $i \leq j$ and put
$$S = \bigcup_{i \in \mathbb{N}} S_{\leq i} \subseteq \mathbf{o}.$$

(ii) We have a natural notion of level for the elements in S, namely we put
$$\ell(s) := \min\{i \in \mathbb{N} : s \in S_{\leq i}\}.$$
We let $S_i := S_{\leq i} - \cup_{j<i} S_j$ be the elements $s \in S$ of level $\ell(s) = i$.

(iii) We are all set to define our generalized van der Put-basis $\{e_s : s \in S\} \subseteq \mathscr{C}^{\mathrm{lc}}(\mathbf{o}, \mathbf{K})$ by the collection of indicator functions $\{e_s : s \in S\}$ defined by
$$e_s := \mathbf{1}_{s+\pi^{\ell(s)}\mathbf{o}}.$$

Remark. We remark that because $\mathbf{K}$ is locally compact, we find $\mathbf{o}$ to be compact and hence every $S_{\leq i}$ for $i \in \mathbb{N}$ to be finite.

Definition 5.3.

(i) Let $s \in S$ with $i = \ell(s) \geq 1$. Then its **preceding** element $s^- \in S$ is defined as the unique element in $S_{\leq i^-}$ such that $s^- \equiv s \bmod \pi^{i^-}\mathbf{o}$ with $i^- = i - 1$. We denote $\delta(s) = s - s^-$.

(ii) Let $s, t \in S$. We write $s \leq t$ if $\ell(s) \leq \ell(t)$ and $t \equiv s \bmod \pi^{\ell(s)}\mathbf{o}$. (That is, if and only if $e_s(t) = 1$.)

Corollary 5.4. *We find*

$$\|e_s(* - s)^k\|_{\mathscr{C}^r} = c^{(\ell(s)-1)(r-k)} \text{ with } c = 1/|\pi|.$$

PROOF: By Lemma 5.1 and the translation invariance of $\|\cdot\|_{\mathscr{C}^r}$. □

LEMMA. *The family $\{e_s : s \in \mathrm{S}\}$ is a basis of the $\mathbf{K}$-vector space $\mathscr{C}^{\mathrm{lc}}(\mathbf{o}, \mathbf{K})$.*

PROOF: For this it suffices to see that $\{e_s : s \in \mathrm{S}_{\leq \mathrm{I}}\}$ spans $\mathscr{C}(\mathbf{o}/\mathbf{o}_{\leq \mathrm{I}}, \mathbf{K})$ for $\mathrm{I} \in \mathbb{N}$. We make the following observation: Let $s_0 \in \mathrm{S}_{\leq \mathrm{I}}$ and $\ell(s_0) = i_0 = \mathrm{I} - 1$. Then

$$\mathbf{1}_{s_0 + \mathbf{o}_{\leq i_0}} = e_{s_0} - \sum_{s \in \mathrm{S}_{\leq \mathrm{I}} \text{ with } s \equiv s_0 \bmod \mathbf{o}_{i_0}} e_s.$$

Thence we deduce inductively that given $s_0 \in \mathrm{S}_{\leq \mathrm{I}}$ and $\ell(s_0) = i_0 < \mathrm{I}$, we find

$$\mathbf{1}_{s_0 + \mathbf{o}_{\leq \mathrm{I}}} = e_{s_0} - [\sum_{i = i_0+1, \ldots, \mathrm{I}} \sum_{s \in \mathrm{S}_i \text{ with } s \equiv s_0 \bmod \mathbf{o}_{\leq i-1}} e_s].$$

Therefore the canonical basis $\{\mathbf{1}_{s + \mathbf{o}_{\leq \mathrm{I}}} : s \in \mathrm{S}\}$ of $\mathscr{C}^{\mathrm{lc}}(\mathbf{o}/\mathbf{o}_{\leq i}, \mathbf{K})$ lies in the span of $\{e_s : s \in \mathrm{S}_{\leq \mathrm{I}}\}$. □

COROLLARY. *The family $\{e_s(* - s)^i : s \in \mathrm{S}, i \in \{0, \ldots, d\}\}$ constitutes a basis of the $\mathbf{K}$-vector space $\mathscr{C}^{\mathrm{lp} \leq d}(\mathbf{o}, \mathbf{K})$.*

Lemma 5.5. *Let $f \in \mathscr{C}^{\mathrm{lp} \leq d}(\mathbf{o}, \mathbf{K})$ be a locally polynomial function of degree $\leq d$. Write*

$$f = f_0 + \breve{f}$$

with $f_0 = \sum_{s \in \mathrm{S}} \lambda_s e_s$ locally constant. Then we have

$$\breve{f}(s) - \breve{f}(s^-) = \sum_{i=1,\ldots,d} \mathrm{D}_i f(s^-)(s - s^-)^i \text{ for all nonzero } s \in \mathrm{S}.$$

PROOF: We make the following observations:

(i) Let $g \in \mathscr{C}^{\mathrm{lp} \leq d}(\mathbf{o}, \mathbf{K})$ and $s_0 \in \mathrm{S}$. Write

$$g = \sum_{i=0,\ldots,d} \sum_{s \in \mathrm{S}} \lambda_{s,i} e_s(* - s)^i$$

We define

$$g_{\leq s_0} := \sum_{i=0,\ldots,d} \sum_{s \leq s_0} \lambda_{s,i} e_s(* - s)^i.$$

Then it holds by definition of $\{e_s : s \in \mathrm{S}\}$ that $g(s_0) - g(s_0^-) = g_{\leq s_0}(s_0) - g_{\leq s_0}(s_0^-)$.

(ii) Let $\mathrm{U} \subseteq \mathbf{o}$ be open, $g \in \mathscr{C}(\mathbf{o}, \mathbf{K})$ such that $f_{|\mathrm{U}}$ is a polynomial function of degree d. Let $x, x_0 \in \mathrm{U}$. Then by the uniqueness of the Taylor polynomial expansion

$$g(x) - g(x_0) = \sum_{i=0,\ldots,d} \mathrm{D}_i g(x_0)(x - x_0)^i.$$

(iii) Let $g \in \mathscr{C}(\mathbf{o}, \mathbf{K})$ be locally constant. Then $\mathrm{D}_i g = 0$ for all $i \geq 1$.

Combining these observations we obtain:

$$\begin{aligned} \check{f}(s) - \check{f}(s^-) &= \check{f}_{\leq s}(s) - \check{f}_{\leq s}(s^-) \\ &= \sum_{i=1,\ldots,d} \mathrm{D}_i \check{f}_{\leq s}(s)(s - s^-)^i \\ &= \sum_{i=1,\ldots,d} \mathrm{D}_i \check{f}(s)(s - s^-)^i \\ &= \sum_{i=1,\ldots,d} \mathrm{D}_i f(s)(s - s^-)^i. \end{aligned}$$ □

Corollary 5.6. *Let $f = f_0 + \cdots + f_d \in \mathscr{C}^{\mathrm{lp}\leq d}(\mathbf{o}, \mathbf{K})$ with $f_i = \sum_{s\in \mathrm{S}} \lambda_{s,i} e_s(* - s)^i$ for $i = 0, \ldots, d$. We have $\lambda_{0,i} = \mathrm{D}_i f(0)$ and*

$$\lambda_{s,i} = \mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s-s^-)^j \binom{i+j}{i} \mathrm{D}_{i+j} f(s^-)] \text{ for all nonzero } s \in \mathrm{S}.$$

Proof: Let $i \in \{0, \ldots, d\}$. It clearly holds $\lambda_{0,i} = \mathrm{D}_i f(0)$. Let $s \in \mathrm{S}$ be nonzero. We find $\mathrm{D}_i f = \mathrm{D}_i f_i + \mathrm{D}_i \check{f}$ with $\mathrm{D}_i f_i = \sum_{s\in\mathrm{S}} \lambda_{s,i} e_s$ locally constant. Thence by the preceding Lemma

$$\begin{aligned} \lambda_{s,i} &= \mathrm{D}_i f_i(s) - \mathrm{D}_i f_i(s^-) \\ &= (\mathrm{D}_i f - \mathrm{D}_i \check{f})(s) - (\mathrm{D}_i f - \mathrm{D}_i \check{f})(s^-) \\ &= \mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \mathrm{D}_j \mathrm{D}_i f(s^-)] \\ &= \mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathrm{D}_{j+i} f(s^-)]. \end{aligned}$$ □

Corollary 5.7. *Let $f \in \mathscr{C}^{\mathrm{lp}\leq d}(\mathbf{o}, \mathbf{K})$ and $\lambda_{s,i} \in \mathbf{K}$ for nonzero $s \in \mathrm{S}$ be as above. Then we have*

$$\lambda_{s,i} = (\mathrm{D}_i f)^{[d+1-i]}(s, s^-, \ldots, s^-)(s - s^-)^{d+1-i}.$$

Proof: By [Sch84, Theorem 29.4] we have

$$\begin{aligned} &(\mathrm{D}_i f)^{[d+1-i]}(s, s^-, \ldots, s^-)(s - s^-)^{d+1-i} \\ =& \mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathrm{D}_{i+j} f(s^-)] \\ =& \lambda_{s,i}, \end{aligned}$$

the last equality by the above Corollary. □

Theorem 5.8. *The van der Put-basis $\{e_s(* - s)^i : s \in \mathrm{S}, i = 0, \ldots, \nu\}$ is an orthogonal basis of $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$ with $\|e_s(*-s)^i\|_{\mathscr{C}^r} = |1/\pi|^{(r-i)(\ell(s)-1)}$ and corresponding coefficients*

$$\lambda_{0,i} = (\mathrm{D}_i f)(0) \text{ for } i = 0, \ldots, \nu$$

respectively

$$\lambda_{s,i} = (\mathrm{D}_i f)^{[\nu+1-i]}(s, s^-, \ldots, s^-)(s - s^-)^{\nu+1-i} \textit{ for nonzero } s \in \mathrm{S}, i = 0, \ldots, \nu$$

for any $f \in \mathscr{C}^r(\mathbf{o}, \mathbf{K})$.

PROOF: We already computed in Corollary 5.4 above that $\|e_s(* - s)^i\|_{\mathscr{C}^r} = |1/\pi|^{(r-i)(\ell(s)-1)}$.

Since $\{e_s : s \in \mathrm{S}\}$ is a basis of $\mathscr{C}^{\mathrm{lc}}(\mathbf{o}, \mathbf{K})$, given $f \in \mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o}, \mathbf{K})$, there exists unique $\lambda_{s,i} \in \mathbf{K}$ such that

$$f = \sum_{i=0,\ldots,\nu} \sum_{s\in\mathrm{S}} \lambda_{s,i} e_s(* - s)^i.$$

Therefore $\|f\|_{\mathscr{C}^r} \leq \max\{|\lambda_{s,i}|\|e_s(* - s)^i\|_{\mathscr{C}^r} : s \in \mathrm{S}, i = 0, \ldots, \nu\}$. It rests to prove that

$$|\lambda_{s,i}|\|e_s(* - s)^i\|_{\mathscr{C}^r} \leq \|f\|_{\mathscr{C}^r} \text{ for any } i \in \{0, \ldots, \nu\} \text{ and } s \in \mathrm{S}.$$

By the above Corollary 5.7 we have

$$\begin{aligned} |\lambda_{s,i}| &= |(s - s^-)|^{\nu+1-i} |\mathrm{D}_i f^{[\nu+1-i]}(s, s^-, \ldots, s^-)| \\ &= |\pi|^{(\nu+1-i)(\ell(s)-1)} |\mathrm{D}_i f^{[\nu+1-i]}(s, s^-, \ldots, s^-)| \\ &= \|e_s(* - s)^i\|^{-1}_{\mathscr{C}^{\nu+1}} |\mathrm{D}_i f^{[\nu+1-i]}(s, s^-, \ldots, s^-)|. \end{aligned}$$

the last equality by Corollary 5.4. It follows by [**Sch84**, Lemma 78.1] that

$$\begin{aligned} |\lambda_{s,i}|\|e_s(* - s)^i\|_{\mathscr{C}^r} &= |\mathrm{D}_i f^{[\nu+1-i]}(s, s^-, \ldots, s^-)||\pi|^{(1-\rho)(\ell(s)-1)} \\ &= |\sum_{i=1,\ldots,\nu+1} f^{[\nu+1]}(\underbrace{s, \ldots, s}_{i\text{-times}}, s^-, \ldots, s^-)||s - s^-|^{1-\rho} \\ &\leq |f^{[r]}|(s, s^-, \ldots, s^-) \\ &\leq \|f\|_{\mathscr{C}^r}, \end{aligned}$$

where the penultimate equality is a direct consequence of Corollary 3.4. □

COROLLARY (AMICE-VÉLU-VISHIK). *Let* $\mu\colon \mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o}, \mathbf{K}) \to \mathbf{K}$ *be a linear form. Then* μ *extends uniquely to a continuous linear form on* $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$ *if and only if there is a constant* $\mathrm{C} > 0$ *such that*

$$|\mu(1_{a+\pi^i\mathbf{o}}(* - a)^j)| \leq \mathrm{C}|\pi|^{(r-j)i}$$

for all $a \in \mathbf{o}, i \in \mathbb{N}$ *and* $j = 0, \ldots, \nu$.

PROOF: We firstly observe that by the density of the space locally polynomial functions of degree ν inside $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$, we find μ to extend to a continuous linear form on $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$ if and only if it is bounded on $\mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o}, \mathbf{K})$ with respect to the $\mathscr{C}^r$-norm:

Indeed, if μ extends to a continuous linear function on $\mathscr{C}^r(\mathbf{o}, \mathbf{K})$ it is in particular bounded on $\mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o}, \mathbf{K})$ and such a constant $\mathrm{C} > 0$ exists. Vice versa, if such $\mathrm{C} > 0$ exists, then μ is in particular uniformly continuous on $\mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o}, \mathbf{K}) \to \mathbf{K}$.

Hence by the density of $\mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o},\mathbf{K}) \subseteq \mathscr{C}^r(\mathbf{o},\mathbf{K})$ and completeness of the range $\mathbf{K}$, we find that μ be uniquely extended to a continuous (linear) function on $\mathscr{C}^r(\mathbf{o},\mathbf{K})$.

Finally by Theorem 5.8 there is an orthogonal subset inside $\mathscr{C}^{\mathrm{lp}\leq\nu}(\mathbf{o},\mathbf{K})$ given by indicator functions (The van der Put-basis that is). Hence the boundedness on $\mathscr{C}^{\mathrm{lp}\leq d}(\mathbf{o},\mathbf{K})$ is equivalent to the one on the indicator functions above. □

6. The Mahler basis

We want to relate the original definition of an $\mathscr{C}^r$-function $f\colon \mathbb{Z}_p \to \mathbf{K}$ given in [BB10] to the one given here. We will in the following assume $|p| = p^{-1}$.

Let $\mathscr{C}(\mathbb{Z}_p,\mathbf{o})^*$ be the continuous dual of the space $\mathscr{C}(\mathbb{Z}_p,\mathbf{o})$ of continuous functions $f\colon \mathbb{Z}_p \to \mathbf{o}$. Then $\mathscr{C}^{\mathrm{lc}}(\mathbb{Z}_p,\mathbf{o}) := \{f\colon \mathbb{Z}_p \to \mathbf{o} \text{ loc. cst. }\} \subseteq \mathscr{C}(\mathbb{Z}_p,\mathbf{o})$ is dense and corresponding to $\mathscr{C}^{\mathrm{lc}}(\mathbb{Z}_p,\mathbf{o}) = \cup_{n\geq 0}\mathbf{o}[\mathbb{Z}/p^n\mathbb{Z}]$, we have $\mathscr{C}(\mathbb{Z}_p,\mathbf{o})^* = \mathbf{o}[[\mathbb{Z}_p]] := \varprojlim \mathbf{o}[\mathbb{Z}/p^n\mathbb{Z}]$. The *Iwasawa* isomorphism

$$\begin{aligned} \mathbf{o}[[\mathbb{Z}_p]] &\to \mathbf{o}[[X]] \\ \mathbf{1} &\mapsto 1 + X \end{aligned}$$

gives an isomorphism of topological rings where the left-hand side is endowed with the topology of pointwise convergence. Define an $\mathbf{o}$-Banach space to be a complete normed torsionfree $\mathbf{o}$-module. Given an $\mathbf{o}$-Banach space V let

$$V^d := \mathrm{Hom}_{\mathbf{o}-\mathrm{cts.}}(V,\mathbf{o})$$

be the compact torsionfree $\mathbf{o}$-module of $\mathbf{o}$-linear continuous functionals with its topology of pointwise convergence. Given a torsionfree compact $\mathbf{o}$-module M let

$$M^d := \mathrm{Hom}_{\mathbf{o}}(M,\mathbf{o})$$

be the $\mathbf{o}$-Banach space whose norm is given by $\|\cdot\|_{\sup}$. Then these functors are by [Sch95] (quasi-)inverses between the categories

$$\{ \text{ torsionfree compact } \mathbf{o}\text{-modules } \} \leftrightarrow \{ \mathbf{o}\text{-Banach spaces V with } |\mathbf{o}| \supseteq \|V\| \}.$$

Let $\mathscr{C}_0(\mathbb{N},\mathbf{o})$ be the zero sequences with entries in $\mathbf{o}$. We conclude

$$\begin{array}{ccccc} \mathscr{C}(\mathbb{Z}_p,\mathbf{o}) & & & & \mathscr{C}_0(\mathbb{N},\mathbf{o}) \\ \downarrow \cdot^d & & & & \downarrow \cdot^d \\ \mathscr{C}(\mathbb{Z}_p,\mathbf{o})^d & \tilde{\to} & \mathbf{o}[[\mathbb{Z}_p]] & \tilde{\to} & \mathbf{o}[[X]] \\ \downarrow \cdot^d & & & & \downarrow \cdot^d \\ \mathscr{C}(\mathbb{Z}_p,\mathbf{o}) = \mathscr{C}(\mathbb{Z}_p,\mathbf{o})^{dd} & & \leftarrow & & \mathscr{C}_0(\mathbb{N},\mathbf{o}) = \mathscr{C}_0(\mathbb{N},\mathbf{o})^{dd}. \end{array}$$

The bottom isomorphism is then given by $e_n := (\ldots,0,1,0\ldots) \mapsto \binom{*}{n}$ with e_n being the sequence whose sole nonzero entry is 1 at the n-th position. We can therefore write $f = \sum_n a_n\binom{*}{n} \in \mathscr{C}(\mathbb{Z}_p,\mathbf{K})$ and call $\{a_n\}$ its *Mahler coefficients.*

In the following we will identify the condition on the Mahler coefficients of the subset $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K}) \subseteq \mathscr{C}^0(\mathbb{Z}_p^d, \mathbf{K})$ under the isomorphism given by this orthogonal basis.

Lemma 6.1. *Let $f\colon \mathbb{Z}_p \to \mathbf{K}$ and $a_0, a_1, \ldots$ its Mahler Coefficients. For $x_1, \ldots, x_\nu \in \mathbb{Z}_{\geq 1}$ and $y \in \mathbb{Z}_{\geq 0}$, put $z = (x_1 + \cdots + x_\nu + y, \ldots, x_1 + y, y) \in \nabla\mathbb{Z}_{\geq 0}^{\nu+1}$. Then*

$$f^{]\nu[}(z)$$
$$= \sum_{j\geq 0} \sum_{m_1,\ldots,m_\nu \geq 1} \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1})\cdots(m_\nu + \cdots + m_1)} \binom{x_1-1}{m_1-1} \cdots \binom{x_\nu - 1}{m_\nu - 1}\binom{y}{j}.$$

Proof: By induction on $\nu \geq 0$. See [**Sch84**, Proof of Theorem 54.1]. □

Lemma 6.2. *Let $f\colon \mathbb{Z}_p \to \mathbf{K}$ have Mahler expansion $f = \sum_{i=n,\ldots,\mathrm{N}} a_n\binom{*}{i}$ with $n \geq \nu$. Let $x_1 = \cdots = x_\nu = p^{\lfloor \log_p(n/\nu)\rfloor}$ and $y = n - (x_1 + \cdots + x_\nu)$. Put $z = (x_1 + \cdots + x_\nu + y, \ldots, x_1 + y, y) \in \mathbb{Z}_{\geq 0}^{[\nu]}$. Then*

$$f^{]\nu[}(z) = \frac{a_n}{\nu! p^{\nu\lfloor \log_p(n/\nu)\rfloor}}.$$

Proof: We make the following observation: Given $\alpha, \beta \in \mathrm{N}$ we have $\binom{k}{l} = 0$ if $k < l$. Fix $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$. Then for any tuple $\beta_1, \ldots, \beta_k \in \mathbb{N}$ different from $(\alpha_1, \ldots, \alpha_k)$ such that $\beta_1 + \cdots + \beta_k \geq \alpha_1 + \cdots + \alpha_k$, there is necessarily $i \in \{1, \ldots, k\}$ such that $\beta_i > \alpha_i$. Therefore the above formula in Lemma 6.1 reduces for $z = (x_1 + \cdots + x_\nu + y, \ldots, x_1 + y, y)$ with $x_1 = \cdots = x_\nu = p^{\lfloor \log_p(n/\nu)\rfloor}$ and $y = n - (x_1 + \cdots + x_\nu)$ to the above equality. □

Lemma 6.3. *Let $r \in \mathbb{R}_{\geq 0}$. There is positive constants $c \leq 1 \leq \mathrm{C}$ such that:*

(i) *We have $c \cdot p^{r\lfloor \log_p(n)\rfloor} \leq \|\binom{*}{n}\|_{\mathscr{C}^r} \leq \mathrm{C} \cdot p^{r\lfloor \log_p(n)\rfloor}$ for all $n \in \mathbb{N}$.*
(ii) *We have $\|\sum_{i=n,\ldots,\mathrm{N}} a_i\binom{*}{i}\|_{\mathscr{C}^r} \geq c \cdot |a_n| \|\binom{*}{n}\|_{\mathscr{C}^r}$ for all $n, \mathrm{N} \in \mathbb{N}$.*

Proof: Ad 1.: We may assume $n \geq \nu$. Because $\|\binom{*}{i}\|_{\sup} \leq 1$, we infer directly by Lemma 6.1 that

$$\|\binom{*}{i}\|_{\mathscr{C}^r} \leq 1/|p^{r\lfloor \log_p(n)\rfloor}| \leq \mathrm{C} \cdot p^{r\lfloor \log_p(n)\rfloor}$$

with $\mathrm{C} = 1$. On the other hand, by the above Lemma 6.2, we have

$$\|\binom{*}{i}\|_{\mathscr{C}^r} \geq \frac{1}{|(\nu+1)!||p^{\lfloor \log_p(n/(\nu+1))\rfloor}|^r} \geq p^{r\lfloor \log_p(n/(\nu+1))\rfloor} \geq c \cdot p^{r\lfloor \log_p(n)\rfloor}$$

with $c = 1/p^{r\lceil \log_p(\nu+1)\rceil}$.

Ad 2.: As we established above that the Mahler polynomials build an orthogonal basis of the $\mathbf{K}$-Banach space of continuous functions, we have $\|\sum_{i\geq 0} a_i\binom{*}{i}\|_{\mathscr{C}^r} \geq \|\sum_{i\geq 0} a_i\binom{*}{i}\|_{\sup} \geq \max\{|a_n| : n \geq 0\}$. We may thus assume $n \geq \nu$. Let $x_1 = \cdots = x_{\nu+1} = p^{\lfloor \log_p(n/(\nu+1))\rfloor}$ and $y = n - (x_1 + \cdots + x_{\nu+1})$. Put

$z = (x_1 + \cdots + x_{\nu+1} + y, \ldots, x_1 + y, y) \in \mathbb{Z}_{\geq 0}^{[\nu+1]}$. By the previous Lemma 6.2 and Step 1., we obtain

$$\begin{aligned}
\|\sum_{i=n,\ldots,N} a_i \binom{*}{i}\|_{\mathscr{C}^r} &\geq |(\sum_{i=n,\ldots,N} a_i \binom{*}{i})^{]r[}|(z) \\
&= \frac{|a_n|}{|(\nu+1)!||p^{\lfloor \log_p(n/(\nu+1)) \rfloor}|^r} \\
&\geq c \cdot p^{r \log_p(n)} \\
&\geq C^{-1} c \cdot \|\binom{*}{i}\|_{\mathscr{C}^r}
\end{aligned}$$
□

Lemma 6.4. *A countable subset* $\{e_1, e_2, \ldots\}$ *of a* **K***-Banach space is orthogonal if and only if*

$$\|\sum_{i=m,\ldots,n} \lambda_i e_i\| \geq |\lambda_m| \|e_m\| \quad \textit{for all } \lambda_m, \ldots, \lambda_n \in \mathbf{K}.$$

PROOF: See [**Sch84**, Proposition 50.4]. □

Corollary 6.5. *The Mahler basis* $\{\binom{*}{i} : i \in \mathbb{N}\}$ *constitutes an orthogonal family such that there are positive constants* $c \leq 1 \leq C$ *with* $c \cdot n^r \leq \|\binom{*}{i}\| \leq C \cdot n^r$.

PROOF: The orthogonality follows directly from the previous Lemma 6.3.2 by applying the preceding Lemma 6.4. The existence of positive constants $c \leq 1 \leq C$ such that $c \cdot n^r \leq \|\binom{*}{i}\| \leq C \cdot n^r$ follows from Lemma 6.3.1, by noting that $p^{-1} n^r \leq p^{\lfloor \log_p(n) \rfloor} \leq n^r$. □

Part 3. Partial fractional differentiability in several variables

7. Definition of partial fractional differentiability

For a functions in d variables, we will give here a definition of $\boldsymbol{r}$-fold differentiability for a tuple $\boldsymbol{r} \in \mathbb{R}_{\geq 0}^d$. Then we use the exponential law for multivariate functions, as established by Helge Glöckner in the recent preprint [GloExpLaw], to obtain corresponding orthogonal bases in the multivariate case.

Let $d \in \mathbb{N}$ and let $\mathbf{X} \subseteq \mathbf{K}^d$ be a subset. We recall that $\mathbf{X}$ is called *cartesian* if $\mathbf{X} = X_1 \times \cdots \times X_d$ with $X_1, \ldots, X_d \subseteq \mathbf{K}$. We also recall that for a subset $X \subseteq \mathbf{K}$ and $n \in \mathbb{N}$, we defined

$$X^{[n]} = X^{\{0,\ldots,n\}} \quad \text{and} \quad X^{]n[} = \nabla X^{[n]} = \{(x_0, \ldots, x_n) : x_i = x_j \text{ only if } i = j\}.$$

DEFINITION. Let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space, let X be a cartesian subset of $\mathbf{K}^d$ and $f \colon X \to \mathbf{E}$. Let $\boldsymbol{n} \in \mathbb{N}^d$. Put

$$X^{]\boldsymbol{n}[} := X_1^{]n_1[} \times \cdots \times X_d^{]n_d[} \quad \text{and} \quad X^{[\boldsymbol{n}]} := X_1^{[n_1]} \times \cdots \times X_d^{[n_d]}.$$

Write elements $x \in X^{[\boldsymbol{n}]}$ as $x = ({}^1x; -; {}^dx)$ with ${}^1x \in X_1^{[n_1]}, \dots, {}^dx \in X_d^{[n_d]}$. Through recursion on $n = |\boldsymbol{n}|$ we define functions $f^{]\boldsymbol{n}[}\colon X^{]\boldsymbol{n}[} \to \mathbf{E}$ by

$$f^{]\mathbf{0}[} = f,$$

and if $\boldsymbol{n}^+ = \boldsymbol{n} + \boldsymbol{e}_k$ for $k \in \{1, \dots, d\}$, then

$$\begin{aligned} &f^{]\boldsymbol{n}^+[}(-; {}^kx_0, {}^kx_1, {}^kx_2, \dots, {}^kx_{n_k+1}; -) \\ =&\frac{f^{]\boldsymbol{n}[}(-; {}^kx_0, {}^kx_2, \dots, {}^kx_{n_k+1}; -) - f^{]\boldsymbol{n}[}(-; {}^kx_1, {}^kx_2, \dots, {}^kx_{n_k+1}; -)}{{}^kx_0 - {}^kx_1}; \end{aligned}$$

here the hyphenations to the left and right of the semicolons representing the same omitted arguments ${}^1x; -; {}^{k-1}x$ and ${}^{k+1}x; -; {}^dx$.

Definition 7.1. Let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space, let X be a cartesian subset of $\mathbf{K}^d$ and $f\colon X \to \mathbf{E}$.

(i) Let $N \subseteq \{1, \dots, d\}$. Then, for any $\boldsymbol{n} \in \mathbb{N}^d$, we define the N-th difference operator $\Delta^N f^{]\boldsymbol{n}[}$ of $f^{]\boldsymbol{n}[}$ by recursion on #N as follows: We put

$$\Delta^{\emptyset} f^{]\boldsymbol{n}[} = f^{]\boldsymbol{n}[}$$

and define $\Delta^{N \cup \{k\}} f^{]\boldsymbol{n}[}$ for $N \subseteq \{1, \dots, d\}$ and $k \in \{1, \dots, d\} - N$ by

$$\begin{aligned} &\Delta^{N \cup \{k\}} f^{]\boldsymbol{n}[}(\dots; {}^kx_0, {}^kx_1, {}^kx_2, \dots, {}^kx_{n+1}; \dots) \\ =&\Delta^N f^{]\boldsymbol{n}[}(\dots; {}^kx_0, {}^kx_2, \dots, x_{n+1}; \dots) - \Delta^N f^{]\boldsymbol{n}[}(\dots; {}^kx_1, {}^kx_2, \dots, x_{n+1}; \dots). \end{aligned}$$

(ii) Let $\boldsymbol{r} = \boldsymbol{\nu} + \boldsymbol{\rho} \in \mathbb{R}^d_{\geq 0}$ with $\boldsymbol{\nu} \in \mathbb{N}^d$ and $\boldsymbol{\rho} \in [0, 1[^d$. We let $\lceil \boldsymbol{r} \rceil = (\lceil r_1 \rceil, \dots, \lceil r_d \rceil)$ with $\lceil r \rceil = \min\{n \in \mathbb{N} : n \geq n\}$ for $r \in \mathbb{R}$ and put $N_{\boldsymbol{\rho}} = \{k \in \{1, \dots, d\} : \rho_k > 0\}$. We define $|f^{]\boldsymbol{r}[}|\colon X^{]\lceil \boldsymbol{r} \rceil[} \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\boldsymbol{r}[}|(\boldsymbol{x}) = \frac{|\Delta^{N_{\boldsymbol{\rho}}} f^{]\boldsymbol{\nu}[}(\boldsymbol{x})|}{\prod_{k \in N_{\boldsymbol{\rho}}} |{}^kx_0 - {}^kx_1|^{\rho_k}}.$$

We will in the following endow $\mathbb{N}^d$ with the partial lexicographic ordering, given by $\boldsymbol{n} \leq \boldsymbol{m}$ if $n_1 \leq m_1, \dots, n_d \leq m_d$.

Definition 7.2. Let $\boldsymbol{r} = \boldsymbol{\nu} + \boldsymbol{\rho} \in \mathbb{R}^d_{\geq 0}$ with $\boldsymbol{\nu} \in \mathbb{N}^d$ and $\boldsymbol{\rho} \in [0, 1[^d$. Let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space, let X be a cartesian subset of $\mathbf{K}^d$ and $f\colon X \to \mathbf{E}$. Then f will be a $\mathscr{C}^{\boldsymbol{r}}$**-function** if the following holds:

- For every $\boldsymbol{n} \leq \boldsymbol{\nu}$ we find $f^{]\boldsymbol{n}[}$ to extend to a continuous function $f^{[\boldsymbol{n}]}$ on $X^{[\boldsymbol{n}]}$.
- Put $N_{\boldsymbol{\rho}} = \{k \in \{1, \dots, d\} : \rho_k > 0\}$. If $\boldsymbol{n} \leq \boldsymbol{\nu}$ with $N_{\boldsymbol{\rho}}(\boldsymbol{n}) = \{k \in N_{\boldsymbol{\rho}} : n_k = \nu_k\} \neq \emptyset$, define $\boldsymbol{\rho}(\boldsymbol{n})$ by $\rho(\boldsymbol{n})_k = \rho_k$ if $k \in N_{\boldsymbol{\rho}}(\boldsymbol{n})$ and having vanishing coordinate entries otherwise. Then we find

$$|f^{]\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})[}|\colon X^{]\lceil \boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n}) \rceil[} \to \mathbb{R}_{\geq 0}$$

to extend to a continuous function $|f^{[\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})]}|$ on $X^{[\lceil \boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n}) \rceil]}$ vanishing for all $\boldsymbol{x} \in X^{[\lceil \boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n}) \rceil]}$ with ${}^kx_0 = {}^kx_1$ for some $k \in N_{\boldsymbol{\rho}}(\boldsymbol{n})$.

We will denote the set of all $\mathscr{C}^{\boldsymbol{r}}$-functions $f\colon \mathrm{X} \to \mathbf{E}$ by $\mathscr{C}^{\boldsymbol{r}}(\mathrm{X}, \mathbf{E})$.

Definition 7.3. Let $\mathrm{X} \subseteq \mathbf{K}^d$ be compact. We define the norm on $\mathscr{C}^{\boldsymbol{r}}(\mathrm{X}, \mathbf{E})$ by

$$\begin{aligned}\|f\|_{\mathscr{C}^{\boldsymbol{r}}} &= \max\{\|f^{[\boldsymbol{n}]}\|_{\sup} : \boldsymbol{n} \leq \boldsymbol{\nu}\}\\ &\quad \vee \max\{\||f^{[\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})]}|\|_{\sup} : \boldsymbol{n} \leq \boldsymbol{\nu} \text{ with } \mathrm{N}_{\boldsymbol{\rho}}(\boldsymbol{n}) \neq \emptyset\}.\end{aligned}$$

8. The exponential law and Tensor products

In the following it will be useful to give another characterization of $|f^{]\boldsymbol{r}[}|$ through a function $f^{]\boldsymbol{r}[}\colon \mathrm{X}^{][\boldsymbol{r}][} \to \tilde{\mathbf{E}}$ for a $\mathbf{K}$-Banach space $\tilde{\mathbf{E}} \supseteq \mathbf{E}$ such that $\||f^{]\boldsymbol{r}[}|\|$ closely mimics the properties of $|f^{]\boldsymbol{r}[}|$.

To this end, we can define for every field $\mathbf{K}$ such that $\rho \in v(\mathbf{K})$, say $v(c_\rho) = \rho$, the exponential mapping $\cdot^{\rho}$ on $\mathbf{K}^*$ by

$$x \mapsto c_\rho^{\lfloor v_{\mathbf{K}}(x) \rfloor}.$$

It is readily verified that $\cdot^{\rho}$ is continuous and it holds $|\cdot|^{\rho} \leq |\cdot^{\rho}| \leq \mathrm{C}|\cdot|^{\rho}$ with $\mathrm{C} = |1/c_\rho| > 0$.

Let $\tilde{\mathbf{K}} \supseteq \mathbf{K}$ be a complete valued field extension such that $\rho_1, \ldots, \rho_d \in v(\tilde{\mathbf{K}})$. There is at least one such, the field of analytic elements on the circle of polyradius $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_d)$. It is given as the completion of the field of rational functions in d variables $\mathbf{K}(t_1, \ldots, t_d)$ with respect to the additive value defined by $v(\sum a_{\boldsymbol{i}} t_1^{i_1} \cdots t_d^{i_d}) = \min\{v(a_{\boldsymbol{i}}) + \rho_1 i_1 + \cdots + \rho_d i_d : \boldsymbol{i} \in \mathbb{N}^d\}$ for any $f = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} t_1^{i_d} \cdots t_d^{i_d} \in \mathbf{K}[t_1, \ldots, t_d]$. We let $\tilde{\mathbf{E}} = \mathbf{E} \hat{\otimes}_{\mathbf{K}} \tilde{\mathbf{K}}$ be the topological tensor product.

Definition. Let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space, let X be a cartesian subset of $\mathbf{K}^d$ and $f\colon \mathrm{X} \to \mathbf{E}$. Let us keep the notations of Definition 7.1. Then we define $f^{]\boldsymbol{r}[}\colon \mathrm{X}^{][\boldsymbol{r}][} \to \tilde{\mathbf{E}}$ by

$$f^{]\boldsymbol{r}[}(\boldsymbol{x}) = \frac{\Delta^{\mathrm{N}_{\boldsymbol{\rho}}} f^{]\boldsymbol{\nu}[}(\boldsymbol{x})}{\prod\limits_{k \in \mathrm{N}_{\boldsymbol{\rho}}} ({}^k x_0 - {}^k x_1)^{\rho_k}}.$$

Let us assume $f \in \mathscr{C}^0(\mathrm{X}, \mathbf{E})$. Then because of the continuity of f and of the functions $\cdot^{\rho_1}, \ldots, \cdot^{\rho_d}$ and the sandwiching $|f^{]\boldsymbol{r}[}| \leq \|f^{]\boldsymbol{r}[}\| \leq \mathrm{C} \cdot |f^{]\boldsymbol{r}[}|$ with $\mathrm{C} = |1/c_{\rho_1} \cdots c_{\rho_d}| > 0$, we find

$$|f^{]\boldsymbol{r}[}|\colon \mathrm{X}^{][\boldsymbol{r}][} \to \mathbb{R}_{\geq 0}$$

to extend to a continuous function $|f^{[\boldsymbol{r}]}|$ on $\mathrm{X}^{[[\boldsymbol{r}]]}$ vanishing for all $\boldsymbol{x} \in \mathrm{X}^{[[\boldsymbol{r}]]}$ such that ${}^k x_0 = {}^k x_1$ for some $k \in \mathrm{N}_{\boldsymbol{\rho}}$ if and only if the same holds true for

$$f^{]\boldsymbol{r}[}\colon \mathrm{X}^{][\boldsymbol{r}][} \to \tilde{\mathbf{E}}.$$

We therefore find the following definition of a $\mathscr{C}^r$-function to be equivalent to the one of Definition 7.2 above.

Definition. Let us keep the notation of Definition 7.2. Then f will be a $\mathscr{C}^r$-**function** if the following holds:

- For every $\boldsymbol{n} \leq \boldsymbol{\nu}$ we find $f^{]\boldsymbol{n}[}$ to extend to a continuous function $f^{[\boldsymbol{n}]}$ on $X^{[\boldsymbol{n}]}$.
- We find

$$f^{]\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})[}\colon X^{][\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})][} \to \tilde{\mathbf{E}}$$

to extend to a continuous function $f^{[\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})]}$ on $X^{[\lceil \boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})\rceil]}$ vanishing for all $\boldsymbol{x} \in X^{[\lceil \boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})\rceil]}$ with ${}^k x_0 = {}^k x_1$ for some $k \in N_{\boldsymbol{\rho}}(\boldsymbol{n})$.

Likewise, once again as a direct consequence of the above sandwiching $|f^{]\boldsymbol{r}[}| \leq \|f^{]\boldsymbol{r}[}\| \leq C \cdot |f^{]\boldsymbol{r}[}|$ for any $f \in \mathscr{C}^0(X, \mathbf{E})$, the topology of $\mathscr{C}^{\boldsymbol{r}}(X, \mathbf{E})$ can equivalently be given through the following norm:

Definition. Let $X \subseteq \mathbf{K}^d$ be compact. We define the norm on $\mathscr{C}^{\boldsymbol{r}}(X, \mathbf{E})$ by

$$\begin{aligned} \|f\|_{\mathscr{C}^{\boldsymbol{r}}} = \max\{&\|f^{[\boldsymbol{n}]}\|_{\sup} : \boldsymbol{n} \leq \boldsymbol{\nu}\} \\ &\vee \max\{\|f^{[\boldsymbol{n}+\boldsymbol{\rho}(\boldsymbol{n})]}\|_{\sup} : \boldsymbol{n} \leq \boldsymbol{\nu} \text{ with } N_{\boldsymbol{\rho}}(\boldsymbol{n}) \neq \emptyset\}. \end{aligned}$$

We will extend results obtained in [Glö13] for partially differentiable functions with integral order of differentiability to case of fractional order of differentiability. Due to the already abundant technicalities involved in loc. cit. we will restrict to the most simple case when the functions' domains are compact cartesian with factors free of isolated points. Having the concrete application of many-variable functions with arguments in the ring of integers $\mathbf{o}_{\mathbf{K}}$ of a local field $\mathbf{K}$ in mind, this will prove sufficient for our purposes.

Theorem 8.1. *Let $d, e \in \mathbb{N}$ and $r \in \mathbb{R}^d_{\geq 0}$, $s \in \mathbb{R}^e_{\geq 0}$. Let $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ be compact cartesian subsets with factors free of isolated points and $\mathbf{E}$ be a $\mathbf{K}$-Banach space. Then we have a well-defined natural mapping defined by*

$$\begin{aligned} \mathscr{C}^{(r,s)}(X \times Y, \mathbf{E}) &\to \mathscr{C}^r(X, \mathscr{C}^s(Y, \mathbf{E})) \\ f &\mapsto [x \mapsto f(x, \cdot)] \end{aligned}$$

which is an isomorphism of topological vector spaces.

Proof: Owed to the fact that $\mathbb{R}_{\geq 0}$ is not a non-Archimedean $\mathbf{K}$-Banach space, it will be more expedient to use the above equivalent definition of the topological $\mathbf{K}$-vector space of $\mathscr{C}^r$-function. This being agreed upon, we make the following general observation: Let $d \in \mathbb{N}$ and $r \in \mathbb{R}^d_{\geq 0}$. Let $X \subseteq \mathbf{K}^d$ be compact open. Then the topological vector space $\mathscr{C}^r(X, \mathbf{E})$ is canonically isomorphic to the topological vector subspace

$$\{(g_n) \in \prod_{n \leq \nu} \mathscr{C}^0(\mathrm{X}^{[n]}, \mathbf{E}) \times \prod_{\substack{n \leq \nu \text{ with} \\ \mathrm{N}_\rho(n) \neq \emptyset}} \mathscr{C}^0(\mathrm{X}^{[\lceil n+\rho(n) \rceil]}, \tilde{\mathbf{E}}) \colon \text{ There is } f \colon \mathrm{X} \to \mathbf{E} \text{ with:}$$

- For all $n \leq \nu$ holds $g_{n|\mathrm{X}^{]n[}} = f^{]n[}$

- For all $n \leq \nu$ with $\mathrm{N}_\rho(n) \neq \emptyset$ holds $g_{n|\mathrm{X}^{]\lceil n+\rho(n) \rceil[}} = |f^{]n+\rho(n)[}|$

  and $g_n(x) = 0$ for all $x \in \mathrm{X}^{[\lceil n+\rho(n) \rceil]}$ with ${}^k x_0 = {}^k x_1$ for some $k \in \mathrm{N}_\rho(n)\}$

$$\subseteq \prod_{n \leq \nu} \mathscr{C}^0(\mathrm{X}^{[n]}, \mathbf{E}) \times \prod_{\substack{n \leq \nu \text{ with} \\ \mathrm{N}_\rho(n) \neq \emptyset}} \mathscr{C}^0(\mathrm{X}^{[\lceil n+\rho(n) \rceil]}, \tilde{\mathbf{E}}).$$

For later use, let us introduce the following notation for the set collecting all indices through which the above two products are running:

$$\begin{aligned} \mathrm{I}_r = \quad & \{n \in \mathbb{N}^d : n \leq \nu\} \\ & \cup \{u \in \mathbb{R}^d_{\geq 0} : u = n + \rho(n) \text{ for } n \in \mathbb{N}^d \text{ with } n \leq \nu \text{ and } \mathrm{N}_\rho(n) \neq \emptyset\}. \end{aligned}$$

Let us return to the situation considered in our proposition: Write $r = \nu + \rho$ and $s = \mu + \sigma$ with ν, μ having entries in $\mathbb{N}$ and ρ, σ in $[0, 1[$. By [**Glö13**, Proposition B.15], we have an isomorphism ϕ of topological vector spaces

$$\begin{aligned} & \prod_{(n,m) \leq (\nu,\mu)} \mathscr{C}^0(\mathrm{X}^{[n]} \times \mathrm{Y}^{[m]}, \mathbf{E}) \times \prod_{\substack{(n,m) \leq (\nu,\mu) \text{ with} \\ \mathrm{N}_{(\rho,\sigma)}((n,m)) \neq \emptyset}} \mathscr{C}^0(\mathrm{X}^{[\lceil n+\rho(n) \rceil]} \times \mathrm{Y}^{[\lceil m+\sigma(m) \rceil]}, \tilde{\mathbf{E}}) \\ \tilde{\to} \quad & \prod_{(n,m) \leq (\nu,\mu)} \mathscr{C}^0(\mathrm{X}^{[n]}, \mathscr{C}^0(\mathrm{Y}^{[m]}, \mathbf{E})) \\ & \times \prod_{\substack{(n,m) \leq (\nu,\mu) \text{ with} \\ \mathrm{N}_{(\rho,\sigma)}((n,m)) \neq \emptyset}} \mathscr{C}^0(\mathrm{X}^{[\lceil n+\rho(n) \rceil]}, \mathscr{C}^0(\mathrm{Y}^{[\lceil m+\sigma(m) \rceil]}, \tilde{\mathbf{E}})) \\ \tilde{\to} \quad & \prod_{n \leq \nu} \mathscr{C}^0(\mathrm{X}^{[n]}, \prod_{m \leq \mu} \mathscr{C}^0(\mathrm{Y}^{[m]}, \mathbf{E}) \times \prod_{\substack{m \leq \mu \text{ with} \\ \mathrm{N}_\sigma(m) \neq \emptyset}} \mathscr{C}^0(\mathrm{Y}^{[\lceil m+\sigma(m) \rceil]}, \tilde{\mathbf{E}})) \\ & \times \prod_{\substack{n \leq \nu \text{ with} \\ \mathrm{N}_\rho(n) \neq \emptyset}} \mathscr{C}^0(\mathrm{X}^{[\lceil n+\rho(n) \rceil]}, \prod_{m \leq \mu} \mathscr{C}^0(\mathrm{Y}^{[m]}, \mathbf{E}) \times \prod_{\substack{m \leq \mu \text{ with} \\ \mathrm{N}_\sigma(m) \neq \emptyset}} \mathscr{C}^0(\mathrm{Y}^{[\lceil m+\sigma(m) \rceil]}, \tilde{\mathbf{E}})); \end{aligned}$$

where the second isomorphism is a rearrangement of factors using the canonical isomorphism $\mathscr{C}^0(\mathrm{A}, \mathrm{B}) \times \mathscr{C}^0(\mathrm{A}, \mathrm{C}) \tilde{\to} \mathscr{C}^0(\mathrm{A}, \mathrm{B} \times \mathrm{C})$ for topological spaces A, B and C.

Using these identifications, it rests to check that ϕ respects the restrictions cutting out the subspaces $\mathscr{C}^{(r,s)}(\mathrm{X} \times \mathrm{Y}, \mathbf{E}) \subseteq \mathrm{domain}(\phi)$ in its domain respectively $\mathscr{C}^r(\mathrm{X}, \mathscr{C}^s(\mathrm{Y}, \mathbf{E}))$ in its range, that is, $\phi(\mathscr{C}^{(r,s)}(\mathrm{X} \times \mathrm{Y}, \mathbf{E})) = \mathscr{C}^r(\mathrm{X}, \mathscr{C}^s(\mathrm{Y}, \mathbf{E}))$. Then ϕ will be a linear isomorphism and, as the topologies are given by the respective subspace topologies, automatically a homeomorphism. To this end, we check that:

(i) For $\boldsymbol{f} \in \operatorname{domain}(\phi)$, it holds for all $(u,v) \in \mathrm{I}_{(r,s)}$ that

$$(\phi(\boldsymbol{f})_u(x))_v(y) = \boldsymbol{f}_{u,v}(x,y) \quad \text{for all } (x,y) \in \mathrm{X}^{]\lceil u\rceil[} \times \mathrm{Y}^{]\lceil v\rceil[}.$$

(ii) Let $\varphi\colon \mathscr{C}^0(\mathrm{X}\times\mathrm{Y},\mathbf{E}) \xrightarrow{\sim} \mathscr{C}^0(\mathrm{X},\mathscr{C}^0(\mathrm{Y},\mathbf{E}))$. For $f \in \mathscr{C}^0(\mathrm{X}\times\mathrm{Y},\mathbf{E})$, it holds for all $(u,v) \in \mathrm{I}_{(r,s)}$ that

$$(\varphi(f)^{]u[}(x))^{]v[}(y) = f^{]u,v[}(x,y) \quad \text{for all } (x,y) \in \mathrm{X}^{]\lceil u\rceil[} \times \mathrm{Y}^{]\lceil v\rceil[}.$$

(It is here that we make use of the equivalent description of $\mathscr{C}^r$-functions. Indeed, we find $\varphi(f)^{]u[}\colon \mathrm{X}^{]\lceil u\rceil[} \to \mathscr{C}^0(\mathrm{Y},\tilde{\mathbf{E}})$, thus ensuring $(\varphi(f)^{]u[}(x))^{]v[}$ to be well defined).

Let $\boldsymbol{f} \in \operatorname{domain}(\phi)$. It follows that, given $f \in \mathscr{C}^0(\mathrm{X}\times\mathrm{Y},\mathbf{E})$ and $(u,v) \in \mathrm{I}_{(r,s)}$, we find

$$\boldsymbol{f}_{u,v} = f^{]u,v[} \quad \text{on } \mathrm{X}^{]\lceil u\rceil,\lceil v\rceil[}$$

if and only if $\varphi(f) \in \mathscr{C}^0(\mathrm{X},\mathscr{C}^0(\mathrm{Y},\mathbf{E}))$ satisfies

$$(\phi(\boldsymbol{f})_u)(x)_v(y) = (\phi(f)^{]u[}(x))^{]v[}(y) \quad \text{for all } x \in \mathrm{X}^{]\lceil u\rceil[} \text{ and } y \in \mathrm{Y}^{]\lceil v\rceil[}.$$

This shows that $\phi(\mathscr{C}^{(r,s)}(\mathrm{X}\times\mathrm{Y},\mathbf{E})) = \mathscr{C}^r(\mathrm{X},\mathscr{C}^s(\mathrm{Y},\mathbf{E}))$. □

THEOREM 8.2. *Let $r \in \mathbb{R}_{\geq 0}$ and $\mathrm{X} \subseteq \mathbf{K}$ compact without isolated points. Then the natural map*

$$\begin{aligned} \mathscr{C}^r(\mathrm{X},\mathbf{K}) \times \mathbf{E} &\to \mathscr{C}^r(\mathrm{X},\mathbf{E}) \\ (f,e) &\mapsto f \otimes e := [x \mapsto f(x)\cdot e] \end{aligned}$$

induces an isomorphism of $\mathbf{K}$-Banach spaces $\mathscr{C}^r(\mathrm{X},\mathbf{K})\hat{\otimes}_{\mathbf{K}}\mathbf{E} \xrightarrow{\sim} \mathscr{C}^r(\mathrm{X},\mathbf{E})$.

PROOF: By the criterion of [vR78, Comment following Cor. 4.31], it suffices to check the following:

(i) The mapping Ψ is bilinear and norm-nonincreasing.
(ii) Let $0 < t \leq 1$. If $f_1,\dots,f_n \in \mathscr{C}^r(\mathrm{X},\mathbf{K})$ are t-orthogonal, then for any $e_1,\dots,e_n \in \mathbf{E}$ we find $f_1\otimes e_1,\dots,f_n\otimes e_n \in \mathscr{C}^r(\mathrm{X},\mathbf{E})$ to be t-orthogonal.
(iii) The $\mathbf{K}$-linear span of $\operatorname{im}\Psi$ is dense in $\mathscr{C}^r(\mathrm{X},\mathbf{E})$.

Indeed, Criterion 1. readily shows Ψ to extend by bilinearity to a linear mapping on $\mathscr{C}^r(\mathrm{X},\mathbf{K}) \otimes_{\mathbf{K}} \mathbf{E}$ and by continuity to one onto $\mathscr{C}^r(\mathrm{X},\mathbf{K})\hat{\otimes}_{\mathbf{K}}\mathbf{E}$. By Criterion 2. this map is an homeomorphism onto its image. In particular its image is closed and by Criterion 3. we can conclude it to be surjective.

Criteria 1. and 2. are readily checked to hold true by definition. Regarding Criterion 3. we find by Proposition 4.4 the subset $\mathscr{C}^{\mathrm{lp}\leq r}(\mathrm{X},\mathbf{E})$ to be dense inside $\mathscr{C}^r(\mathrm{X},\mathbf{E})$. Because $\mathscr{C}^{\mathrm{lp}\leq r}(\mathrm{X},\mathbf{E}) = \mathscr{C}^{\mathrm{lp}\leq r}(\mathrm{X},\mathbf{K})\otimes_{\mathbf{K}}\mathbf{E} \subseteq \operatorname{im}\Psi$, in particular $\operatorname{im}\Psi \subseteq \mathscr{C}^r(\mathrm{X},\mathbf{E})$ is dense. □

Corollary 8.3. *Let $r \in \mathbb{R}^d_{\geq 0}$ and $X = X_1 \times \cdots \times X_d \subseteq \mathbf{K}^d$ a compact cartesian subset with factors free of isolated points. Then the natural map*

$$\mathscr{C}^{r_1}(X_1, \mathbf{K}) \times \cdots \times \mathscr{C}^{r_d}(X_d, \mathbf{K}) \to \mathscr{C}^r(X, \mathbf{K})$$
$$(f_1, \ldots, f_d) \mapsto f \otimes \cdots \otimes f_d := [(x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)]$$

induces an isomorphism of **K***-Banach spaces*

$$\mathscr{C}^{r_1}(X_1, \mathbf{K}) \hat{\otimes} \cdots \hat{\otimes} \mathscr{C}^{r_d}(X_d, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}^r(X, \mathbf{K}).$$

Proof: By induction on d. If $d = 1$ there is nothing to show. Let $d > 1$. We put $r = (s, t)$ with $s = r_1 \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}^{d-1}_{\geq 0}$ and $X = Y \times Z$ with $Y = X_1$ and $Z = X_2 \times \cdots \times X_d$. Then by Theorem 8.1 we find

$$\mathscr{C}^r(X, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}^s(Y, \mathbf{E})$$

with $\mathbf{E} = \mathscr{C}^t(Z, \mathbf{K})$. By Theorem 8.2

$$\mathscr{C}^s(Y, \mathbf{E}) \xrightarrow{\sim} \mathscr{C}^s(Y, \mathbf{K}) \hat{\otimes}_{\mathbf{K}} \mathbf{E} = \mathscr{C}^s(Y, \mathbf{K}) \hat{\otimes}_{\mathbf{K}} \mathscr{C}^t(Z, \mathbf{K}).$$

The induction hypothesis provides an isomorphism

$$\mathscr{C}^t(Z, \mathbf{K}) = \mathscr{C}^{r_2}(X_2, \mathbf{K}) \hat{\otimes}_{\mathbf{K}} \cdots \hat{\otimes}_{\mathbf{K}} \mathscr{C}^{r_d}(X_d, \mathbf{K}).$$

We conclude

$$\mathscr{C}^r(X, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}^{r_1}(X_1, \mathbf{K}) \hat{\otimes}_{\mathbf{K}} \mathscr{C}^{r_2}(X_2, \mathbf{K}) \hat{\otimes}_{\mathbf{K}} \cdots \hat{\otimes}_{\mathbf{K}} \mathscr{C}^{r_d}(X_d, \mathbf{K}).$$

□

9. The multivariate van der Put- and Mahler basis

We define the multivariate van der Put-basis and Mahler basis via the tensor products of the one-variable functions. Let us keep the notations of Definition 5.2.

Definition 9.1. Let $\boldsymbol{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d_{\geq 0}$. The **van der Put-basis**

$$\{e_{(s_1, i_1), \ldots, (s_d, i_d)} : ((s_1, i_1), \ldots, (s_d, i_d)) \in S \times \{0, \ldots, \lfloor r_1 \rfloor\} \times \cdots \times S \times \{0, \ldots, \lfloor r_d \rfloor\}\}$$

of $\mathscr{C}^{\boldsymbol{r}}(\mathbf{o}^d, \mathbf{K})$ is defined by

$$e_{(s_1, i_1), \ldots, (s_d, i_d)} = e_{s_1, i_1} \otimes \cdots \otimes e_{s_d, i_d} := [(x_1, \ldots, x_d) \mapsto e_{s_1, i_1}(x_1) \cdots e_{s_d, i_d}(x_d)].$$

Definition 9.2. Let $\boldsymbol{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d_{\geq 0}$. The **Mahler basis**

$$\{\binom{*}{\boldsymbol{i}} : \boldsymbol{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d\}$$

of $\mathscr{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ is defined by

$$\binom{*}{\boldsymbol{i}} = \binom{*}{i_1} \otimes \cdots \otimes \binom{*}{i_d} := [(x_1, \ldots, x_d) \mapsto \binom{x_1}{i_1} \cdots \binom{x_d}{i_d}].$$

We refer the reader to for example, [**Nag13**] for a definition of r-fold differentiability in many variables for functions taking values in non-Archimedanly valued Banach spaces for a real number $r \geq 0$.

Corollary 9.3. *We establish the following orthogonal bases inside fractionally differentiable function spaces.*

(i) *Let us regard the van der Put basis:*
  (a) *Let* $r \in \mathbb{R}^d_{\geq 0}$. *The van der Put-basis* $\{e_{(\boldsymbol{s},\boldsymbol{i})} : (\boldsymbol{s},\boldsymbol{i}) \in \prod_{k=1,\dots,d} \mathrm{S} \times \{0,\dots,\lfloor r_k \rfloor\}\}$ *and Mahler basis are orthogonal bases of* $\mathscr{C}^r(\mathbf{o}^d, \mathbf{K})$.
  (b) *Let* $r \in \mathbb{R}_{\geq 0}$. *The van der Put-basis* $\{e_{(\boldsymbol{s},\boldsymbol{i})} : (\boldsymbol{s},\boldsymbol{i}) \in \mathrm{S}^d \times \mathbb{N}^d$ *with* $i_1 + \cdots + i_d \leq r\}$ *is an orthogonal basis of* $\mathscr{C}^r(\mathbf{o}^d, \mathbf{K})$.

(ii) *Let us regard the Mahler basis: Let* $r \in \mathbb{R}_{\geq 0}$ *or* $r \in \mathbb{R}^d_{\geq 0}$. *Then the Mahler basis* $\{\binom{*}{\boldsymbol{i}} : \boldsymbol{i} \in \mathbb{N}^d\}$ *is an orthogonal basis of* $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$.

Proof: Ad 1.a): This follows directly by Theorem 5.8 and Theorem 8.2, for example, through [**Nag11**, Corollary 3.34]. Ad 1.b): Because V is in particular by Step 1.a) orthogonal in each $\mathscr{C}^{\boldsymbol{r}}(\mathbf{o}^d, \mathbf{K})$ for $\boldsymbol{r} \in \mathbb{N}^d_{=r}$, a quick check, carried out in [**Nag13**, Lemma 29.(i)], shows V to be orthogonal in $\mathscr{C}^r(\mathbf{o}^d, \mathbf{K})$. Since in particular $e_{s,i}$ for $s \in \mathrm{S}, i \in \{0,\dots,\lfloor r \rfloor\}$ is a basis of $\mathscr{C}^{\mathrm{lp} \leq r}(\mathbf{o}, \mathbf{K})$, we find

$$\mathrm{V} := \{e_{(\boldsymbol{s},\boldsymbol{i})} : (\boldsymbol{s},\boldsymbol{i}) \in \mathrm{S}^d \times \mathbb{N}^d \text{ with } i_1 + \cdots + i_d \leq r\}$$

to be a basis of $\mathscr{C}^{\mathrm{lp} \leq r}(\mathbf{o}^d, \mathbf{K})$. Thus by [**Nag13**, Proposition II.40] we find the $\mathbf{K}$-linear span of V to be dense inside $\mathscr{C}^r(\mathbf{o}^d, \mathbf{K})$. By [**Nag13**, Proposition II.33] we find $\mathscr{C}^r(\mathbf{o}^d, \mathbf{K})$ to be the initial vector space

$$\mathscr{C}^r(\mathbf{o}^d, \mathbf{K}) = \bigcap_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} \mathscr{C}^{\boldsymbol{r}}(\mathbf{o}^d, \mathbf{K})$$

with $\mathbb{N}^d_{=r} :=$

$\{\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$ such that $r_1 + \cdots + r_d = r$ and all except at most one entry are in $\mathbb{N}\}$.

By [**Sch84**, Exercise 50.F] we conclude V to be an orthogonal basis of $\mathscr{C}^r(\mathbf{o}^d, \mathbf{K})$.

Ad 2.: If $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$, this follows directly by Theorem 5.8 and Theorem 8.2, for example, through [**Nag11**, Corollary 3.34]. Let $r \in \mathbb{R}_{\geq 0}$. Then as above we find $\mathrm{M} := \{\binom{*}{\boldsymbol{i}} : \boldsymbol{i} \in \mathbb{N}^d\}$ to be orthogonal inside $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. Its $\mathbf{K}$-linear span is the set of polynomial functions $p \colon \mathbb{Z}_p^d \to \mathbf{K}$ which is by [**Nag13**, Corollary II.42] dense inside $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. We conclude by [**Sch84**, Exercise 50.F] that V is an orthogonal basis of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. □

Remark. We make this explicit in the following example: Let $f \in \mathscr{C}^{\boldsymbol{r}}(\mathrm{X}, \mathbf{E})$. For $\boldsymbol{i} \leq \boldsymbol{\nu}$, we denote by $\mathrm{D}_{\boldsymbol{i}} f : \mathrm{X} \to \mathbf{E}$ the function given by

$$\mathrm{D}_{\boldsymbol{i}} f(a) = f^{[\boldsymbol{i}]}(a_1,\dots,a_1;\dots;a_d,\dots,a_d).$$

We obtain

$$\|e_{(s,i),(t,j)}\|_{\mathscr{C}^{r,s}} = |1/\pi|^{(r-i)(\ell(s)-1)+(t-j)(\ell(t)-1)},$$

and letting $f = \sum \lambda_{(s,i),(t,j)} e_{(s,i),(t,j)} \in \mathscr{C}^{r,s}(\mathbf{o} \times \mathbf{o}, \mathbf{K})$, it holds

$$\lambda_{(0,i),(0,j)} = \mathrm{D}_{i,j} f(0,0) \text{ for } i = 0, \dots, \nu$$

and

$$\lambda_{(s,i),(0,j)} = \mathrm{D}_{i,0} f(s,0) - \sum_{k=0,\dots,\nu-i} (s-s^-)^k \binom{i+k}{i} \mathrm{D}_{i+k,0} f(s^-,0)$$

respectively

$$\lambda_{(0,i),(t,j)} = \mathrm{D}_{0,j} f(0,t) - \sum_{l=0,\dots,\mu-j} (t-t^-)^l \binom{j+l}{j} \mathrm{D}_{0,j+l} f(0,t^-)$$

and, for nonzero $s, t \in \mathrm{S}$,

$$\begin{aligned}\lambda_{(s,i)(t,j)} = {} & \mathrm{D}_{i,j} f(s,t) \\ & - \sum_{k=0,\dots,\nu-i} \binom{i+k}{i} \mathrm{D}_{i+k,j} f(i^-,j)(s-s^-)^k \\ & - \sum_{l=0,\dots,\mu-j} \binom{j+l}{j} \mathrm{D}_{i,j+l} f(i,j^-)(t-t^-)^l \\ & + \sum_{\substack{k=0,\dots,\nu-i,\\ l=0,\dots,\mu-j}} \binom{i+k}{i}\binom{j+l}{j} \mathrm{D}_{i+k,j+l} f(s^-,t^-)(s-s^-)^k(t-t^-)^l.\end{aligned}$$

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