Fractional non-Archimedean calculus in one variable

By Enno Nagel

Let ν be a natural number. A function $f\colon \mathbb{Z}_p \to \mathbf{K}$ into a non-Archimedeanly valued complete field $\mathbf{K} \supseteq \mathbb{Q}_p$ is ν-times continuously differentiable if and only if its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ obey $|a_n|n^\nu \to 0$ as $n \to \infty$. For a real number $r \geq 0$, this suggests the ad hoc definition by [BB10] of a *$\mathscr{C}^r$-function* $f\colon \mathbb{Z}_p \to \mathbf{K}$ by asking its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ to satisfy $|a_n|n^r \to 0$ as $n \to \infty$.

We will present for functions $f\colon X \to \mathbf{K}$ on subsets $X \subseteq \mathbf{K}$ without isolated points a general pointwise notion of r-fold differentiability through iterated difference quotients, subsequently shown on the domain $X = \mathbb{Z}_p$ to coincide with the one given above. For functions on open domains, we prove this notion to admit a handier characterization by its Taylor polynomial up to degree $\lfloor r \rfloor$.

Contents

Introduction

Let $\mathbf{K} \supseteq \mathbb{Q}_p$ be a valued field extension. There is a distinguished (see Section 2) orthogonal basis of the continuous $\mathbf{K}$-valued functions $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ relating to the domain's cyclic topological group structure. Cf. Definition 2.36, it is given by the *Mahler polynomials* $\binom{*}{i}$ for

$i \in \mathbb{N}$, and we accordingly call a function's coefficients with respect to this basis its *Mahler coefficients*.

In [Sch84, Section 29 ff.] the author introduced for $\nu \in \mathbb{N}$ the concept of ν-fold iterated differentiability of a function of one variable on a non-Archimedean domain and proved in the following (cf. [Sch84, Section 53]) a ν-times differentiable function $f\colon \mathbb{Z}_p \to \mathbf{K}$ to be equivalently characterizable by demanding its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ to fulfill $|a_n|n^\nu \to 0$ as $n \to \infty$. This observation then underlaid in [BB10, Section 4] the introduction of the notion of a $\mathscr{C}^r$-function $f\colon \mathbb{Z}_p \to \mathbf{K}$ for any $r \in \mathbb{R}_{\geq 0}$ by asking its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ to satisfy $|a_n|n^r \to 0$ as $n \to \infty$. In the realms of the p-adic Langlands programme, this class of functions has then been used to give rise to those $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space representations corresponding to certain "crystalline" two-dimensional Galois representations.

This article aims at extending the concept of an r-times differentiable function to the more general domain of a subset without isolated points $\mathrm{X} \subseteq \mathbf{K}$ for a complete non-Archimedeanly nontrivially valued field $\mathbf{K}$ by building onto the classic concept of integral differentiability through iterated difference quotients as introduced in [Sch84].

So for integral values $\nu \in \mathbb{N}$, we will adopt the differentiability notion of [Sch84, Section 26 ff.] by iterated difference quotients. By lack of an intermediate value theorem for continuous functions over non-Archimedeanly valued domains, to have nonetheless a good notion of iterated differentiability, it will not suffice to just demand a function's successive derivatives (in the common Archimedean sense) to be differentiable at each iteration step, but rather one has to keep track of the difference quotients themselves: Namely a function $f\colon \mathrm{X} \to \mathbf{K}$ is *ν-times differentiable* at a point $a \in \mathrm{X}$ if its *ν-th iterated difference quotient*, recursively defined by $f^{]0[} := f$ and

$$f^{]\nu[}(x_0, \ldots, x_\nu) := \frac{f^{]\nu-1[}(x_0, x_2, \ldots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \ldots, x_\nu)}{x_0 - x_1},$$

defined on all $(x_0, \ldots, x_\nu) \in \mathrm{X}^{\{0,\ldots,\nu\}}$ with pairwise distinct coordinates, extends continuously to the point $\vec{a} := (a, \ldots, a)$. (See Remark 1.1 for a precise definition of this condition.) To account for the fractional part of the differentiability condition, we introduce the notion of a $\mathscr{C}^\rho$-point for $\rho \in [0, 1[$ as follows: Let U and V be metric spaces, $\mathrm{A} \subseteq \mathrm{U}$ and $f\colon \mathrm{A} \to \mathrm{V}$. Then f is $\mathscr{C}^\rho$ at the point $a \in \mathrm{U}$ if for every $\varepsilon > 0$ there is a neighborhood $\mathrm{U}_\varepsilon \ni a$ in U with

$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x, y)^\rho \quad \text{for all } x, y \in \mathrm{A} \cap \mathrm{U}_\varepsilon.$$

(Note that this amounts to a tightened Hölder condition, where we demand the difference quotient to asymptotically vanish at the point a instead of the usual boundedness condition around it.)
Now write $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu \in \mathbb{N}$ and $\rho \in [0, 1[$. Then for $f\colon \mathrm{X} \to \mathbf{K}$ to be a *$\mathscr{C}^r$-function*, we demand its ν-th iterated difference quotient $f^{]\nu[}$ not merely to extend continuously, but $\mathscr{C}^\rho$-wise at $\vec{a}$.

In Section 1 we will introduce the concept of a $\mathscr{C}^\rho$-function for $\rho \in [0, 1[$, the natural locally convex topology on the corresponding function space and state some of its basic properties.

This is then employed in Section 2, where we give the definition of a $\mathscr{C}^r$-function for $r \in \mathbb{R}_{\geq 0}$ and introduce the basic definitions and properties around this concept of r-fold differentiability.

In Section 2 we will discuss the question wether the definition of r-fold iterated differentiability can also more conveniently be characterized by the convergence of the Taylor polynomial, a function in solely two arguments (in the sense that the polynomial's unknown as well as the expansion point are considered as variables) as opposed to the $\nu + 1$ of them taken by the ν-th iterated difference quotient.

For a function $f \colon X \to \mathbf{K}$ on a general domain $X \subseteq \mathbf{K}$, we show the convergence of the Taylor polynomials of the function's derivatives $f, \mathscr{D}_1 f, \ldots, \mathrm{D}_\nu f$ to be a sufficient and necessary criterion for f to be a $\mathscr{C}^r$-function. (Where $\mathrm{D}_i f = f^{(i)}/i!$ for the usual Archimedean derivative $f^{(i)}$ in case $\operatorname{char} \mathbf{K} = 0$. See Corollary 2.5 for the general definition.) Here the crucial observation is that $f^{]\nu[}$ can be expressed as a convex combination of the Taylor polynomial's rest terms $\mathrm{R}_{\nu-i} \mathrm{D}_i f$ and the derivatives $\mathscr{D}_i f$ for $i = 0, \ldots, \nu$.

On domains $X \subseteq \mathbf{K}$ with locally sufficiently many points — such as open ones — all lower derivatives' rest terms $\mathrm{R}_{\nu-i} \mathrm{D}_i f$ for $i = 0, \ldots, \nu - 1$ can then in turn be estimated by the one of highest order $\mathrm{R}_\nu f$, reducing the question of r-fold differentiability of f indeed to one of the convergence behavior of its ν-th Taylor polynomial.

All of this generalizes results already obtained in [Sch78] for the case of integral differentiability $r = \nu \in \mathbb{N}$, where the question wether this Taylor polynomial criterion is sufficient to characterize differentiability was also discussed for the first time.

Section 2 will then show how to characterize a $\mathscr{C}^r$-function $f \colon \mathbb{Z}_p \to \mathbf{K}$ by its Mahler coefficients $(a_n)_{n \in \mathbb{N}}$ so as to recover the initial notion of r-fold differentiability on $\mathbb{Z}_p$ given in [BB10].

For this, we firstly want to show the Mahler polynomials to constitute an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ and determine their $\mathscr{C}^\rho$-norms. Because we can rather quickly check these to form an orthogonal family and compute their $\mathscr{C}^\rho$-norms, the main content of this subsection consists in showing that their $\mathbf{K}$-linear span is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Let $\mathbf{o}_{\leq \varepsilon} := \{x \in \mathbf{K} : |x| \leq \varepsilon\}$ and $\mathbf{o} := \mathbf{o}_{\leq 1}$. Let $\mathscr{C}^{\mathrm{cst}}_n(\mathbb{Z}_p, \mathbf{o}/\mathbf{o}_{\leq \varepsilon})$ denote the functions on $p^n \mathbb{Z}_p$-cosets with values in $\mathbf{o}/\mathbf{o}_{\leq \varepsilon}$ for $\varepsilon > 0$. We will then find $\varepsilon(n) > 0$ depending on n such that the functions $\{\binom{*}{0}, \ldots, \binom{*}{p^n - 1}\}$ have a well-defined image in $\mathscr{C}^{\mathrm{cst}}_n(\mathbb{Z}_p, \mathbf{o}/\mathbf{o}_{\leq \varepsilon(n)})$.

We can without loss of generality assume that $\mathbf{K}$ is discretely valued and $v(\mathbf{K}) \ni \rho$. For a Banach space E, define $\mathrm{E}_{\leq 1} := \{f \in \mathrm{E} : |f| \leq 1\}$, accordingly $\mathrm{E}_{<1}$ and put $\overline{\mathrm{E}} := \mathrm{E}_{\leq 1}/\mathrm{E}_{<1}$. Then by the previous result we can infer $\{\overline{\binom{*}{0}}, \ldots, \overline{\binom{*}{p^n - 1}}\}$ to lie in $\overline{\mathscr{C}^{\mathrm{cst}}_n(\mathbb{Z}_p, \mathbf{K})} \subseteq \overline{\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})}$ and hence by orthogonality $\overline{e}_i := \overline{\lambda_i \binom{*}{i}}$ for suitable scalars $\lambda_i \in \mathbf{K}$ with $i = 0, \ldots, p^n - 1$ constitute a basis of this subspace for all $n \geq 0$. In particular the $\overline{e_i}$ span $\overline{\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})}$. This is by a well-known criterion equivalent to the density of the $\mathbf{K}$-linear span of $\binom{*}{i}$ inside $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

We then infer a mapping $f \colon \mathbb{Z}_p^d \to \mathbf{K}$ to be a $\mathscr{C}^\rho$-function if and only if their Mahler coefficients $(a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ fulfill $|a_{\boldsymbol{n}}||\boldsymbol{n}|^\rho \to 0$ as $|\boldsymbol{n}| \to \infty$ with $|\boldsymbol{n}| := n_1 + \cdots + n_d$. This rests on the observation that $\mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$ is just the intersection of completed tensor products where one factor is given by $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ and the other ones being copies of $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$. As we have shown the Mahler polynomials in one variable to constitute an orthogonal basis of either of these function spaces, we deduce the Mahler polynomials in many variables $\binom{*}{i}$, by definition given as tensor products of the Mahler polynomials in one variable, to constitute an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$. Their $\mathscr{C}^\rho$-norms are then readily computed and yield the convergence behavior of a $\mathscr{C}^\rho$-function's Mahler coefficients claimed above.

Let $f\colon \mathbb{Z}_p \to \mathbf{K}$ be a function. Then in [Sch84, Section 53] it was already shown by combinatorial means via studying the Mahler expansion of the ν-th difference quotient $f^{]\nu[]}\colon \mathbb{Z}_p^{\{0,\dots,\nu\}} \to \mathbf{K}$ itself, that f is a $\mathscr{C}^\nu$-function if and only if its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ fulfill $|a_n|n^\nu \to 0$ as $n \to \infty$. Now because f is a $\mathscr{C}^r$-function for $r = \nu + \rho$ if and only if $f^{]\nu[}$ extends to a $\mathscr{C}^\rho$-function $f^{[\nu]}$, by applying our above attained description of the Mahler coefficients of $\mathscr{C}^\rho$-functions of many variables, we can generalize this to obtain f to be a $\mathscr{C}^r$-function if and only if its Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ satisfy $|a_n|n^r \to 0$ as $n \to \infty$. (In other words the definition given in [BB10] by Mahler coefficients and the one given here by iterated difference quotients coincide on $\mathbb{Z}_p$.) We finally remark that recently in [Col10] a different definition of a $\mathscr{C}^r$-function on $\mathbb{Z}_p$ was given, which can be seen by Proposition 2.32 to coincide with the one employed here (which uses the equivalent characterization of r-fold differentiability through Taylor polynomials by Corollary 2.30). There the same characterization of $\mathscr{C}^r$-functions on $\mathbb{Z}_p$ through their Mahler coefficients was obtained through a different approach by the explicit description of the transition behavior of the coefficients between different orthogonal bases of $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ and comparison of the norms of the locally analytic and $\mathscr{C}^r$-function spaces on $\mathbb{Z}_p$.

Notations and Conventions

Throughout this paper $\mathbf{K}$ will denote a complete non-Archimedeanly valued field whose valuation v is nontrivial. If we fix a positive real constant $c_v < 1$, we obtain a norm $|x| := c_v^{v(x)}$. Define $\mathbf{o}_{<\lambda} = \{x \in \mathbf{K} : |x| < \lambda\}$ respectively $\mathbf{o}_{\leq\lambda} = \{x \in \mathbf{K} : |x| \leq \lambda\}$ for $\lambda \in \mathbb{R}_{\geq 0}$; put $\mathbf{o} = \mathbf{o}_{\leq 1}$ and $\mathbf{k} = \mathbf{o}/\mathbf{o}_{<1}$. If the residue field $\mathbf{k}$ of $\mathbf{K}$ has positive characteristic p, we will always put $c_v = p^{-1}$. Then $v(p) > 0$ and if this value is finite, we will assume $v(p) = 1$.

Cartesian products

Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be a finite cartesian product of sets. Then we will call a subset $\mathrm{A} \subseteq \mathrm{X}$ **cartesian** if $\mathrm{A} = \mathrm{A}_1 \times \cdots \times \mathrm{A}_d$ with $\mathrm{A}_1 \subseteq \mathrm{X}_1, \dots, \mathrm{A}_d \subseteq \mathrm{X}_d$. If $\mathrm{X}_1, \dots, \mathrm{X}_d \ni \{0, 1\}$, we will denote by $\boldsymbol{e}_k$ the tuple whose sole nonzero entry is 1 at the k-th place.

Let $\mathrm{A} \subseteq \mathrm{X}^\mathrm{I}$ for a set X and an index set I. We will denote by $\triangle\mathrm{A}$ the diagonal subset

$$\triangle\mathrm{A} = \{(x, \dots, x) \in \mathrm{A} : x \in \mathrm{X}\}$$

and by $\triangledown\mathrm{A}$ its subset of tuples with pairwise distinct coordinates

$$\triangledown\mathrm{A} = \{(x_i)_{i\in\mathrm{I}} \in \mathrm{A} : x_{i'} \neq x_{i''} \text{ if } i', i'' \in \mathrm{I} \text{ distinct }\}.$$

If $d = 1$, then $\triangle\mathrm{A} = \triangledown\mathrm{A} = \mathrm{A}$.

Metric and normed spaces

We will throughout assume all seminorms to be non-Archimedean. All normed respectively metric spaces are implicitly assumed to be endowed with a norm $\|\cdot\|$ respectively metric d, through whose arguments it will be clear whereon it is defined. Every normed space gives rise to a metric $\mathrm{d}(x, y) := \|x - y\|$.

Let the set $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be the cartesian product of normed respectively metric spaces $\mathrm{X}_1, \ldots, \mathrm{X}_d$ with correspondingly indexed norms respectively metrics. Then we endow X with the structure of a normed respectively metric space through the norm

$$\|x\| = \max\{\|x_1\|_1, \ldots, \|x_d\|_d\}$$

respectively metric

$$\mathrm{d}(x, y) = \max\{\mathrm{d}_1(x_1, y_1), \ldots, \mathrm{d}_d(x_d, y_d)\}.$$

We will then call X a **cartesian** normed respectively metric space.

If X is an arbitrary set and Y a normed space, we define a *quasinorm* $\|\cdot\|_{\sup}$ (a map with image in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying all axioms of a norm) on the mappings $f : \mathrm{X} \to \mathrm{Y}$ by

$$\|f\|_{\sup} = \begin{cases} \sup_{x \in \mathrm{X}} \|f(x)\|, & \text{if this supremum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

For a subset $\mathrm{A} \subseteq \mathrm{X}$, we define $\|f\|_{\mathrm{A}} := \|f_{|\mathrm{A}}\|_{\sup}$.

Let X be a metric space. Then for a subset $\mathrm{A} \subset \mathrm{X}$, we define its *diameter* by $\operatorname{dia} \mathrm{A} := \sup\{\mathrm{d}(x, y) : x, y \in \mathrm{A}\}$. If $\varepsilon \geq 0$ and $x_0 \in \mathrm{X}$, we define the *ball of radius* ε *around* x_0 by $\mathrm{B}_{\leq \varepsilon}(x_0) := \{x \in \mathrm{X} : \mathrm{d}(x_0, x) \leq \varepsilon\}$.

Notational conventions

- We denote by $\mathbb{N} = \{0, 1, 2, \cdots\}$ the set of nonnegative integers.
- We might abbreviate $\min\{a, b\}$ respectively $\max\{a, b\}$ for two real numbers a and b by the associative logical conjunction respectively disjunction operator $a \wedge b$ respectively $a \vee b$.

1 $\mathscr{C}^\rho$-functions for $\rho \in [0, 1[$

Assumption. Throughout this section, we will fix a real number $\rho \in [0, 1[$.

We introduce the general concept of a $\mathscr{C}^\rho$-function for $\rho \in [0, 1[$, endow the space of $\mathscr{C}^\rho$-functions with a natural locally convex topology and state its most basic properties. This will be used to explain the fractional part in the definition of a $\mathscr{C}^r$-function for $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu \in \mathbb{N}$ later on.

Definition of $\mathscr{C}^\rho$-functions

Definition. Let X be a metric space, Y a complete metric space, $f\colon A \to Y$ a mapping defined on a subset $A \subseteq X$ and a some point in X; we will say that f is $\mathscr{C}^\rho$ **at** a, if for every $\varepsilon > 0$ there is a neighborhood $U \ni a$ in X such that

$$d(f(x), f(y)) \leq \varepsilon \cdot d(x, y)^\rho \quad \text{for all } x, y \in U \cap A.$$

Then f will be a $\mathscr{C}^\rho$**-function** if f is $\mathscr{C}^\rho$ at all points $a \in A$, where we note that this notion is independent of the ambient space X. We will denote the set of all $\mathscr{C}^\rho$-functions $f\colon A \to Y$ by $\mathscr{C}^\rho(A, Y)$.

We emphasize that we also defined what it means for a point $a \in X$ not in the function's domain A to be $\mathscr{C}^\rho$. If there is a neighborhood of a disjoint to A, then this condition will be void. The interesting case occurs whenever a is a boundary point of A in X.

Remark 1.1. Keeping the notations above, let us assume that $a \in X$ is a boundary point in $\partial A = \bar{A} - A \subseteq X$. Then by completeness of Y, a function f is $\mathscr{C}^0$ at a if and only if there is a *unique* limit $f(a) \in Y$ such that for every $\varepsilon > 0$, there is a neighborhood $U \ni a$ in X such that

$$d(f(x), f(a)) \leq \varepsilon \quad \text{for all } x \in U \cap A.$$

If even $a \in A$, then a function $f\colon A \to Y$ will be $\mathscr{C}^0$ at a if and only if it will be continuous at a.

The next Proposition 1.3 tells us that we can at least assume all functions to be defined on their set of $\mathscr{C}^\rho$-points in the boundary of A in X. Note that in case $\rho = 0$, the above definition is also meaningful whenever X is merely a topological space.

Lemma 1.2. *Let* X *be a topological space,* (Y, d) *a complete metric space and* $f\colon A \to Y$ *a continuous mapping defined on a subset* $A \subseteq X$*. Let* $A \subseteq B \subseteq \bar{A} \subseteq X$ *denote the* $\mathscr{C}^0$*-points of* f*. Then* f *extends uniquely to a continuous mapping* $F\colon B \to Y$.

Proof: This is a well-known fact in general topology. □

Proposition 1.3. *Let* X *be a metric space,* Y *a complete metric space and* $f\colon A \to Y$ *a* $\mathscr{C}^\rho$*-function defined on subset* $A \subseteq X$*. Let* $A \subseteq B \subseteq \bar{A} \subseteq X$ *denote the* $\mathscr{C}^\rho$*-points of* f*. Then* f *extends uniquely to a* $\mathscr{C}^\rho$*-function* $F\colon B \to Y$.

Proof: Through the foregoing Lemma 1.2, we know that f extends to a continuous function $F\colon B \to Y$. We want to show that F is even $\mathscr{C}^\rho$ there. For this, choose $a \in B$ and fix $\varepsilon > 0$. As f is $\mathscr{C}^\rho$ at a, we find a neighborhood $U \ni a$ in X such that

$$d(F(x), F(y)) \leq \tilde{\varepsilon} \cdot d(x, y)^\rho \quad \text{for all } x, y \in A \cap U, \tag{$*$}$$

with $\tilde{\varepsilon} := \varepsilon / C^2$ and $C := 1 + 2^\rho \geq 1$. It remains to show that this inequality also holds in case x or y in $(B - A) \cap U$ with $\tilde{\varepsilon}$ replaced by ε. We firstly assume that $x \notin A$, but y so. Then x lies in the boundary of A and we can find $x' \in A$ so close to x that $d(F(x), F(x')) \leq \tilde{\varepsilon}_{x,y}$

with $\tilde{\varepsilon}_{x,y} := \tilde{\varepsilon} \cdot \mathrm{d}(x,y)^\rho$ by the continuity of F. Convergency towards x does not harm, so we may as well assume that $x' \in \mathrm{U}$ and $\mathrm{d}(x',x) \leq \mathrm{d}(x,y)$.

It follows

$$\begin{aligned}
\mathrm{d}(\mathrm{F}(x),\mathrm{F}(y)) &\leq \mathrm{d}(\mathrm{F}(x),\mathrm{F}(x')) + \mathrm{d}(\mathrm{F}(x'),\mathrm{F}(y)) \\
&\leq \tilde{\varepsilon}_{x,y} + \tilde{\varepsilon} \cdot \mathrm{d}(x',y)^\rho \\
&\leq \tilde{\varepsilon}_{x,y} + \tilde{\varepsilon} \cdot (\mathrm{d}(x',x) + \mathrm{d}(x,y))^\rho \\
&\leq \mathrm{C}\, \tilde{\varepsilon} \cdot \mathrm{d}(x,y)^\rho \leq \varepsilon \cdot d(x,y)^\rho.
\end{aligned}$$

By symmetry, this shows that Inequality $(*)$ likewise holds in case that not both of x and y lie in $(\mathrm{B} - \mathrm{A}) \cap \mathrm{U}$.

If x and y lie in $(\mathrm{B}-\mathrm{A}) \cap \mathrm{U}$, we will reduce to the first case by inserting an element $z \in \mathrm{A} \cap \mathrm{U}$ in between: Since x is in the boundary of A, we find z such that $\mathrm{d}(x,z) \leq \mathrm{dia}_{x,y} := \mathrm{d}(x,y)$. Thence by the cases already considered,

$$\begin{aligned}
\mathrm{d}(\mathrm{F}(x),\mathrm{F}(y)) &\leq \mathrm{d}(\mathrm{F}(x),\mathrm{F}(z)) + \mathrm{d}(\mathrm{F}(z),\mathrm{F}(y)) \\
&\leq \mathrm{C}\tilde{\varepsilon} \cdot \mathrm{d}(x,z)^\rho + \mathrm{C}\tilde{\varepsilon} \cdot \mathrm{d}(y,z)^\rho \\
&\leq \mathrm{C}\tilde{\varepsilon} \cdot \mathrm{d}(x,z)^\rho + \mathrm{C}\tilde{\varepsilon} \cdot (\mathrm{d}(x,z) + \mathrm{d}(x,y))^\rho \\
&\leq \mathrm{C}^2\, \tilde{\varepsilon} \cdot \mathrm{d}(x,y)^\rho \leq \varepsilon \cdot \mathrm{d}(x,y)^\rho.
\end{aligned}$$

This completes the proof of the remaining case, so Inequality $(*)$ holds for all $x, y \in \mathrm{B} \cap \mathrm{U}$ which was left to show. □

The locally convex topology on $\mathscr{C}^\rho$-functions

For the remainder of this subsection we let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space.

Definition. Let X be a metric space and $f\colon \mathrm{X} \to \mathbf{E}$ a mapping thereon. We define $|f^{]\rho[}|\colon \triangledown\mathrm{X}\times \mathrm{X} \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\rho[}|(x,y) = \frac{\|f(x) - f(y)\|}{\mathrm{d}(x,y)^\rho}.$$

Then the mapping $f\colon \mathrm{X} \to \mathrm{E}$ is $\mathscr{C}^\rho$ if and only if the function $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle\mathrm{X}\times\mathrm{X}$. Therefore the following definition is meaningful.

Definition. For every compact $\mathrm{C} \subseteq \mathrm{X}$, we define the seminorm $\|\cdot\|_{\mathscr{C}^\rho,\mathrm{C}}$ on $\mathscr{C}^\rho(\mathrm{X},\mathbf{E})$ by

$$\|f\|_{\mathscr{C}^\rho,\mathrm{C}} = \|f_{|\mathrm{C}}\|_{\sup} \vee \||(f_{|\mathrm{C}})^{[\rho]}|\|_{\sup}.$$

We equip the $\mathbf{K}$-vector space $\mathscr{C}^\rho(\mathrm{X},\mathbf{E})$ with the locally convex topology given by the set of seminorms $\{\|\cdot\|_{\mathscr{C}^\rho,\mathrm{C}} : \mathrm{C} \subseteq \mathrm{X} \text{ compact}\}$.

If X itself is compact, then we will turn $\mathscr{C}^\rho(\mathrm{X},\mathbf{E})$ into a normed $\mathbf{K}$-vector space by endowing it with the norm $\|\cdot\|_{\mathscr{C}^\rho} := \|\cdot\|_{\mathscr{C}^\rho,\mathrm{X}}$.

Remark.

(i) The locally convex $\mathbf{K}$-vector space $\mathscr{C}^\rho(X, \mathbf{E})$ is the initial locally convex $\mathbf{K}$-vector space with respect to all restriction mappings

$$\mathscr{C}^\rho(X, \mathbf{E}) \to \mathscr{C}^\rho(C, \mathbf{E}),$$
$$f \mapsto f_{|C},$$

with C running through the family of all compact $C \subseteq X$.

(ii) The locally convex $\mathbf{K}$-vector space $\mathscr{C}^\rho(X, \mathbf{E})$ is complete.

Properties of the space of $\mathscr{C}^\rho$-functions

Definition. Let X and Y be metric spaces, $f : X \to Y$ a mapping on X and a some point in X; we will say that f **is** $\mathscr{C}^{\text{lip}}$ or is **locally Lipschitzian** at a if there is a constant $C > 0$ and a neighborhood $U \ni a$ such that

$$d(f(x), f(y)) \leq C \cdot d(x, y) \text{ for all } x, y \in U.$$

Then f will be a $\mathscr{C}^{\text{lip}}$**-function** or **a locally Lipschitzian function** if f is $\mathscr{C}^{\text{lip}}$ at all points $a \in X$. We will denote the set of all $\mathscr{C}^{\text{lip}}$-functions $f : X \to Y$ by $\mathscr{C}^{\text{lip}}(X, Y)$.

Proposition 1.4. *Let* X, Y *and* Z *be metric spaces. Then the* $\mathscr{C}^\rho$*-functions are closed under composition with locally Lipschitzian functions, that is, if* $g: X \to Y$ *and* $f : Y \to Z$*, then if one of these functions will be* $\mathscr{C}^\rho$ *and the other one* $\mathscr{C}^{\text{lip}}$*, then* $f \circ g \in \mathscr{C}^\rho(X, Z)$.

More generally we can prove $f \circ g$ to be a $\mathscr{C}^{\rho\eta}$-function if f is $\mathscr{C}^\rho$ and g is $\mathscr{C}^\eta$, but nothing more.

Definition. Let X be a metric spaces and Y a set; a mapping $g: X \to Y$ will be called δ**-constant** if $d(x, y) \leq \delta$ implies $g(x) = g(y)$.

Lemma 1.5. *Let* X *be a metric space and* $f : X \to \mathbf{E}$ *a mapping such that for fixed* $\varepsilon > 0$*, there is* $0 < \delta \leq 1$ *such that* $d(x, y) \leq \delta$ *implies* $\|f(x) - f(y)\| \leq \varepsilon \cdot d(x, y)^\rho$ *for all* $x, y \in X$*. Then there is a* δ*-constant function* $g: X \to \mathbf{E}$ *with* $\|f - g\|_{\mathscr{C}^\rho, C} \leq \varepsilon$ *for all* $C \subseteq X$ *compact.*

Proof: Because $\mathbf{E}$ is non-Archimedean, we can partition X into finitely many equivalence classes U_i by declaring

$$x \sim y \quad \text{if} \quad \|f(x) - f(y)\| \leq \varepsilon\delta^\rho.$$

By assumption on f, two points x and y will be equivalent if $d(x, y) \leq \delta$. In particular every U_i is open.

We now choose an element a_i from each U_i and define δ-constant $g: X \to \mathbf{E}$ by

$$g(x) := f(a_i) \quad \text{if} \quad x \in U_i.$$

Then $\|f-g\|_{\sup} \le \varepsilon\delta^\rho \le \varepsilon$ and

$$\begin{aligned}
&|||(f-g)^{[\rho]}|||_{\sup} \\
=&|||(f-g)^{[\rho]}|||_{\{(x,y)\in X^2:d(x,y)\le\delta\}} \vee |||(f-g)^{[\rho]}|||_{\{(x,y)\in X^2:d(x,y)>\delta\}} \\
\le \quad & |||f^{[\rho]}|||_{\{(x,y)\in X^2:d(x,y)\le\delta\}} \vee |||g^{[\rho]}|||_{\{(x,y)\in X^2:d(x,y)\le\delta\}} \\
&\vee \max_{x,y\in X:d(x,y)>\delta} \left(\frac{\|f(x)-g(x)\|}{d(x,y)^\rho} \vee \frac{\|f(y)-g(y)\|}{d(x,y)^\rho}\right) \\
\le&\varepsilon \vee 0 \vee \varepsilon\delta^\rho/\delta^\rho = \varepsilon.
\end{aligned}$$

□

Corollary 1.6. *Let X be a compact metric space. Then the locally constant functions $g\colon X \to \mathbf{E}$ are dense in $\mathscr{C}^\rho(X, \mathbf{E})$.*

Proof: Fix $\varepsilon > 0$ and let $f \in \mathscr{C}^\rho(X, \mathbf{E})$. Then $|f^{[\rho]}|\colon X^2 \to \mathbb{R}_{\ge 0}$ is by compactness of X^2 a uniformly continuous function vanishing on $\triangle X^2$. Hence we find a $0 < \delta \le 1$ such that in particular for all $(a, a) \in X^2$,

$$||f^{[\rho]}|(x,y) - |f^{[\rho]}|(a,a)| = |f^{[\rho]}|(x,y) \le \varepsilon \text{ for all } x, y \in X \text{ with } d((x,y),(a,a)) \le \delta.$$

By the triangle inequality, we will have $\delta(\{(x,y)\} \cup \triangle X^2) \le \delta$ if $d(x,y) \le \delta$ for any $x, y \in X$. Thus

$$\|f(x) - f(y)\| \le \varepsilon \cdot d(x,y)^\rho \quad \text{for all } x, y \in X \text{ with } d(x,y) \le \delta.$$

By Lemma 1.5, we find δ-constant g with $\|f-g\|_{\mathscr{C}^\rho} \le \varepsilon$. In particular the locally constant functions are dense in $\mathscr{C}^\rho(X, \mathbf{E})$. □

2 Fractional differentiability in one variable

Assumption. Throughout this section, we will fix a real number $r = \nu + \rho \in \mathbb{R}_{\ge 0}$ with integral part $\nu = \lfloor r \rfloor = \max\{n \in \mathbb{N} : n \le r\} \in \mathbb{N}$ and fractional one $\rho = \{r\} = r - \lfloor r \rfloor \in [0, 1[$.

$\mathscr{C}^r$-functions for $r \in \mathbb{R}_{\ge 0}$

We now specialize to the case that the function's domain X is a nonempty subset of **K** without isolated points and takes values in **K** and want to give a general definition of fractional differentiability under these circumstances. A good hint of the strong dependence of the common differentiability notion over the real numbers on the intermediate value theorem is given by the proof of the completeness of the continuously differentiable real-valued functions $\mathscr{C}^1(I, \mathbb{R})$ defined on an open interval I, which already uses the fundamental theorem of calculus. This shows that over general base fields we have to put stronger assumptions on our class of continuously differentiable functions to yield for example, their completeness.

Definition of $\mathscr{C}^r$-functions. Following [Sch84, Section 29 ff.], we recall the notion of the iterated difference quotient of a function on a non-Archimedeanly valued domain.

Definition. Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{K}$ a mapping thereon. For $\nu \in \mathbb{N}$ put

$$X^{[\nu]} = X^{\{0,\dots,\nu\}} \quad \text{and} \quad X^{]\nu[} := \triangledown X^{[\nu]} = \{(x_0, \dots, x_\nu) : x_i = x_j \text{ only if } i = j\}.$$

The **ν-th difference quotient** $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ of a function $f : X \to \mathbf{K}$ is inductively given by $f^{]0[} := f$ and for $n \in \mathbb{N}$ and $(x_0, \dots, x_\nu) \in X^{]\nu[}$ by

$$f^{]\nu[}(x_0, \dots, x_\nu) := \frac{f^{]\nu-1[}(x_0, x_2, \dots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \dots, x_\nu)}{x_0 - x_1}.$$

Having already defined $\mathscr{C}^\rho$-functions for $\rho \in [0, 1[$, we add up our definitions to obtain our notion of fractional differentiability over (non-Archimedeanly valued) complete fields:

Definition 2.1. Fix $r = \nu + \rho \in \mathbb{R}_{\geq 0}$. Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{K}$ a mapping thereon.

(i) We will say that f is $\mathscr{C}^r$ (or r **times continuously differentiable**) at a point $a \in X$ if $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a, \dots, a) \in X^{[\nu]}$.

(ii) Then f will be a **$\mathscr{C}^r$-function** (or an **r-times continuously differentiable function**) if f is $\mathscr{C}^r$ at all points $a \in X$. The set of all $\mathscr{C}^r$-functions $f : X \to \mathbf{K}$ will be denoted by $\mathscr{C}^r(X, \mathbf{K})$.

Lemma 2.2. *Let $X \subseteq \mathbf{K}$. Then a function $f : X \to \mathbf{K}$ is $\mathscr{C}^r$ at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is a neighborhood $U \ni a$ in X such that*

$$|f^{]\nu[}(x_0, x_1, \dots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \dots, x_\nu)| \leq \varepsilon |x_0 - \tilde{x}_0|^\rho$$

for distinct $x_0, \tilde{x}_0, x_1, \dots, x_\nu \in U$.

Proof: By symmetry of $f^{]\nu[}$, see [Nag11, Lemma 2.1]. □

Remark.

(i) We observe that the differentiability at some point a may vanish if the function's domain expands in $\mathbf{K}$ - as long as there is no neighborhood U of a in $\mathbf{K}$ lying in the domain.

(ii) Let a be some accumulation point in X. Then $\vec{a}$ is an accumulation point of $X^{]\nu[}$. As $\mathbf{K}$ is complete, we find by Remark 1.1 that $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ is $\mathscr{C}^0$ at $\vec{a} \in X^{[\nu]}$ if and only if their exists a limit $D_\nu f(a) \in \mathbf{K}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f^{]\nu[}(x) - D_\nu f(a)| \leq \varepsilon \quad \text{for all } x \in X^{]\nu[} \text{ with } |x_0 - a|, \dots, |x_\nu - a| \leq \delta.$$

(iii) The previous point shows that our notion coincides with the common notion of ν-fold differentiability of f at an accumulation point a in the domain of f, as considered for example, in [Sch84, Section 29] in case $r = \nu \in \mathbb{N}$.

Remark. We remark that the function $f^{]\nu[}$ is symmetric in its $\nu + 1$ arguments for any function $f : X \to \mathbf{K}$.

Properties of $\mathscr{C}^r$-functions.

Lemma 2.3. *Let $X \subseteq \mathbf{K}$ be a subset, a some point in X and $f: X \to \mathbf{K}$ a mapping thereon. If f is $\mathscr{C}^r$ at a, then f will be $\mathscr{C}^s$ at a for every $s \leq r$.*

Proof: Immediate by Lemma 2.2. For details see [Nag11, Lemma 2.3]. □

Proposition 2.4. *Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points and $f: X \to \mathbf{K}$ a mapping thereon. Then $f \in \mathscr{C}^r(X, \mathbf{K})$ if and only if $f^{]\nu[}: X^{]\nu[} \to \mathbf{K}$ extends to a $\mathscr{C}^\rho$-function $f^{[\nu]}: X^{[\nu]} \to \mathbf{K}$.*

Proof: By an induction on $\nu \in \mathbb{N}$ and the previous Lemma 2.3, one obtains that $f^{]\nu[}$ is a $\mathscr{C}^\rho$-function. Because $X^{]\nu[} = \nabla X^{[\nu]} \subseteq X^{[\nu]}$ is a dense subset, one can then apply Proposition 1.3. For details, see [Nag11, Proposition 2.5]. □

Corollary 2.5. *Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points and $f \in \mathscr{C}^r(X, \mathbf{K})$. Then the functions*

$$\mathrm{D}_i f(a) := f^{[i]}(\vec{a}) \quad \text{for } a \in X$$

are in $\mathscr{C}^{r-i}(X, \mathbf{K})$ for $i = 0, \ldots, \nu$.

Proof: By [Sch84, Lemma 78.1], it holds for all $x = (x_0, \ldots, x_i) \in X^{]i[}$ that

$$(\mathrm{D}_{\nu-i} f)^{]i[}(x) = \sum_{y \in \mathrm{S}_{i,\nu}} f^{[\nu]}(y),$$

where $\mathrm{S}_\nu(x)$ is the set of all tuples $(x_{m_0}, \ldots, x_{m_\nu}) \in X^{[\nu]}$ for which $m_0 \leq \ldots \leq m_\nu$ and $\{m_0, \ldots, m_\nu\} = \{0, \ldots, i\}$. Because each tuple $y(x) \in \mathrm{S}_\nu(x)$ as a function $X^{]i[} \to X^{[\nu]}$ is just repetition of coordinates, it is locally Lipschitzian, and Proposition 1.4(i),(ii) tells us that the equation's right-hand side defines a $\mathscr{C}^\rho$-function on $X^{]i[}$, yielding $\mathrm{D}_{\nu-i} f \in \mathscr{C}^{i+\rho}(X, \mathbf{K})$. In other words $\mathrm{D}_i f \in \mathscr{C}^{r-i}(X, \mathbf{K})$ for $i = 0, \ldots, \nu$. □

The locally convex **K***-algebra of $\mathscr{C}^r$-functions.* In the following, we want to endow the **K**-vector space of r-times continuously differentiable functions with a complete locally convex topology.

Definition. Let $f \in \mathscr{C}^r(X, \mathbf{K})$. Then $f^{]0[}, \ldots, f^{]\nu-1[}$ and $f^{]\nu[}$ extend by Lemma 2.3 and Proposition 2.4 to continuous functions $f^{[0]}, \ldots, f^{[\nu-1]}$ and a $\mathscr{C}^\rho$-function $f^{[\nu]}$. For a compact subset $C \subseteq X$, we can thence define the seminorm $\|\cdot\|_{\mathscr{C}^r, C}$ on $\mathscr{C}^r(X, \mathbf{K})$ by

$$\|f\|_{\mathscr{C}^r, C} := \|f^{[0]}\|_C \vee \cdots \vee \|f^{[\nu-1]}\|_{C^{[\nu-1]}} \vee \|f^{[\nu]}\|_{\mathscr{C}^\rho, C^{[\nu]}}.$$

We provide $\mathscr{C}^r(X, \mathbf{K})$ with the locally convex topology induced through this family of seminorms $\{\|\cdot\|_{\mathscr{C}^r, C}\}$ with C running through all compact subsets $C \subseteq X$.

Lemma 2.6. *We have for $s \leq r$ a norm-nonincreasing inclusion of locally convex* **K***-vector spaces $\mathscr{C}^r(X, \mathbf{K}) \subseteq \mathscr{C}^s(X, \mathbf{K})$.*

We will denote a locally convex $\mathbf{K}$-vector space which is a $\mathbf{K}$-algebra whose multiplication is continuous a *locally convex* $\mathbf{K}$*-algebra*.

Proposition 2.7. *Let* X *be a nonempty subset of* $\mathbf{K}$ *without isolated points. The space* $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ *is a complete locally convex* $\mathbf{K}$*-algebra.*

Description through iterated difference quotients. We give a more direct description of the locally convex $\mathbf{K}$-vector space of $\mathscr{C}^r$-functions, better suited for the calculations to come up in the succeeding Section 2.

Proposition 2.8. *Let* $f : \mathrm{X} \to \mathbf{K}$ *be a mapping defined on a nonempty subset* $\mathrm{X} \subseteq \mathbf{K}$ *without isolated points. Define a function* $|f^{]r[}| : \mathrm{X}^{]\nu+1[} \to \mathbb{R}_{\geq 0}$ *by*

$$|f^{]r[}|(x_0, \tilde{x}_0, x_1, \ldots, x_\nu) := \frac{|f^{]\nu[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \ldots, x_\nu)|}{|x_0 - \tilde{x}_0|^\rho}.$$

Then $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K})$ *if and only if* $|f^{]r[}| : \mathrm{X}^{]\nu+1[} \to \mathbb{R}_{\geq 0}$ *extends to a continuous function* $|f^{[r]}| : \mathrm{X}^{[\nu+1]} \to \mathbb{R}_{\geq 0}$ *which will vanish if* $x_0 = \tilde{x}_0$*. In this case,* $\|f^{[\nu]}\|_{\mathscr{C}^\rho, \mathrm{C}^{[\nu]}} = \|f^{[\nu]}\|_{\mathrm{C}^{[\nu]}} \vee \||f^{[r]}|\|_{\mathrm{C}^{[\nu+1]}}$ *for compact* $\mathrm{C} \subseteq \mathrm{X}$*.*

Proof: By symmetry of $f^{]\nu[}$, see [Nag11, Corollary 2.11] for details. □

Lemma 2.9. *For any permutation* σ *of mutually distinct* $x_0, x_1, \ldots, x_{\nu+1} \in \mathrm{X}$*, we find*

$$|f^{]r[}|(x^\sigma) = \left|\frac{x_0^\sigma - x_1^\sigma}{x_0 - x_1}\right|^{1-\rho} |f^{]r[}|(x) \quad \textit{for } x := (x_0, \ldots, x_{\nu+1}) \in \mathrm{X}^{]\nu+1[}.$$

Proof: Let σ swap the indices $0, 1$ with $i, j \in \{0, \ldots, \nu + 1\}$. We notice

$$|f^{]r[}|(x) = |f^{]\nu+1[}(x)||x_0 - x_1|^{1-\rho}.$$

By symmetry of the latter function therefore holds

$$\begin{aligned} |f^{]r[}|(x^\sigma) =& |f^{]\nu+1[}(x^\sigma)||x_i - x_j|^{1-\rho} = |f^{]\nu+1[}(x)||x_0 - x_1|^{1-\rho} \frac{|x_i - x_j|^{1-\rho}}{|x_0 - x_1|^{1-\rho}} \\ =& |f^{]r[}|(x) \frac{|x_i - x_j|^{1-\rho}}{|x_0 - x_1|^{1-\rho}} = |f^{]r[}|(x) \left|\frac{x_0^\sigma - x_1^\sigma}{x_0 - x_1}\right|^{1-\rho}. \end{aligned}$$ □

Remark 2.10. Let $(x_0, x_1, \ldots, x_{\nu+1}) \in \mathrm{X}^{]\nu+1[}$. Let σ be the mapping on $\mathrm{X}^{]\nu+1[}$ swapping the entries with coordinate indices $0, 1$ with those with coordinate indices $i, j \in \{0, \ldots, \nu + 1\}$. Then we find $|f^{]r[}|(x) \leq |f^{]r[}|(x^\sigma)$ if $|x_0 - x_1| \leq |x_i - x_j|$. In particular if $|x_0 - x_1| = \mathrm{dia}\{x_0, x_1, \ldots, x_{\nu+1}\}$, then $|f^{]r[}(x)| \geq |f^{]r[}|(x^\sigma)$ for any permutation σ of $(x_0, x_1, \ldots, x_{\nu+1}) \in \mathrm{X}^{]\nu+1[}$.

Characterization through Taylor polynomials

We give different characterizations of r-fold differentiability through the function's Taylor polynomial. The strength of the conditions imposed on the Taylor polynomial to yield an equivalent characterization of r-fold differentiability is then shown to depend on the structure of the function's domain.

Assumption. Throughout this subsection $X \subseteq \mathbf{K}$ will denote a nonempty subset without isolated points, if not explicitly mentioned otherwise.

The Taylor polynomial of $\mathscr{C}^r$-functions. In this subsection, we turn to the Taylor expansion of a $\mathscr{C}^r$-function. By a straightforward induction over $\nu \geq 0$, we find that all $\mathscr{C}^r$-functions have a unique Taylor-polynomial expansion:

Corollary 2.11 (Taylor-polynomial)**.** *Let $f \in \mathscr{C}^r(X, \mathbf{K})$. Then*

$$f(x) = \sum_{i=0,\ldots,\nu-1} D_i f(y)(x-y)^i + f^{[\nu]}(x, y, \ldots, y)(x-y)^\nu \quad \textit{for all } x, y \in X$$

with $\mathscr{C}^{r-i}$-functions $D_i f : X \to \mathbf{K}$ for $i = 0, \ldots, \nu - 1$ given in Corollary 2.5 and a $\mathscr{C}^\rho$-function $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$.

Proof: This is proved by induction on $\lfloor r \rfloor \geq 0$, the case $\lfloor r \rfloor = 0$ being trivial. □

Let $f \in \mathscr{C}^r(X, \mathbf{K})$. We define the Taylor-polynomial's scaled rest-function by

$$\Delta_\nu f(x, y) := f^{[\nu]}(x, y, \ldots, y) - D_\nu f(y) \quad \text{for } x, y \in X.$$

Then $\Delta_\nu f : X \times X \to \mathbf{K}$ is a $\mathscr{C}^\rho$-function vanishing on the diagonal. In particular we find by definition that for every $\varepsilon > 0$ and $a \in X$, there is a neighborhood $U \ni (a, a)$ such that

$$|\Delta_\nu f(x, y) - \Delta_\nu f(y, y)| \leq \varepsilon |(x, y) - (y, y)|^\rho \quad \text{for all } (x, y), (y, y) \in U^2.$$

This yields to $|\Delta_\nu f(x, y)| \leq \varepsilon |x - y|^\rho$ for all $x, y \in U$. Thus in particular Corollary 2.11 entails:

Corollary (2.11')**.** *Let $f \in \mathscr{C}^r(X, \mathbf{K})$. Then there is a polynomial of degree ν whose coefficients are functions $D_0 f, \ldots, D_\nu f : X \to \mathbf{K}$ such that*

$$f(x + y) = \sum_{i=0,\ldots,\nu} D_i f(x) y^i + R_\nu f(x + y, x) \quad \textit{for all } x + y, x \in X,$$

and for every $a \in X$ and $\varepsilon > 0$ exists a neighborhood $U \ni a$ such that

$$|R_\nu f(x + y, x)| \leq \varepsilon |y|^r \quad \textit{for all } x + y, x \in U.$$

Characterizing $\mathscr{C}^r$-functions through Taylor polynomials on a general domain. In the following we will study the relation of the functions admitting a Taylor-polynomial expansion and the continuously differentiable ones. We will see in the following subsection that the property of Corollary 2.11' is equivalent to being $\mathscr{C}^r$ on a large class of subsets $X \subseteq K$. In this subsection however, in the chase of a general domain $X \subseteq \mathbf{K}$ without isolated points, we have to impose further conditions on the Taylor polynomial:

Definition 2.12. A function $f : X \to \mathbf{K}$ will be in $\mathscr{C}^r_{T+}(X, \mathbf{K})$ if there are *continuous* functions $D_i f : X \to \mathbf{K}$ for $i = 0, \ldots, \nu$ such that if one defines $R_\nu f : X \times X \to \mathbf{K}$ by

$$R_\nu f(x, y) := f(x) - \sum_{i=0,\ldots,\nu} D_i f(y)(x - y)^i,$$

then for every point $a \in X$ and any $\varepsilon > 0$, there will exist a neighborhood $U \ni a$ with

$$|R_\nu f(x, y)| \leq \varepsilon |x - y|^r \quad \text{for all } x, y \in U.$$

Since $R_\nu f : X \times X \to \mathbf{K}$ vanishes on the diagonal $\triangle X \times X$, we see that $f = D_0 f$. Keeping $y = y_0$ fixed, the convergence condition shows that $D_0 f$ is in any case continuous (and by a more elaborate argument $D_1 f$, too). Moreover the continuity of $D_0 f, \ldots, D_\nu f : X \to K$ implies the continuity of $R_\nu f : X \times X \to \mathbf{K}$.

Lemma 2.13. *The functions* $D_0 f, \ldots, D_\nu f : X \to \mathbf{K}$ *in Definition 2.12 are unique.*

Proof: This is proved by induction on $\nu \geq 0$. □

Definition 2.14.

(i) Let $f \in \mathscr{C}^r_{T+}(X, \mathbf{K})$. We define functions $\Delta_\nu f : \nabla X \times X \to \mathbf{K}$ and $|\Delta_r f| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ by putting

$$\Delta_\nu f(x, y) := \frac{R_\nu f(x, y)}{(x - y)^\nu} \quad \text{and} \quad |\Delta_r f|(x, y) := \frac{|R_\nu f(x, y)|}{|x - y|^r}.$$

Since $f \in \mathscr{C}^r_{T+}(X, \mathbf{K})$, we can by definition extend these functions onto $X \times X$ such that they continuously vanish on the diagonal $\triangle X \times X$ and denote these prolongations likewise. By the comment following Definition 2.12, they are also continuous on $X \times X - \triangle X \times X$ and thus on all of $X \times X$.

(ii) By Lemma 2.13, the functions $D_0 f, \ldots, D_\nu f : X \to \mathbf{K}$ of Definition 2.12 are uniquely determined continuous functions. So it makes sense to endow the space $\mathscr{C}^r_{T+}(X, \mathbf{K})$ with the locally convex topology induced from the family of seminorms $\{\|\cdot\|_{\mathscr{C}^r_{T+}, C}\}$ running through all compact subsets $C \subseteq X$ defined by

$$\|f\|_{\mathscr{C}^r_{T+}, C} := \|D_0 f\|_C \vee \ldots \vee \|D_\nu f\|_C \vee \||\Delta_r f|\|_{C \times C}.$$

Under our hitherto imposed assumptions, the inverse of Corollary 2.11' turns out to be true for $r \leq 2$, see [Sch84, Proposition 28.4]. But a counterexample for $r = 3$ is given in [Example 83.2, loc.cit.].

Definition. For a subset $C \subseteq \mathbf{K}$, we define

$$C^d_{\leq 1} := \{x = (x_1, \ldots, x_d) \in C^d : \operatorname{dia}\{x_1, \ldots, x_d\} \leq 1\}.$$

Lemma 2.15. *Let $f \in \mathscr{C}^r(X, \mathbf{K})$. Then for all $(x_0, \ldots, x_\nu), (y_0, \ldots, y_\nu) \in X^{[\nu]}$, we have*

$$\begin{aligned} &|f^{[\nu]}(x_0, \ldots, x_\nu) - f^{[\nu]}(y_0, \ldots, y_\nu)| \\ \leq &\max_{i=0,\ldots,\nu} |x_i - y_i|^\rho |f^{[r]}|(y_i, x_i, y_0, \ldots, y_{i-1}, x_{i+1}, \ldots, x_\nu). \end{aligned}$$

Proof: By the symmetry of $f^{[\nu]} \colon X^{[\nu]} \to \mathbf{K}$. ☐

Lemma 2.16. *Let $f \in \mathscr{C}^r(X, \mathbf{K})$. For any compact subset $C \subseteq X$ holds*

$$\|f\|_{\mathscr{C}^r, C} = \|D_0 f\|_C \vee \ldots \vee \|D_\nu f\|_C \vee \||f^{[r]}|\|_{C^{[\nu+1]}_{\leq 1}}.$$

Proof: We only have to prove that $\|f\|_{\mathscr{C}^r, C}$ is not greater than the right-hand side. By Proposition 2.8, we have $\|f\|_{\mathscr{C}^r, C} = \|f^{[0]}\|_C \vee \ldots \vee \|f^{[\nu]}\|_{C^{[\nu]}} \vee \||f^{[r]}|\|_{C^{[\nu+1]}}$. Firstly, we prove by downward induction on $n = \nu, \ldots, 0$ that

$$\|f^{[n]}\|_{C^{[n]}_{\leq 1}} \leq \|f^{[\nu]}\|_{C^{[\nu]}_{\leq 1}} \vee \|D_\nu f\|_C \vee \ldots \vee \|D_n f\|_C.$$

In case $n = \nu$ there is nothing to show. Let $n < \nu$. Then for any $(x_0, \ldots, x_n) \in C^{[n]}$ with $|x_i - x_j| \leq 1$ for all i, j we have

$$\begin{aligned} |f^{[n]}(x_0, \ldots, x_n)| \leq &|f^{[n]}(x_0, \ldots, x_n) - f^{[n]}(x_0, \ldots, x_0)| \vee |f^{[n]}(x_0, \ldots, x_0)| \\ = &|\sum_{j=1,\ldots,n} (x_j - x_0) f^{[n+1]}(x_0, \ldots, x_0, x_j, \ldots, x_n)| \vee |D_n f(x_0)| \\ \leq &\|f^{[n+1]}\|_{C^{[n+1]}_{\leq 1}} \vee \|D_n f\|_C; \end{aligned}$$

the middle equality by [Sch84, Lemma 29.2(iii)]. Thus $\|f^{[n]}\|_{C^{[n]}_{\leq 1}} \leq \|f^{[n+1]}\|_{C^{[n+1]}_{\leq 1}} \vee \|D_n f\|_C$, and the induction hypothesis for $n+1$ yields the desired inequality.

Now for any $(x_0, \ldots, x_\nu) \in C^{[\nu]}$ with $|x_i - x_j| \leq 1$ for all i, j, we have

$$\begin{aligned} |f^{[\nu]}(x_0, \ldots, x_\nu)| \leq &|f^{[\nu]}(x_0, \ldots, x_\nu) - f^{[\nu]}(x_0, \ldots, x_0)| \vee |f^{[\nu]}(x_0, \ldots, x_0)| \\ \leq &\max_{j=1,\ldots,\nu} |x_j - x_0|^\rho |f^{[r]}|(x_0, x_j, x_0, \ldots, x_0, x_{j+1}, \ldots, x_\nu)| \vee |D_\nu f(x_0)| \\ \leq &\||f^{[r]}|\|_{C^{[\nu+1]}_{\leq 1}} \vee \|D_\nu f\|_C; \end{aligned}$$

the middle inequality by the preceding Lemma 2.15. We thence see that $\|f^{[\nu]}\|_{C^{[\nu]}_{\leq 1}} \leq \||f^{[r]}|\|_{C^{[\nu+1]}_{\leq 1}} \vee \|D_\nu f\|_C$. Plugging both results together, we saw for $n = 0, \ldots, \nu$ that

$$\begin{aligned} \|f^{[n]}\|_{C^{[n]}_{\leq 1}} \leq &\|D_0 f\|_C \vee \ldots \vee \|D_{\nu-1} f\|_C \vee \|f^{[\nu]}\|_{C^{[\nu]}_{\leq 1}} \\ \leq &\|D_0 f\|_C \vee \ldots \vee \|D_{\nu-1} f\|_C \vee \|D_\nu f\|_C \vee \||f^{[r]}|\|_{C^{[\nu+1]}_{\leq 1}}. \end{aligned}$$

By [Sch78, Theorem 8.3], we find $\max_{n=0,\dots,\nu}\|f^{[n]}\|_{C^{[n]}} = \max_{n=0,\dots,\nu}\|f^{[n]}\|_{C^{[n]}_{\le 1}}$ and so

$$\max_{n=0,\dots,\nu}\|f^{[n]}\|_{C^{[n]}} \le \|D_0 f\|_C \vee \dots \vee \|D_\nu f\|_C \vee |||f^{[r]}|||_{C^{[\nu+1]}_{\le 1}}.$$

It solely remains to show $|||f^{[r]}|||_{C^{[\nu+1]}} \le \|f^{[\nu]}\|_{C^{[\nu]}} \vee |||f^{[r]}|||_{C^{[\nu+1]}_{\le 1}}$. To this end, let $x = (x_0, \tilde{x}_0, x_1, \dots, x_\nu) \in C^{[\nu+1]}$ with $\mathrm{dia}\{x_0, \tilde{x}_0, x_1, \dots, x_\nu\} > 1$. By continuous extension of Remark 2.10, we find

$$|f^{[r]}|(x) = \max_{\sigma\in\{\text{permutations of } x\}} |f^{[r]}|(x^\sigma) \quad \text{iff} \quad |x_0 - \tilde{x}_0| = \mathrm{dia}\{x_0, \tilde{x}_0, x_1, \dots, x_\nu\}.$$

We may therefore assume $|x_0 - \tilde{x}_0| > 1$. By the definition and continuous extension, we find

$$|f^{[r]}|(x) = |f^{[\nu]}(x_0, x_1, \dots, x_\nu) - f^{[\nu]}(\tilde{x}_0, x_1, \dots, x_\nu)|/|x_0 - \tilde{x}_0| \le \|f^{[\nu]}\|_{C^{[\nu]}}.$$

Lemma 2.17. *Let $f \in \mathscr{C}^r(X, \mathbf{K})$. Then for compact $C \subseteq X$ holds*

$$\|f\|_{\mathscr{C}^r, C} = \max_{n=0,\dots,\nu}(\|D_n f\|_C \vee |||\Delta_{r-n} D_n f|||_{C^2}).$$

Proof: Foremost for this statement to be meaningful, we note that by Corollary 2.5 and Corollary 2.11, we find $D_n f \in \mathscr{C}^{r-n}(X, \mathbf{K}) \subseteq \mathscr{C}^{r-n}_{T^+}(X, \mathbf{K})$ for $n = 0, \dots, \nu$. So the above right-hand side is well defined.

By the preceding Lemma 2.16, it suffices to prove $|||f^{[r]}|||_{C^{[\nu+1]}} \le |||\Delta_{r-n} D_n f|||_{C^2}$ for $n = 0, \dots, \nu$. Then we put

$$\varphi_n f(x, y) := f^{[\nu]}(\underbrace{x, \dots, x}_{n-\text{times}}, y, \dots, y) \qquad \text{for all distinct } x, y \in X.$$

By [Sch84, Lemma 78.3], we have for distinct $x, y \in X$

$$\begin{pmatrix} (D_{\nu-1} f)^{[1]}(x, y) \\ \vdots \\ (D_0 f)^{[\nu]}(x, y, \dots, y) \end{pmatrix} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & & \cdots & \binom{\nu-1}{0} \\ & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{\nu-1}{1} \\ & & \ddots & & \vdots \\ & & & \binom{\nu-2}{\nu-2} & \binom{\nu-1}{\nu-2} \\ & & & & \binom{\nu-1}{\nu-1} \end{pmatrix} \begin{pmatrix} \varphi_\nu f(x, y) \\ \vdots \\ \varphi_1 f(x, y) \end{pmatrix}.$$

Denote the upper $\nu \times \nu$-square matrix by M. We note that inductively

$$\binom{i}{i} + \binom{i+1}{i} + \dots + \binom{\nu}{i} = \binom{\nu}{i+1} + \binom{\nu}{i} = \binom{\nu+1}{i+1}.$$

Therefore

$$M \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \binom{\nu}{1} \\ \vdots \\ \binom{\nu}{\nu} \end{pmatrix} = \begin{pmatrix} \binom{\nu}{\nu-1} \\ \vdots \\ \binom{\nu}{0} \end{pmatrix}.$$

Because M has determinant 1, it is invertible in $\mathbb{Z}$, and thus

$$\begin{pmatrix}\varphi_\nu f(x,y)-D_\nu f(y)\\ \vdots \\ \varphi_1 f(x,y)-D_\nu f(y)\end{pmatrix} = M^{-1}\cdot\begin{pmatrix}(D_{\nu-1}f)^{[1]}(x,y)\\ \vdots \\ (D_0 f)^{[\nu]}(x,y,\dots,y)\end{pmatrix} - \begin{pmatrix}D_\nu f(y)\\ \vdots \\ D_\nu f(y)\end{pmatrix}$$
$$= M^{-1}\cdot\begin{pmatrix}(D_{\nu-1}f)^{[1]}(x,y)-\binom{\nu}{\nu-1}D_\nu f(y)\\ \vdots \\ (D_0 f)^{[\nu]}(x,y,\dots,y)-\binom{\nu}{0}D_\nu f(y)\end{pmatrix}$$
$$= M^{-1}\cdot\begin{pmatrix}(D_{\nu-1}f)^{[1]}(x,y)-D_1D_{\nu-1}f(y)\\ \vdots \\ (D_0 f)^{[\nu]}(x,y,\dots,y)-D_\nu D_0 f(y)\end{pmatrix}$$
$$= M^{-1}\cdot\begin{pmatrix}\Delta_1 D_{\nu-1}f(x,y)\\ \vdots \\ \Delta_\nu D_0 f(x,y)\end{pmatrix};$$

the penultimate equality by [Sch84, Theorem 78.2]. So for short, we see that we may express $\varphi_n f(x,y)-D_\nu f(y)$ for $n=1,\dots,\nu$ as a $\mathbb{Z}$-linear combination of the values $\Delta_1 D_{\nu-1}f(x,y),\dots,\Delta_\nu D_0 f(x,y)$.

By [Sch78, Lemma 8.18], we may express $f^{]\nu[} = f^{[\nu]}_{|\nabla X^{\nu+1}}$ at some $x\in\nabla X^{\nu+1}$ as a convex combination of the $\varphi_1 f,\dots,\varphi_\nu f\colon \nabla X\times X\to \mathbf{K}$. More exactly

$$f^{]\nu[}(x_0,\dots,x_\nu) = \sum_{\substack{n=1,\dots,\nu,\\ i,j=0,\dots,\nu \text{ s.t. } i\neq j}} \lambda^{(n)}_{i,j}(x)\varphi_n f(x_i,x_j)$$

for some elements $\lambda^{(n)}_{i,j}(x)\in\mathbf{K}$ for distinct $i,j=0,\dots,\nu$ and $n=1,\dots,\nu$ such that

$$\sum_{\substack{n=1,\dots,\nu,\\ i,j=0,\dots,\nu \text{ s.t. } i\neq j}} \lambda^{(n)}_{i,j}(x)=1 \quad\text{and}\quad |\lambda^{(n)}_{i,j}(x)|\leq 1 \quad \text{for all these } i,j \text{ and } n.$$

For notational convenience, let $(x_0,x_1,\dots,x_\nu)$ and $(\tilde{x}_0,x_1,\dots,x_\nu)$ in $X^{]\nu[}$ be denoted by x and x'. We obtain $\lambda^{(n)}_{i,j}(x)$ respectively $\lambda^{(n')}_{i',j'}(x')$ in $\mathbf{K}$ such that

$$\begin{aligned} &f^{]\nu[}(x_0,x_1,\dots,x_\nu)-f^{]\nu[}(\tilde{x}_0,x_1,\dots,x_\nu)\\ =&\sum_{\substack{n=1,\dots,\nu,\\ (i,j)\in\nabla\{0,\dots,\nu\}^2}} \lambda^{(n)}_{i,j}(x)\varphi_n f(x_i,x_j) - \sum_{\substack{n'=1,\dots,\nu,\\ (i',j')\in\nabla\{0,\dots,\nu\}^2}} \lambda^{(n')}_{i',j'}(x')\varphi_{n'} f(x'_{i'},x'_{j'}). \end{aligned} \tag{$*$}$$

As $\sum_{n,i,j} \lambda_{i,j}^{(n)}(x) = 1$ and likewise $\sum_{n',i',j'} \lambda_{i',j'}^{(n')}(x') = 1$, this equals

$$\begin{aligned}
&\sum_{\substack{n,n'=1,\dots,\nu,\\ (i,j),(i',j')\in\nabla\{0,\dots,\nu\}^2}} \lambda_{i,j}^{(n)}(x)\lambda_{i',j'}^{(n')}(x')\varphi_n f(x_i,x_j)\\
&- \sum_{\substack{n',n=1,\dots,\nu,\\ (i',j'),(i,j)\in\nabla\{0,\dots,\nu\}^2}} \lambda_{i',j'}^{(n')}(x')\lambda_{i,j}^{(n)}(x)\varphi_{n'} f(x'_{i'},x'_{j'})\\
=& \sum_{n,n'=1,\dots,\nu} \Big(\sum_{(i,j),(i',j')\in\nabla\{0,\dots,\nu\}^2} \lambda_{i,j}^{(n)}(x)\lambda_{i',j'}^{(n')}(x')(\varphi_n f(x_i,x_j) - \varphi_{n'} f(x'_{i'},x'_{j'}))\Big).
\end{aligned}$$

Let $n \in \{0,\dots,\nu\}$ and denote the $\mathbb{Z}$-coefficients of $\Delta_1 D_{\nu-1} f, \dots, \Delta_\nu D_0 f$ summing to $\varphi_n f(x,y) - D_\nu f(y)$ by $\mu_1^{(n)}, \dots, \mu_\nu^{(n)}$. We find

$$\begin{aligned}
&\varphi_n f(x,y) - \varphi_{n'} f(x',y')\\
=&(\varphi_n f(x,y) - D_\nu f(y)) + (D_\nu f(y) - D_\nu f(y')) - (\varphi_{n'} f(x',y') - D_\nu f(y'))\\
=& \sum_{l=1,\dots,\nu} \mu_l^{(n)} \Delta_l D_{\nu-l} f(x,y) + (D_\nu f(y) - D_\nu f(y')) - \sum_{l'=1,\dots,\nu} \mu_{l'}^{(n')} \Delta_{l'} D_{\nu-l'} f(x',y').
\end{aligned}$$

Plugging this into Equation $(*)$ and noting $|\lambda_{i,j}^{(n)}|, |\mu_l^{(n)}| \le 1$, we find

$$\begin{aligned}
&|f^{]\nu[}(x_0,x_1,\dots,x_\nu) - f^{]\nu[}(\tilde{x}_0,x_1,\dots,x_\nu)|\\
\le& \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\dots,x_\nu\}} (|D_\nu f(x) - D_\nu f(y)| \vee \max_{n=1,\dots,\nu} |\Delta_n D_{\nu-n} f(x,y)|).
\end{aligned}$$

Let $x = (x_0, \tilde{x}_0, x_1, \dots, x_\nu) \in X^{]\nu+1[}$. By Remark 2.10, we find for any coordinate permutation map $\sigma\colon X^{]\nu+1[} \to X^{]\nu+1[}$ with $|x'_0 - \tilde{x}'_0| = \operatorname{dia}\{x_0, \tilde{x}_0, x_1, \dots, x_\nu\}$, where we put $x^\sigma = (x'_0, \tilde{x}'_0, x'_1, \dots, x'_\nu)$, that $|f^{]r[}|(x) \le |f^{]r[}|(x^\sigma)$. Hence for such a mapping σ, we find

$$\begin{aligned}
|f^{]r[}|(x) \le& |f^{]r[}|(x^\sigma)\\
\le& \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\dots,x_\nu\}} \max_{n=1,\dots,\nu} \frac{|\Delta_n D_{\nu-n} f(x,y)|}{|x-y|^\rho} \vee \frac{|D_\nu f(x) - D_\nu f(y)|}{|x-y|^\rho}\\
=& \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\dots,x_\nu\}} \max_{n=0,\dots,\nu} |\Delta_{r-n} D_n f|(x,y).
\end{aligned}$$

This extends continuously to

$$|f^{[r]}|(x) \le \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\dots,x_\nu\}} \max_{n=0,\dots,\nu} |\Delta_{r-n} D_n f|(x,y) \quad \text{for all } x \in X^{[\nu+1]}.$$

In particular for all compact $C \subseteq X$ holds $|||f^{[r]}|||_{C^{[\nu+1]}} \le \max_{n=0,\dots,\nu} |||\Delta_{r-n} D_n f|||_{C^2}$. □

Definition. We will define a map $f\colon X \to \mathbf{K}$ to lie in $\mathscr{C}^r_{T^{++}}(X, \mathbf{K})$ if there are functions $D_0 f, \dots, D_\nu f\colon X \to \mathbf{K}$ such that $\binom{n}{n} D_n f, \binom{n+1}{n} D_{n+1} f, \dots, \binom{\nu}{n} D_\nu f$ prove $D_n f$ to be in $\mathscr{C}^{r-n}_{T^+}(X, \mathbf{K})$ for $n = 0, \dots, \nu$. We endow the space $\mathscr{C}^r_{T^{++}}(X, \mathbf{K})$ with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\mathscr{C}^r_{T^{++}},C}\}$ on each compact subset $C \subseteq X$ defined by

$$\|f\|_{\mathscr{C}^r_{T^{++}},C} := \max_{n=0,\dots,\nu} (\|D_n f\|_C \vee \||\Delta_{r-n} D_n f|\|_{C^2})$$

for any $f \in \mathscr{C}^r_{T^{++}}(X, \mathbf{K})$.

Remark 2.18. In this case the functions $f = D_0 f, D_1 f, \dots, D_\nu f : X \to \mathbf{K}$ are automatically continuous.

Proof: This is proved by downward induction on $n = \nu, \dots, 0$. See [Nag11, Remark 2.21] for details. □

Definition. Let $X \subseteq \mathbf{K}$ be a subset. We will write $\mathscr{C}^{\mathrm{pol}}_{\leq g}(X, \mathbf{K})$ for the $\mathbf{K}$-vector space of all locally polynomial functions $f : X \to \mathbf{K}$ of degree at most g.

Lemma 2.19. *For $i \in \mathbb{Z}$, define $*^i : X \to \mathbf{K}$ by the monomial function $x \mapsto x^i$ if $i \in \mathbb{N}$ and $*^i = 0$ if $i < 0$. Then $*^i \in \mathscr{C}^\nu(X, \mathbf{K})$ for any nonnegative integer ν with $D_\nu *^i = \binom{i}{\nu} *^{i-\nu}$.*

Proof: By induction on i, using the product rule [Sch84, Lemma 29.2(v)]. □

Corollary 2.20. *Let $p : X \to \mathbf{K}$ be a polynomial function of degree at most i. If $j \geq i$, then $R_j p \equiv 0$.*

Proof: By the preceding Lemma 2.19 and the binomial expansion

$$(x + y)^i = \sum_{n=0,\dots,i} \binom{i}{n} x^{i-n} y^i = 0.$$

Remark. For the following, we note that by Lemma 2.19 every monomial function $*^i : X \to \mathbf{K}$ is arbitrarily often continuously differentiable. As being $\mathscr{C}^r$ is a $\mathbf{K}$-linear local property, this extends to all locally polynomial functions.

Then by Corollary 2.11 and [Sch84, Theorem 78.2], there is an inclusion of locally convex $\mathbf{K}$-vector spaces $\mathscr{C}^r(X, \mathbf{K}) \subseteq \mathscr{C}^r_{T^{++}}(X, \mathbf{K})$. Therefore $\mathscr{C}^{\mathrm{pol}}(X, \mathbf{K}) \subseteq \mathscr{C}^r_{T^{++}}(X, \mathbf{K})$.

Lemma 2.21. *The locally polynomial functions of degree at most ν lie dense in the $\mathscr{C}^r_{T^{++}}$-functions.*

Proof: Fix $f \in \mathscr{C}^r_{T^{++}}(X, \mathbf{K})$ and $\varepsilon > 0$. We find a covering $\{U_\alpha\}$ of X with $\operatorname{dia} U_\alpha \leq 1$ such that

$$|\Delta_{r-0} D_0 f(x, y)|, \dots, |\Delta_{r-\nu} D_\nu f(x, y)| \leq \varepsilon \quad \text{for all } x, y \in U_\alpha.$$

Since X is totally disconnected, we can refine this covering to one whose sets are pairwise disjoint, again denoted by $\{U_\alpha\}$. We choose $a_\alpha \in U_\alpha$ and define the locally polynomial function $g : X \to \mathbf{K}$ by putting

$$g(x) = f(a_\alpha) + (x - a_\alpha) D_1 f(a_\alpha) + \dots + (x - a_\alpha)^\nu D_\nu f(a_\alpha) \quad \text{if } x \in U_\alpha.$$

By Lemma 2.19, we have for $x \in U_\alpha$ that

$$D_n g(x) = D_n f(a_\alpha) + \binom{n+1}{n} D_{n+1} f(a_\alpha)(x - a_\alpha) + \cdots + \binom{\nu}{n} D_\nu f(a_\alpha)(x - a_\alpha)^{\nu - n}.$$

Therefore $D_n f(x) - D_n g(x) = R_{\nu-n} D_n f(x, a_\alpha) = (x - a_\alpha)^{\nu-n} \Delta_{\nu-n} D_n f(x, a_\alpha)$. Hence

$$\|D_n f - D_n g\|_{U_\alpha} \leq \||\Delta_{r-n} D_n f(x, a_\alpha)|\|_{U_\alpha^2} (\operatorname{dia} U_\alpha)^{r-n} \leq \varepsilon (\operatorname{dia} U_\alpha)^{r-n}. \qquad (*)$$

As the U_α cover X with $\operatorname{dia} U_\alpha \leq 1$, we see $\|D_n f - D_n g\|_X \leq \varepsilon$ for $n = 0, \ldots, \nu$.

By Corollary 2.20, we find $R_{\nu-n}(D_n g_{|U_\alpha}) \equiv 0$ and consequently

$$\||\Delta_{r-n} D_n (f - g)|\|_{U_\alpha^2} = \||\Delta_{r-n} D_n f|\|_{U_\alpha^2} \leq \varepsilon.$$

Since $X = \cup_{\alpha,\beta} U_\alpha \times U_\beta$, it remains to show that $\||\Delta_{r-n} D_n(f-g)|\|_{U_\alpha \times U_\beta} \leq \varepsilon$ in case $\alpha \neq \beta$. So let $x \in U_\alpha, y \in U_\beta$. By disjointness, we have $|x - y| \geq \operatorname{dia} U_\alpha \vee \operatorname{dia} U_\beta =: \delta > 0$. It follows

$$\begin{aligned}
&|\Delta_{r-n} D_n (f-g)(x,y)| \\
=&|R_{\nu-n} D_n f(x,y) - R_{\nu-n} D_n g(x,y)| / |x-y|^{r-n} \\
=&|(D_n f(x) - D_n g(x)) - \sum_{i=0,\ldots,\nu-n} \binom{i+n}{n} (D_{i+n} f(y) - D_{i+n} g(y))(x-y)^i| / |x-y|^{r-n} \\
\leq&(\|D_n f - D_n g\|_{U_\alpha} \vee \max_{i=0,\ldots,\nu-n} \|D_{i+n} f - D_{i+n} g\|_{U_\beta} (\operatorname{dia} U_\beta)^i) / \delta^{r-n} \\
\leq&(\varepsilon (\operatorname{dia} U_\alpha)^{r-n} \vee \max_{i=0,\ldots,\nu-n} \varepsilon (\operatorname{dia} U_\beta)^{r-(i+n)}) (\operatorname{dia} U_\beta)^i / \delta^{r-n} \qquad \text{(by Inequality } (*)) \\
\leq&\varepsilon.
\end{aligned}$$

□

Theorem 2.22. *The canonical inclusion $\mathscr{C}^r(X, \mathbf{K}) \hookrightarrow \mathscr{C}^r_{T++}(X, \mathbf{K})$ is an isomorphism of locally convex $\mathbf{K}$-algebras with $\|\cdot\|_{\mathscr{C}^r, C} = \|\cdot\|_{\mathscr{C}^r_{T++}, C}$ for all compact $C \subseteq X$.*

Proof: Foremost, the inclusion map $\iota : \mathscr{C}^r(X, \mathbf{K}) \hookrightarrow \mathscr{C}^r_{T++}(X, \mathbf{K})$ is an injective homomorphism of $\mathbf{K}$-vector spaces. By Lemma 2.17, it satisfies $\|\iota(\cdot)\|_{\mathscr{C}^r_{T++}, C} = \|\cdot\|_{\mathscr{C}^r, C}$ on $\mathscr{C}^r(X, \mathbf{K})$ for all compact $C \subseteq X$ and is therefore an isomorphism of locally convex $\mathbf{K}$-vector spaces onto its image. It therefore suffices to prove its surjectivity. By Lemma 2.21, we have a dense inclusion $\mathscr{C}^{pol}_{\leq g}(X, \mathbf{K}) \subseteq \mathscr{C}^r_{T++}(X, \mathbf{K})$ and we are hence reduced to showing that the image $\mathscr{C}^r(X, \mathbf{K}) \subseteq \mathscr{C}^r_{T++}(X, \mathbf{K})$ is closed with respect to the locally convex topology of $\mathscr{C}^r_{T++}(X, \mathbf{K})$.

By Proposition 2.7, it is complete with respect to the locally convex topology of $\mathscr{C}^r(X, \mathbf{K})$. Because ι is an isomorphism of topological $\mathbf{K}$-vector spaces onto its image, $\mathscr{C}^r(X, \mathbf{K})$ is also complete with respect to the subspace topology in $\mathscr{C}^r_{T++}(X, \mathbf{K})$. Therefore $\mathscr{C}^r(X, \mathbf{K}) \subseteq \mathscr{C}^r_{T++}(X, \mathbf{K})$ is closed as $\mathscr{C}^r_{T++}(X, \mathbf{K})$ is Hausdorff. □

Sufficiency of the Taylor polynomial expansion on B_ν-sets for $\mathscr{C}^r$-functions. We will see that the property of Corollary 2.11' is indeed equivalent to r-fold differentiability on the following kind of subsets $X \subseteq \mathbf{K}$:

Definition 2.23. We will say that, for a natural number $\nu > 1$, a subset $X \subseteq \mathbf{K}$ has the B_ν**-property** if there is a positive constant $c \le 1$ such that fixing any $x_0 \in X$ and another point x_1 around x_0, a **"c-regular" ν-gon snuggles into the circle spanned by x_1 around** x_0; that is, there are $x_2, \ldots, x_\nu \in B_{\le\delta}(x_0) \subseteq X$ with $\delta := |x_0 - x_1|$ such that

$$c_{(x_0,\ldots,x_\nu)} := \min_{i,j=0,\ldots,\nu \text{ distinct}} |x_i - x_j| \ge c \cdot \delta.$$

By convention, every subset $X \subseteq \mathbf{K}$ has the property B_0 and B_1. We will say that a subset $X \subseteq \mathbf{K}$ has the **local** B_ν**-property** if it can be covered by open B_ν-sets.

Remark. The definition of a $B_{\nu+1}$-set in [Sch84, Section 83] implies our notion of a B_ν-set: Let x_0, x_1 be distinct points in X and $\delta = |x_0 - x_1|$. Then there is a constant $C \ge 1$ and points $x_2, \ldots, x_\nu$ such that $|x_i - x_j| \le C|x_k - x_l|$ for all $(i,j),(k,l) \in \nabla\{1,\ldots,\nu\}$. This means $|x_k - x_l| \ge c \cdot \mathrm{dia}\{x_1,\ldots,x_\nu\} \ge c \cdot \delta$ with $c := C^{-1} \le 1$.

Lemma 2.24. *All balls in* $\mathbf{K}$ *have the* B_ν*-property and consequently all open subsets of* $\mathbf{K}$ *have the local* B_ν*-property for every* $\nu \in \mathbb{N}$.

Proof: We firstly prove that for any complete nontrivially non-Archimedeanly valued field $\mathbf{K}$ and natural number $\nu > 1$ exists a positive constant $c \le 1$ such that some c-regular ν-gon snuggles into $\mathbf{o}$, the circle of the closed unit disc: Fix distinct x_0 and x_1 therein and put $\delta := |x_0 - x_1|$. Up to scaling, we may assume $|x_0| = 1$. Because $|\cdot|$ is nontrivial, we find positive $c \le 1$ so small that $\#\mathbf{o}/\mathbf{o}_{\le c} \ge \nu$. Then we find $\tilde{c} \le c$ such that also $x_0 \not\equiv x_1 \mod o_{\le\tilde{c}}$. We then choose $x_0, \ldots, x_\nu \in \mathbf{o}$ in different residue classes of $\mathbf{o}/\mathbf{o}_{\le\tilde{c}}$. Then $x_0, \ldots, x_\nu$ constitute a c-regular ν-gon. This proves the first proposition.

Let $\nu > 1$ and assume $B \subseteq \mathbf{K}$ to be a ball. Fix a point $x_0 \in B$ and distinct $x_1 \in B$. Let $D := B_{\le\delta}(x_0) \subseteq B$ be a closed disc around x_0 with $\delta = |x_0 - x_1|$. Since B a ball, we have $D = B_{\le\delta}(x_0) \subseteq \mathbf{K}$. Now there is the homothety $(x_1 - x_0) \cdot \_$ and the translation $x_0 + \_$ which transform the closed unit disc $B_{\le 1}(0)$ into $B_{\le\delta}(x_0)$. We apply their composed affine linear map to the ν-gon $\{y_0, \ldots, y_\nu\}$ with $y_0 := 0$ and $y_1 := 1$ in the unit disc found above, yielding the ν-gon $\{x_0, \ldots, x_\nu\} \subseteq B$. Then $c_{(y_0,\ldots,y_\nu)} = \delta \cdot c_{(x_0,\ldots,x_\nu)} \ge c \cdot \delta$ and therefore B has the B_ν-property. □

Definition 2.25. Let $X \subseteq \mathbf{K}$ be a local B_ν-subset without isolated points. A function $f\colon X \to \mathbf{K}$ will be in $\mathscr{C}^r_T(X,\mathbf{K})$ if there are functions $D_i f\colon X \to \mathbf{K}$ for $i = 0,\ldots,\nu$ such that if one defines $R_\nu f\colon X \times X \to \mathbf{K}$ by

$$R_\nu f(x,y) := f(x) - \sum_{i=0,\ldots,\nu} D_i f(y)(x-y)^i,$$

then for every point $a \in X$ and any $\varepsilon > 0$, there will exist a neighborhood $U \ni a$ with

$$|R_\nu f(x,y)| \le \varepsilon |x-y|^r \quad \text{for all } x, y \in U.$$

Notice that — in comparison to $\mathscr{C}^r_{T^+}(X,\mathbf{K})$ of Definition 2.12 - together with the plus, we have dropped the continuity assumption on $D_0 f, \ldots, D_\nu f\colon X \to \mathbf{K}$. In the following we want to show that this is automatically implied by the B_ν-property of X. For this, we will investigate these functions more closely.

Remember that any polynomial of degree ν is determined by $\nu + 1$ values of it. The next Lemma 2.26 makes this somewhat more explicit.

Lemma 2.26. *Let $x_0, \ldots, x_\nu \in \mathbf{K}$ be pairwise distinct points. Let $\mathrm{R} =< x_0, \ldots, x_\nu >\subseteq \mathbf{K}$ be the subring in $\mathbf{K}$ generated by $x_0, \ldots, x_\nu$. Then we can find coefficients $c_{0,i}, \ldots, c_{\nu,i}$ in the principal fractional ideal $1/\prod_{k\neq l}(x_k - x_l) \cdot \mathrm{R}$ for $i = 0, \ldots, \nu$ such that for any polynomial $\mathrm{P}(\mathrm{X}) = \sum_{i=0,\ldots,\nu} a_i \mathrm{X}^i \in \mathbf{K}[\mathrm{X}]$ of degree ν,*

$$a_i \mathrm{X}^i = c_{0,i}\mathrm{P}(x_0\mathrm{X}) + c_{1,i}\mathrm{P}(x_1\mathrm{X}) + \cdots + c_{\nu,i}\mathrm{P}(x_\nu\mathrm{X}).$$

Proof: Let W and D denote the $(\nu + 1) \times (\nu + 1)$-square matrices

$$\mathrm{W} := \begin{pmatrix} 1 & x_0\mathrm{X} & \cdots & (x_0\mathrm{X})^\nu \\ 1 & x_1\mathrm{X} & \cdots & (x_1\mathrm{X})^\nu \\ \vdots & & & \vdots \\ 1 & x_\nu\mathrm{X} & \cdots & (x_\nu\mathrm{X})^\nu \end{pmatrix} \quad \text{and} \quad \mathrm{D} := \begin{pmatrix} 1 & & & \\ & \mathrm{X}^{-1} & & \\ & & \ddots & \\ & & & \mathrm{X}^{-\nu} \end{pmatrix}.$$

Denote by $\mathrm{V} = \mathrm{W} \cdot \mathrm{D}$ the product of these. This is the matrix with coefficients $(x_i^j)_{i,j=0,\ldots,\nu}$ in $\mathbf{K}$, which is invertible as can be seen by its Vandermonde-determinant

$$\det \mathrm{V} = \prod_{\substack{i,j\in\{0,\ldots,\nu\} \\ \text{with } i>j}} (x_i - x_j) \neq 0.$$

Because

$$\mathrm{W}\begin{pmatrix} a_0 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} 1 & x_0\mathrm{X} & \cdots & (x_0\mathrm{X})^\nu \\ 1 & x_1\mathrm{X} & \cdots & (x_1\mathrm{X})^\nu \\ \vdots & & & \vdots \\ 1 & x_\nu\mathrm{X} & \cdots & (x_\nu\mathrm{X})^\nu \end{pmatrix}\begin{pmatrix} a_0 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} \mathrm{P}(x_0\mathrm{X}) \\ \vdots \\ \mathrm{P}(x_\nu\mathrm{X}) \end{pmatrix},$$

we find by right-multiplication with V^{-1} that

$$\mathrm{V}^{-1}\begin{pmatrix} \mathrm{P}(x_0\mathrm{X}) \\ \mathrm{P}(x_1\mathrm{X}) \\ \vdots \\ \mathrm{P}(x_\nu\mathrm{X}) \end{pmatrix} = \mathrm{D}^{-1}\mathrm{W}^{-1}\begin{pmatrix} \mathrm{P}(x_0\mathrm{X}) \\ \mathrm{P}(x_1\mathrm{X}) \\ \vdots \\ \mathrm{P}(x_\nu\mathrm{X}) \end{pmatrix} = \mathrm{D}^{-1}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1\mathrm{X}^1 \\ \vdots \\ a_\nu\mathrm{X}^\nu \end{pmatrix}.$$

By the Cramer rule and the shape of V, this spelled out is the proposition. □

Corollary 2.27. *Let $\mathrm{P}(\mathrm{X}) = \sum_{i=0,\ldots,\nu} a_i\mathrm{X}^i \in \mathbf{K}[\mathrm{X}]$ be a polynomial of degree ν. Then for pairwise distinct points $x_0, \ldots, x_\nu \in \mathbf{K}$ of maximal norm $\delta \in |\mathbf{K}|$ we find*

$$|a_i| \leq \mathrm{M}(x_0, \ldots, x_\nu)\delta^{-i}(|\mathrm{P}(x_0)| \vee \ldots \vee |\mathrm{P}(x_\nu)|)$$

with an upper bound $\mathrm{M}(x_0, \ldots, x_\nu) := \prod_{i\neq j} \delta/|x_i - x_j| \geq 1$.

Proof: Given these pairwise distinct $x_0, \dots, x_\nu \in \mathbf{K}$ of maximal norm δ, let without loss of generality $|x_0| = \delta$. Then apply the preceding Lemma 2.26 for the $\nu + 1$ points $x_0/x_0, \dots, x_\nu/x_0$ of norm at most 1 and $\mathrm{X} = x_0$. $\square$

Lemma 2.28. *Let* $\mathrm{X} \subseteq \mathbf{K}$ *be a* B_ν*-subset. Then there is a constant* $\mathrm{C} \geq 1$ *such that for all* $f \in \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K})$ *and distinct* $x, y \in \mathrm{X}$ *there is a set* P *of* $\nu + 1$ *points in* $\mathrm{U} := \mathrm{B}_{\leq \delta}(x) \subseteq \mathrm{X}$ *with* $\delta = |x - y|$ *such that*

$$|\mathrm{R}_{\nu-n}\,\mathrm{D}_n f(x, y)| \leq \mathrm{C} \cdot \delta^{-n} \max_{z \in \mathrm{P}}(|\mathrm{R}_\nu f(z, x)| \vee |\mathrm{R}_\nu f(z, y)|) \quad \textit{for } n = 0, \dots, \nu;$$

here $\mathrm{R}_{\nu-n}\,\mathrm{D}_n f$ *is given through the functions* $\binom{n}{n}\mathrm{D}_n f, \binom{n+1}{n}\mathrm{D}_{n+1} f, \dots, \binom{\nu}{n}\mathrm{D}_\nu f$.

Proof: Let $x, x + y$ and $x + y + z \in \mathrm{X}$. We have

$$\begin{aligned}
&\mathrm{R}_\nu f(x + y + z, x) - \mathrm{R}_\nu f(x + y + z, x + y) \\
=&f(x + y + z) - \sum_{k=0,\dots,\nu} \mathrm{D}_k f(x)(y + z)^k - (f(x + y + z) - \sum_{k=0,\dots,\nu} \mathrm{D}_k f(x + y) z^k) \\
=&\sum_{k=0,\dots,\nu} \mathrm{D}_k f(x)(y + z)^k - \sum_{k=0,\dots,\nu} \mathrm{D}_k f(x + y) z^k.
\end{aligned}$$

By the binomial identity and then altering the order of summation we calculate

$$\begin{aligned}
\sum_{k=0,\dots,\nu} \mathrm{D}_k f(x)(y + z)^k &= \sum_{k=0,\dots,\nu} \sum_{i+j=k} \mathrm{D}_k f(x) \binom{k}{i} y^i z^j \\
&= \sum_{j=0,\dots\nu} z^j (\sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathrm{D}_{i+j} f(x) y^i).
\end{aligned}$$

Together this yields

$$\begin{aligned}
&\mathrm{R}_\nu f(x + y + z, x) - \mathrm{R}_\nu f(x + y + z, x + y) \\
=&\sum_{j=0,\dots\nu} z^j (\mathrm{D}_j f(x + y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathrm{D}_{i+j} f(x) y^i). \qquad (*)
\end{aligned}$$

This is a polynomial function $\mathrm{Q}(z)$ of degree ν in z. By Corollary 2.27, we obtain for its coefficients the inequality

$$|\mathrm{D}_j f(x + y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathrm{D}_{i+j} f(x) y^i| \leq \mathrm{M}(z_0, \dots, z_\nu) \delta^{-j} \max_{z \in \mathrm{P}} |\mathrm{Q}(z)|$$

for any collection of $\nu + 1$ points $\tilde{\mathrm{P}} := \{z_0, \dots, z_\nu\}$ in $\mathbf{K}$ of maximal norm δ. If we can find these points such that $x + y + \mathrm{P} \subseteq \mathrm{X}$, then equality $(*)$ will yield

$$\begin{aligned}
&|\mathrm{D}_j f(x + y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathrm{D}_{i+j} f(x) y^i| \\
\leq&\mathrm{M}(z_0, \dots, z_\nu) \delta^{-j} \max_{z \in \tilde{\mathrm{P}}} (|\mathrm{R}_\nu f(x + y + z, x)| \vee |\mathrm{R}_\nu f(x + y + z, x + y)|).
\end{aligned}$$

Now since X satisfies the B_ν-property, we can indeed extend the two distinct points $z_0 := x$ and $z_1 := x + y$ to a collection of $\nu + 1$ points $P := \{z_0, \dots, z_\nu\} \subseteq U := B_{\leq\delta}(x) \subseteq X$ with $c_{(z_0,\dots,z_\nu)} \geq c \cdot \delta$ as in Definition 2.23. Then

$$M(z_0, \dots, z_\nu) = \prod_{i \neq j} \delta/|z_i - z_j| \leq c^{-\binom{\nu+1}{2}} =: C,$$

which will be our sought positive constant. If we let

$$\tilde{P} := P - (x + y) = \{z_0 - (x + y), \dots, z_\nu - (x + y)\},$$

we find therefore

$$\begin{aligned}&|D_j f(x + y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} D_{i+j} f(x) y^i| \\ \leq& C \cdot \delta^{-j} \max_{z \in \tilde{P}}(|R_\nu f(x + y + z, x)| \vee |R_\nu f(x + y + z, x + y)|).\end{aligned}$$

This proves the proposition. □

Definition. Cf. Definition 2.25, we will prove below that $D_0 f, \dots, D_\nu f : X \to \mathbf{K}$ and accordingly $R_\nu f : X \times X \to \mathbf{K}$ are continuous functions. Keeping the notations of Definition 2.14(i), we endow the space $\mathscr{C}^r_T(X, \mathbf{K})$ with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\mathscr{C}^r_T, C}\}$ on $\mathscr{C}^r_T(X, \mathbf{K})$ running through all compact subsets $C \subseteq X$ defined by

$$\|f\|_{\mathscr{C}^r_T, C} := \|D_0 f\|_C \vee \dots \vee \|D_\nu f\|_C \vee \||\Delta_r f|\|_{C^2}.$$

In other words: In case $X \subseteq \mathbf{K}$ is a B_ν-subset, we find $\mathscr{C}^r_T(X, \mathbf{K}) = \mathscr{C}^r_{T^+}(X, \mathbf{K})$ and we give $\mathscr{C}^r_T(X, \mathbf{K})$ the same locally convex topology.

Theorem 2.29. *Let $X \subseteq \mathbf{K}$ be a nonempty local B_ν-subset without isolated points. Then the canonical inclusion $\mathscr{C}^r_{T^{++}}(X, \mathbf{K}) \hookrightarrow \mathscr{C}^r_T(X, \mathbf{K})$ is an isomorphism of $\mathbf{K}$-algebras. It will be an isomorphism of locally convex $\mathbf{K}$-algebras if $\mathbf{K}$ is locally compact.*

Proof: Let $f \in \mathscr{C}^r_T(X, \mathbf{K})$. Firstly, we have to show that $\binom{n}{n} D_n f, \dots, \binom{\nu}{n} D_\nu f$ prove $D_n f$ to be in $\mathscr{C}^{r-n}_{T^+}(X, \mathbf{K})$ for $n = 0, \dots, \nu$. Fix $\varepsilon > 0$ and $a \in X$. Find a B_ν-neighborhood $U \ni a$ such that $|R_\nu f(x, y)| \leq \varepsilon |x - y|^r$ for all $x, y \in U$. If $x \neq y$, we find by Lemma 2.28 a constant $C = C(U) \geq 1$ solely depending on U, a finite subset $P \subseteq B_{\leq\delta}(x) \subseteq U$ with $\delta := |x - y|$ such that

$$\begin{aligned}|R_{\nu-n} D_n f(x, y)| \leq& C|x - y|^{-n} \max_{z_0 = x, y \text{ and } z \in P} |R_\nu f(z, z_0)| \\ \leq& C|x - y|^{-n} \max_{z_0 = x, y \text{ and } z \in P} \varepsilon |z_0 - z|^r \\ \leq& C\varepsilon |x - y|^{r-n};\end{aligned}$$

the last inequality since $|z_0 - z| \leq \delta$, the points $z_0 = x, y$ will both be centers of $B_{\leq\delta}(x)$. If $x = y$, the above inequality will trivially hold. By Remark 2.18, the functions $D_n f$ for $n = 0, \dots, \nu$ are in particular automatically continuous.

Secondly, we prove that $|||\Delta_{r-n} \, \mathrm{D}_n f|||_{\mathrm{K}^2} \le \mathrm{C} \cdot |||\Delta_r f|||_{\mathrm{K}^2}$ for all compact B_ν-subsets $\mathrm{K} \subseteq \mathrm{X}$. By Lemma 2.28, we find for distinct $x, y \in \mathrm{K}$ a finite subset $\mathrm{P} \subseteq \mathrm{B}_{\le\delta}(x) \subseteq \mathrm{K}$ with $\delta := |x - y|$ such that

$$\begin{aligned}
|\Delta_{r-n} \, \mathrm{D}_n f(x, y)| &= \frac{|\mathrm{R}_{\nu-n} \, \mathrm{D}_n f(x, y)|}{|x - y|^{r-n}} \\
&\le \mathrm{C} \cdot \max_{z_0 = x, y \text{ and } z \in \mathrm{P}} |\mathrm{R}_\nu f(z, z_0)| \frac{1}{|x - y|^r} \\
&= \mathrm{C} \cdot \max_{z_0 = x, y \text{ and } z \in \mathrm{P}} |\Delta_r f(z, z_0)| \left| \frac{z_0 - z}{x - y} \right|^r \\
&\le \mathrm{C} \cdot \max_{z_0 = x, y \text{ and } z \in \mathrm{P}} |\Delta_r f(z, z_0)|. \qquad (*)
\end{aligned}$$

If $x = y$, then the left-hand side will vanish as $\mathrm{D}_n f$ was seen to be in $\mathscr{C}_{\mathrm{T}}^{r-n}(\mathrm{X}, \mathbf{K})$. As $\mathrm{P} \subseteq \mathrm{K}$, we find $|||\Delta_{r-n} \, \mathrm{D}_n f|||_{\mathrm{K}^2} \le \mathrm{C} \cdot |||\Delta_r f|||_{\mathrm{K}^2}$ for all compact subsets $\mathrm{K} \subseteq \mathrm{X}$.

Let now $\mathbf{K}$ be locally compact. The continuity of this inclusion of function spaces holds true by definition. Regarding its openness, we observe that any compact subset $\mathrm{K} \subseteq \mathrm{X}$ is contained in a closed, hence compact, ball B inheriting the B_ν-property for a constant $\mathrm{C} \ge 1$. Therefore the locally convex topology on $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ is induced by all seminorms $\|\cdot\|_{\mathscr{C}^r, \mathrm{B}}$ for closed balls B and we have

$$\begin{aligned}
\|f\|_{\mathscr{C}^r, \mathrm{B}} &= \max_{n=0,\dots,\nu} (\|\mathrm{D}_n f\|_{\mathrm{B}} \vee |||\Delta_{r-n} \mathrm{D}_n f|||_{\mathrm{B}^2}) \\
&\le \max_{n=0,\dots,\nu} \|\mathrm{D}_n f\|_{\mathrm{B}} \vee \mathrm{C} \cdot |||\Delta_r f(z, z_0)|||_{\mathrm{B}^2} \le \mathrm{C} \cdot \|f\|_{\mathscr{C}_{\mathrm{T}}^r, \mathrm{B}};
\end{aligned}$$

here the first inequality by B having the B_ν-property and the above estimate $(*)$. □

Corollary 2.30. *Let* $\mathrm{X} \subseteq \mathbf{K}$ *be a nonempty local* B_ν*-subset without isolated points. Then the canonical inclusion* $\mathscr{C}^r(\mathrm{X}, \mathbf{K}) \hookrightarrow \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}, \mathbf{K})$ *is an isomorphism of* $\mathbf{K}$*-algebras. It will be an isomorphism of locally convex* $\mathbf{K}$*-algebras if* $\mathbf{K}$ *is locally compact.*

Proof: By the previous Theorem 2.29 together with Theorem 2.22. □

Another characterization of $\mathscr{C}^r$*-functions on compact sets.* We show another equivalence of differentiability notions: In [Col10], the author gave a definition of r-times differentiable functions on $\mathbb{Z}_p$ (into a closed subfield of $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$). We canonically generalize this to functions on nonempty subsets $\mathrm{X} \subseteq \mathbf{K}$ (into $\mathbf{K}$) and show that these two notions coincide on compact B_ν-subsets $\mathrm{X} \subseteq \mathbf{K}$ without isolated points.

Definition 2.31 (Colmez)**.** If $r \in \mathbb{R}_{\ge 0}$, we will say that $f \colon \mathrm{X} \to \mathbf{K}$ is of class $\tilde{\mathscr{C}}_{\mathrm{T}}^r$, if there are functions $\mathrm{D}_i f \colon \mathrm{X} \to \mathbf{K}$ for $i = 0, \dots, \lfloor r \rfloor$ such that if we define $\mathrm{R}_{\lfloor r \rfloor} f \colon \mathrm{X} \times \mathrm{X} \to \mathbf{K}$ by

$$\mathrm{R}_{\lfloor r \rfloor} f(x, y) = f(x) - \sum_{i=0,\dots,\lfloor r \rfloor} \mathrm{D}_i f(y)(x - y)^i,$$

then

$$\tilde{\mathrm{C}}_r f(\delta) := \sup_{x_0 \in \mathrm{X}} \sup_{y \in \mathrm{B}_{\leq\delta}(x_0)} \frac{|\mathrm{R}_{\lfloor r \rfloor} f(x_0, y)|}{\delta^r}$$

is a well-defined function $\tilde{\mathrm{C}}_r f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ which converges to 0 as δ does. We denote the set of $\tilde{\mathscr{C}}^r_{\mathrm{T}}$-functions $f : \mathrm{X} \to \mathbf{K}$ by $\tilde{\mathscr{C}}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K})$.

Proposition 2.32. *Let r be a nonnegative real number. Then Definition 2.25 and Definition 2.31 coincide on compact B_ν-subsets $\mathrm{X} \subseteq \mathbf{K}$ without isolated points, that is,*

$$\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K}) = \tilde{\mathscr{C}}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K}).$$

Proof: Given a function $f : \mathrm{X} \to \mathbf{K}$, it will suffice to show that the conditions on the functions $\mathrm{D}_i f$ in Definition 2.25 respectively Definition 2.31 for $i = 0, \ldots, \nu := \lfloor r \rfloor$ are equivalent.

Recall by Definition 2.31 that $f \in \tilde{\mathscr{C}}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K})$ if $\tilde{\mathrm{C}}_r f(\delta) \to 0$ for $\delta \to 0$, that is, for any $\varepsilon > 0$ there is a $\delta_0 > 0$ such that $\tilde{\mathrm{C}}'_r f(\delta_0) := \sup_{0<\delta\leq\delta_0} \tilde{\mathrm{C}}_r f(\delta) < \varepsilon$.

On the other hand assume that $f \in \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K})$. Since $\mathrm{X} \subseteq \mathbf{K}$ is a B_ν-subset, we find $f \in \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K}) \subseteq \mathscr{C}^r_{\mathrm{T}^+}(\mathrm{X}, \mathbf{K})$, which holds if and only if $|\Delta_r f| : \nabla\mathrm{X} \times \mathrm{X} \to \mathbb{R}_{\geq 0}$ extends to a continuous function on $\mathrm{X} \times \mathrm{X}$ vanishing on the diagonal. As X is a compact metric space, $|\Delta_r f|$ is continuous on $\mathrm{X} \times \mathrm{X}$ if and only if it is uniformly so. In particular on $\triangle\mathrm{X} \times \mathrm{X}$, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mathrm{C}'_r f(\delta) := \sup_{a \in \mathrm{X}} \sup_{\substack{x, y \in \mathrm{B}_{\leq\delta}(a), \\ x \neq y}} |\Delta_r f(x, y)| < \varepsilon.$$

It suffices to show that $\mathrm{C}'_r f(\delta) = \tilde{\mathrm{C}}'_r f(\delta)$. Plugging in the Definitions of $\mathrm{C}'_r f(\delta)$ respectively $\tilde{\mathrm{C}}'_r f(\delta)$ we thus have to show that

$$\sup_{a \in \mathrm{X}} \sup_{\substack{x, y \in \mathrm{B}_{\leq\delta}(a), \\ x \neq y}} \frac{|\mathrm{R}_\nu f(x, y)|}{|x - y|^r} = \sup_{0<\delta'\leq\delta} \sup_{x \in \mathrm{X}} \sup_{y \in \mathrm{B}_{\leq\delta'}(x)} \frac{|\mathrm{R}_\nu f(x, y)|}{\delta'^r}. \qquad (*)$$

We note that for any $x, y \in \mathrm{X}$ and $\delta > 0$ holds $|x - y| \leq \delta$ if and only if there is an $x_0 \in \mathrm{X}$ such that $\max\{|x - x_0|, |y - x_0|\} \leq \delta$ by the strong triangle inequality. Thus the left-hand side of $(*)$ equals

$$\sup_{x \in \mathrm{X}} \sup_{\substack{y \in \mathrm{B}_{\leq\delta}(x), \\ x \neq y}} \frac{|\mathrm{R}_\nu f(x, y)|}{|x - y|^r}. \qquad (**)$$

Furthermore we note that for any $x, y \in \mathrm{X}$ we have $x \neq y$ and $|x - y| \leq \delta$ if and only if $|x - y| = \delta'$ for some $0 < \delta' \leq \delta$. Thus

$$\sup_{x \in \mathrm{X}} \sup_{\substack{y \in \mathrm{B}_{\leq\delta}(x), \\ x \neq y}} \frac{|\mathrm{R}_\nu f(x, y)|}{|x - y|^r} = \sup_{x \in \mathrm{X}} \sup_{0<\delta'\leq\delta} \sup_{y \text{ s.t. } |x-y|=\delta'} \frac{|\mathrm{R}_\nu f(x, y)|}{\delta'^r}$$

Now keeping x fixed,

$$\sup_{0<\delta'\leq\delta}\ \sup_{y\in X \text{ s.t. } |x-y|=\delta'} \frac{|R_\nu f(x,y)|}{\delta'^r} = \sup_{0<\delta'\leq\delta}\ \sup_{y\in X \text{ s.t } |x-y|\leq\delta'} \frac{|R_\nu f(x,y)|}{\delta'^r}$$

as $R_\nu f(x,y) = 0$ for any $y = x$. But then

$$\sup_{x\in X}\ \sup_{\substack{y\in B_{\leq\delta}(x),\\ x\neq y}} \frac{|R_\nu f(x,y)|}{|x-y|^r} = \sup_{x\in X}\ \sup_{0<\delta'\leq\delta}\ \sup_{y\in B_{\leq\delta'}(x)} \frac{|R_\nu f(x,y)|}{\delta'^r}.$$

This is the claimed equality $(*)$ after substituting the left-hand side by $(**)$. □

The Mahler basis of $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$

This section is devoted to the computation of the Mahler coefficients of $\mathscr{C}^r$-functions. We firstly introduce some terminology and make some general observations.

The Mahler basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. We will throughout this subsection assume that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a normed field.

Definition. We define $\mathrm{l}\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ by $\mathrm{l}(0) = 0$ and otherwise $\mathrm{l}(i)$ as the largest $n \in \mathbb{N}$ such that $p^n \leq i$.

So $\mathrm{l}(i) = \lfloor \log_p i \rfloor$ with $\log_p := \log / \log p$ for $i \geq 1$. Recall that in Section 2, we gave a norm on $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ by

$$\|f\|_{\mathscr{C}^\rho} = \|f\|_{\sup} \vee \||f^{[\rho]}|\|_{\sup}.$$

Here $|f^{[\rho]}|\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{R}_{\geq 0}$ is the continuous function extending the mapping $|f(x) - f(y)|/|x-y|^\rho$, defined for all distinct $x, y \in \mathbb{Z}_p$, by 0 on the diagonal $\triangle\mathbb{Z}_p^2$. Thus

$$\||f^{[\rho]}|\|_{\sup} = \max_{x,y\in\mathbb{Z}_p \text{ distinct}} \frac{|f(x)-f(y)|}{|x-y|^\rho}.$$

Definition. We will denote the normed $\mathbf{K}$-linear subspace of locally constant functions $f\colon \mathbb{Z}_p \to \mathbf{K}$ in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ by $\mathscr{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. Then $\mathscr{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho = \cup_{n\geq 0} \mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$, where $\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$ is the $\mathbf{K}$-Banach subspace of functions $f\colon \mathbb{Z}_p \to \mathbf{K}$ constant on the $p^n\mathbb{Z}_p$-cosets.

Definition. We will call a countable subset $\{e_0, e_1, \ldots\}$ of a $\mathbf{K}$-Banach space $\mathbf{E}$ **orthogonal** if $\|\sum_{i\geq 0} \lambda_i e_i\| = \max_{i\geq 0} |\lambda_i| \|e_i\|$ for all scalars λ_i such that this series converges. It will be called **orthonormal** if it is orthogonal and $\|e_i\| = 1$ for all i. It will be an **orthogonal base** if every $x \in \mathbf{E}$ can be written $x = \sum_i \lambda_i e_i$ for some scalars λ_i.

Lemma 2.33. *A countable subset $\{e_1, e_2, \ldots\}$ of a $\mathbf{K}$-Banach space $\mathbf{E}$ is orthogonal if and only if*

$$\|\sum_{i=m,\ldots,n} \lambda_i e_i\| \geq |\lambda_m| \|e_m\| \quad \textit{for all scalars } \lambda_m, \ldots, \lambda_n.$$

Proof: See [Sch84, Proposition 50.4]. □

Corollary 2.34. *Let* $\{e_0, e_1, \dots\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ *be such that* $e_i(i) = 1$ *and* $e_i(m) = 0$ *for any nonnegative integer* $m < i$ *and* $\|e_i\|_{\mathscr{C}^\rho} = p^{l(i)\rho}$. *Then* $\{e_0, e_1, \dots\}$ *is an orthogonal system of* $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof: By the above Lemma 2.33, we have to show $\|\sum_{i=m,\dots,n} a_i e_i\|_{\mathscr{C}^\rho} \geq |a_m| \|e_m\|_{\mathscr{C}^\rho}$ for any $m \leq n \in \mathbb{N}$ and $a_m, \dots, a_n \in \mathbf{K}$. Firstly, for $m = 0$, we find

$$\| \sum_{i=0,\dots,n} a_i e_i \|_{\mathscr{C}^\rho} \geq \| \sum_{i=0,\dots,n} a_i e_i \|_{\sup} \geq | \sum_{i=0,\dots,n} a_i e_i(0)| = |a_0| = |a_0| \|e_0\|_{\mathscr{C}^\rho}.$$

If instead $m > 0$, we calculate

$$\begin{aligned} \| \sum_{i=m,\dots,n} a_i e_i \|_{\mathscr{C}^\rho} &\geq \| | (\sum_{i=m,\dots,n} a_i e_i)^{[\rho]} | \|_{\sup} \\ &\geq \frac{|\sum_{i=m,\dots,n} a_i (e_i(m) - e_i(m - p^{l(m)}))|}{|p^{l(m)}|^\rho} \\ &= |a_m| p^{l(m)\rho} = |a_m| \|e_m\|_{\mathscr{C}^\rho}. \end{aligned}$$

□

Interlude: The van der Put basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. This brief interlude is motivated by the remark following [Sch84, Theorem 63.2] about the similarity between the description of the $\mathscr{C}^{\mathrm{lip}}$-functions by its coefficients with respect to the Mahler- and van der Put-base.

Definition. We define the **van der Put indicator function** $\mathrm{P}_i \colon \mathbb{Z}_p \to \mathbf{K}$ for $i \in \mathbb{N}$ by $\mathrm{P}_0 \equiv 1$ and for $i \geq 1$ through

$$\mathrm{P}_i(x) = \begin{cases} 1, & \text{if } a_0 + a_1 p + \dots + a_n p^n = i \text{ for some } n, \text{ where } x = \sum_{j \geq 0} a_j p^j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.35.

(i) $\|\mathrm{P}_i\|_{\mathscr{C}^\rho} = p^{l(i)\rho}$ *for all* $i \in \mathbb{N}$.

(ii) The family $\{\mathrm{P}_0, \mathrm{P}_1, \dots\}$ *is orthogonal.*

Proof: Ad (i): Obviously $\|\mathrm{P}_i\|_{\sup} = 1$ for all $i \in \mathbb{N}$. Since P_i is constant on $p^{l(i)}\mathbb{Z}_p$-cosets, we therefore find $\| | \mathrm{P}_i^{[\rho]} | \|_{\sup} \leq p^{l(i)\rho}$. Firstly, if $i = 0$, we thus find $1 = \|\mathrm{P}_0\|_{\sup} \leq \|\mathrm{P}_0\|_{\mathscr{C}^\rho} \leq p^{l(0)\rho} = 1$, that is, $\|\mathrm{P}_0\|_{\mathscr{C}^\rho} = 1$. If instead $i > 0$, we achieve the postulated equality by

$$\frac{|\mathrm{P}_i(i) - \mathrm{P}_i(i - p^{l(i)})|}{|i - (i - p^{l(i)})|^\rho} = \frac{|1 - 0|}{|p^{l(i)}|^\rho} = p^{l(i)\rho}.$$

Ad (ii): Since $\mathrm{P}_i(i) = 1$, as well as $\mathrm{P}_i(m) = 0$ if $m < i$ and we just saw $\|\mathrm{P}_i\|_{\mathscr{C}^\rho} = p^{l(i)\rho}$, Corollary 2.34 applies. □

Proposition. *The family* $\{\mathrm{P}_i\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ *is an orthogonal basis of* $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof: It is a general fact that an orthogonal system of a **K**-Banach space **E** over a nontrivially non-Archimedeanly valued field **K**, whose **K**-linear span is dense therein is an orthonormal basis (cf. [Sch84, Exercise 50.F]). We must therefore show that the **K**-linear span of $\{\mathrm{P}_0, \mathrm{P}_1, \ldots\}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. By definition, we find $\{\mathrm{P}_0, \ldots, \mathrm{P}_{p^n-1}\} \subseteq \mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. By the orthogonality of $\{\mathrm{P}_0, \mathrm{P}_1, \ldots\}$ through Lemma 2.35, the coefficients a_i of each linear combination $\sum_i a_i \mathrm{P}_i$ are unique. That is, $\mathrm{P}_0, \mathrm{P}_1, \ldots$ are all linearly independent. Since $\dim_{\mathbf{K}} \mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho = p^n = \#\{\mathrm{P}_0, \ldots, \mathrm{P}_{p^n-1}\}$, the family $\{\mathrm{P}_0, \ldots, \mathrm{P}_{p^n-1}\}$ is a maximal linearly independent subset and in particular spans $\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. Because $\mathscr{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho = \cup_n \mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$, and this space is by Corollary 1.6 dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$, we are done. □

Definition 2.36. We define the i-th **Mahler polynomial** $\binom{*}{i} \colon \mathbb{Z}_p \to \mathbf{K}$ for $i \in \mathbb{N}$ by

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}.$$

We want to show the Mahler polynomials to constitute an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Because they can quickly be checked to form an orthogonal family, the main content of this subsection consists of showing that their **K**-linear span is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. We will prove this by showing them to behave quite similar to the canonical basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{o}/\mathbf{o}_{<p^{-n}\rho})$ for any $n \geq 0$.

Remark. Let us briefly recall in which respect the orthogonal basis of $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ of Mahler polynomials relates to the domain's topological group structure: Let $\mathbf{o}[[\mathrm{X}]]$ be the topological ring of formal power series endowed with its weak topology, described by a sequence $f_n \to f$ converging in $\mathbf{o}[[\mathrm{X}]]$ if at any fixed index i, the i-th coefficient of f_n for $n \geq 0$ converges to the i-th coefficient of f. Let $\mathbf{o}[[\mathbb{Z}_p]] = \varprojlim \mathbf{o}[\mathbb{Z}/p^n\mathbb{Z}]$ be the completed group algebra. The so called Iwasawa isomorphism $\mathbf{o}[[\mathrm{X}]] \xrightarrow{\sim} \mathbf{o}[[\mathbb{Z}_p]]$ of topological $\mathbf{o}$-algebras is given by $\mathrm{X} \mapsto 1 + \mathbf{1}$; here $\mathbf{1} \in \mathbb{Z}_p$ denoting the canonical generator of the topological abelian group $\mathbb{Z}_p$. Let $\mathscr{D}(\mathbb{Z}_p, \mathbf{o})$ be the continuous $\mathbf{o}$-linear dual of $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ equipped with the topology of pointwise convergence. We obtain a natural identification $\mathbf{o}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathscr{D}(\mathbb{Z}_p, \mathbf{o})$.

We conclude that the Iwasawa isomorphism yields by Schikhof duality (see [Sch95, Theorem 4.6]) the isomorphism of **K**-Banach spaces $c_0(\mathbb{N}, \mathbf{K}) \xrightarrow{\sim} \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ with $c_0(\mathbb{N}, \mathbf{K})$ denoting all zero sequences in **K**.

The Mahler polynomials are then the images in $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ of the canonical orthogonal basis $\{(\ldots 0, 1, 0 \ldots)\} \subseteq c_0(\mathbb{N}, \mathbf{K})$ under the dual of the Iwasawa isomorphism.

Lemma 2.37.

(i) $\|\binom{*}{i}\|_{\mathscr{C}^\rho} = p^{\mathrm{l}(i)\rho}$ *for all* $i \in \mathbb{N}$.

(ii) The family $\{\binom{*}{0}, \binom{*}{1}, \ldots\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ *is orthogonal.*

Proof: Ad (i): By [Sch84, Lemma 47.4], we find

$$\left|\binom{x}{i} - \binom{y}{i}\right| \leq |x-y| p^{\mathrm{l}(i)} \quad \text{for all } x, y \in \mathbb{Z}_p.$$

Since the left-hand side is bounded by 1, this implies $|\binom{x}{i}-\binom{y}{i}| \leq |x-y|^\rho p^{l(i)\rho}$ for $0 \leq \rho \leq 1$. Hence

$$\frac{|\binom{x}{i} - \binom{y}{i}|}{|x-y|^\rho} \leq p^{l(i)\rho} \quad \text{for all distinct } x, y \in \mathbb{Z}_p.$$

We note that by continuity $\|\binom{*}{i}\|_{\sup} \leq 1$ since $\binom{j}{i} \in \mathbb{Z}_{\geq 0}$ for all $j \in \mathbb{Z}$. If $i = 0$, then $1 = \|\binom{*}{0}\|_{\sup} \leq \|\binom{*}{0}\|_{\mathscr{C}^\rho} \leq p^{l(0)\rho} = 1$, that is, $\|\binom{*}{0}\|_{\mathscr{C}^\rho} = 1$. It therefore remains to prove that $\||f^{[\rho]}(x,y)\|| = p^{l(i)\rho}$ for distinct $x, y \in \mathbb{Z}_p$ for $i \geq 1$. Since $\binom{i}{i} - \binom{i-p^{l(i)}}{i} = 1$, we achieve the craved equality by

$$\frac{|\binom{i}{i} - \binom{i-p^{l(i)}}{i}|}{|p^{l(i)}|^\rho} = p^{l(i)\rho}.$$

Ad (ii): Since $\binom{i}{i} = 1$, as well as $\binom{m}{i} = 0$ if $m < i$ and since we just saw $\|\binom{*}{i}\|_{\mathscr{C}^\rho} = p^{l(i)\rho}$, Corollary 2.34 applies. □

Our aim is to prove that $\{\binom{*}{i}\}$ is an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. At this point, by the general criterion [Sch84, Exercise 50.F] already mentioned, it remains to show that the $\mathbf{K}$-linear span of $\{\binom{*}{i}\}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. This will be initially only proved in the special case of a complete non-Archimedeanly nontrivially valued field $\mathbf{K}$ such that $v(\mathbf{K}) \ni \rho$ and $v(\mathbf{K}^*)$ is a discrete subgroup of $\mathbb{R}$. Afterwards this case will be reduced to.

Definition. Let $(\mathrm{E}, \|\cdot\|)$ be a normed $\mathbf{K}$-vector space over a non-Archimedean nontrivially discretely valued field $\mathbf{K}$ such that $\|\mathrm{E}\| \subseteq |\mathrm{K}|$. We define the $\mathbf{o}$-module $\mathrm{E}_{\leq 1} := \{f \in \mathrm{E} : \|f\| \leq 1\}$ and its submodule $\mathrm{E}_{<1} := \{f \in \mathrm{E} : \|f\| < 1\}$. We set $\overline{\mathrm{E}} := \mathrm{E}_{\leq 1}/\mathrm{E}_{<1}$.

Note that $\overline{\mathrm{E}}$ is naturally a $\mathbf{k}$-vector space for the residue field $\mathbf{k}$ of $\mathbf{K}$. The importance of $\overline{\mathrm{E}}$ stems from the following Lemma.

Lemma 2.38. *Let $\mathbf{E}$ be a $\mathbf{K}$-Banach space over a discretely non-Archimedeanly nontrivially valued field $\mathbf{K}$ such that $\|\mathbf{E}\| \subseteq |\mathbf{K}|$. Then a $\mathbf{K}$-linear subspace $\mathrm{D} \subseteq \mathbf{E}$ is dense in $\mathbf{E}$ if and only if $\overline{\mathrm{D}} = \overline{\mathbf{E}}$.*

Proof: This is a well-known fact in p-adic analysis, proved by successive approximation. See for example, [Sch84]. □

Lemma. *Let $\mathbf{K}$ be a discretely non-Archimedeanly nontrivially valued field with $\rho \in v(\mathbf{K})$. Then $\|\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})\|_{\mathscr{C}^\rho} \subseteq |\mathbf{K}|$.*

Proof: Let $f \in \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Since $\mathbb{Z}_p$ is compact, the supremum

$$\|f\|_{\mathscr{C}^\rho} = \|f\|_{\sup} \vee \||f^{[\rho]}\||_{\sup} = \sup_{x \in \mathbb{Z}_p} |f(x)| \vee \sup_{x,y \in \mathbb{Z}_p \text{ distinct}} \frac{|f(x)-f(y)|}{|x-y|^\rho}$$

is attained. If $\|f\|_{\sup} \geq \||f^{[\rho]}\||_{\sup}$, there will be nothing to show. So assume that there are distinct $x, y \in \mathbb{Z}_p$ such that $\|f\|_{\mathscr{C}^\rho} = |f(x) - f(y)|/|x-y|^\rho$. We must show $|x-y|^\rho \in |\mathbf{K}|$ or equivalently $\rho \cdot v(x-y) \in v(\mathbf{K})$. But $\rho \in v(\mathbf{K})$ and $v(x-y) \in \mathbb{Z}$ by assumption. □

Corollary 2.39. *Let* $\mathbf{K}$ *be a discretely non-Archimedeanly nontrivially valued field with* $\rho \in v(\mathbf{K})$*. Let* $\{e_0, e_1, \dots\}$ *be an orthonormal family of* $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$*. If* $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$ *for all* $n \geq 0$*, then* $\{e_0, e_1, \dots\}$ *will be an orthonormal basis of* $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$*.*

Proof: It is a general fact that an orthonormal system of a $\mathbf{K}$-Banach space $\mathbf{E}$ over a complete nontrivially non-Archimedeanly valued field $\mathbf{K}$, whose $\mathbf{K}$-linear span is dense therein is an orthonormal basis (cf. [Sch84, Theorem 50.7]). We must therefore show that the K-linear span of $\{e_0, e_1, \dots\}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. By the preceding Lemma, the conditions on $\mathbf{E} = \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ of Lemma 2.38 apply. We are hence reduced to proving that

$$\overline{\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})} = \overline{\oplus_{i \geq 0} \mathbf{K} \cdot e_i} = \oplus_{i \geq 0} \mathbf{k} \cdot \overline{e_i},$$

where the last equality stems from the orthogonality of $\{e_i\}$. By Corollary 1.6 (and the obvious implication of Lemma 2.38), we find

$$\overline{\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})} = \overline{\mathscr{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho} = \cup_{n \geq 0} \overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}.$$

Let $n \in \mathbb{N}$. By assumption $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$, and it suffices therefore to show that $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\}$ is a basis of this subspace. By orthonormality, the $\overline{e}_0, \dots, \overline{e}_{p^n-1}$ are linearly independent over $\mathbf{k}$. On the other hand, $\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$ is $\mathbf{K}$-linearly generated by p^n-many functions living on the $p^n\mathbb{Z}_p$-cosets, so that $\dim_{\mathbf{k}} \overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho} \leq p^n$. Therefore

$$\dim_{\mathbf{k}} \overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho} = p^n = \#\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\}.$$

Hence $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\}$ is a maximal linearly independent subset of $\overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$, that is, a basis. □

Recall that we firstly prove $\{\binom{*}{i}\}$ to be an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ only in the case of a discretely valued $\mathbf{K}$ with $\rho \in v(\mathbf{K})$. Since $\rho \in v(\mathbf{K})$, the $\binom{*}{i}$ can be rescaled to yield an ortho*normal* system $\{e_i\}$ of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Also, $\mathbf{K}$ will fulfill the assumptions of Corollary 2.39, so that we are reduced to proving that $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathscr{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$. To this end, the following criterion will be helpful.

Definition.

(i) Let $f \in \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Then there is a smallest $n \geq 0$ such that $|f(x) - f(y)| < |x - y|^\rho$ for all distinct x, y in the same $p^n\mathbb{Z}_p$ coset. We denote this unique number by $o(f)$. (For *oscillation index* of f.)

(ii) We have a well defined $\mathbf{o}$-linear reduction map π_n from the $\mathbf{o}$-module of functions $f \in \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{o})$ of oscillation index $o(f) \leq n$ to the finite dimensional $\mathbf{o}/\mathbf{o}_{<p^{-n\rho}}$-module of functions $f \colon \mathbb{Z}_p/p^n\mathbb{Z}_p \to \mathbf{o}/\mathbf{o}_{<p^{-n\rho}}$.

Lemma. *Let* $f \in \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{o})$ *with* $o(f) \leq n$ *for* $n \in \mathbb{N}$*. If* $\pi_n f = 0$*, then* $\overline{f} = 0$*.*

Proof: Firstly $\|f\|_{\sup} < p^{-\rho n} \leq 1$ because $\pi_n f = 0$. Therefore it remains to show that $|||f^{[\rho]}|||_{\sup} < 1$. We calculate

$$|||f^{[\rho]}|||_{\sup} = \max_{x,y\in\mathbb{Z}_p \text{ distinct}} \frac{|f(x)-f(y)|}{|x-y|^\rho}$$
$$= \max_{\substack{x,y\in\mathbb{Z}_p \text{ distinct} \\ \text{s.t. } |x-y|\leq p^{-n}}} \frac{|f(x)-f(y)|}{|x-y|^\rho} \vee \max_{\substack{x,y\in\mathbb{Z}_p \text{ s.t.} \\ |x-y|>p^{-n}}} \frac{|f(x)-f(y)|}{|x-y|^\rho}$$

As $o(f) \leq n$, we find $|f(x)-f(y)| < |x-y|^\rho$ for all distinct $x, y \in \mathbb{Z}_p$ with $|x-y| \leq p^{-n}$, so that the first maximum is less than 1. Secondly,

$$\max_{\substack{x,y\in\mathbb{Z}_p \\ \text{s.t. } |x-y|>p^{-n}}} \frac{|f(x)-f(y)|}{|x-y|^\rho} \leq \max_{\substack{x,y\in\mathbb{Z}_p \\ \text{s.t. } |x-y|>p^{-n}}} \frac{|f(x)| \vee |f(y)|}{|x-y|^\rho} < \|f\|_{\sup}/p^{n\rho} < p^{n\rho}/p^{n\rho} = 1;$$

here we used $1/|x-y|^\rho < p^{n\rho}$ and $\|f\|_{\sup} \leq p^{-n\rho}$. □

Corollary 2.40. *Let $f \in \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})_{\leq 1}$. If $o(f) \leq n$ for $n \in \mathbb{N}$, then $\overline{f} \in \overline{\mathscr{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$.*

Proof: Consider the mapping

$$\pi_n \colon \mathscr{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{o})^\rho \to \{f \colon \mathbb{Z}_p/p^n\mathbb{Z}_p \to \mathbf{o}/\mathbf{o}_{<p^{-n\rho}}\}.$$

It is well-defined and surjective. Therefore $\pi_n f = \pi_n g$, that is, $\pi_n(f-g) = 0$ for some $g \in \mathscr{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{o})^\rho$. By the preceding Lemma $\overline{f} = \overline{g}$. □

We will now prove the oscillation index of the normalized $\binom{*}{i}$ to equal $\mathrm{l}(i)+1$. This will allows us to apply the above Corollary 2.40 for $i = 0, \dots, p^n - 1$.

Lemma 2.41. *Assume $\rho \in v(\mathbf{K})$. Then we can define $e_i = \lambda_i\binom{*}{i}$ for a scalar $\lambda_i \in \mathbf{K}$ such that $\|e_i\|_{\mathscr{C}^\rho} = 1$. Moreover $o(e_0) = 0$ and $o(e_i) = \mathrm{l}(i) + 1$ if $i \geq 1$.*

Proof: Let us find such scalar λ_i: Let $\alpha \in \mathbf{K}$ such that $v(\alpha) = \rho$. Then $\|\binom{*}{i}\|_{\mathscr{C}^\rho} = p^{\mathrm{l}(i)\rho}$ by Lemma 2.37, so that we put $\lambda_i = \alpha^{\mathrm{l}(i)}$.

Clearly $o(e_0) = 0$. Let $i \geq 1$. We now want to prove that $o(e_i) = \mathrm{l}(i)+1$, that is, $n = \mathrm{l}(i)+1$ is the smallest $n \in \mathbb{N}$ such that

$$|e_i(x) - e_i(y)| < |x-y|^\rho \quad \text{for all distinct } x, y \text{ with } |x-y| \leq p^{-n}.$$

Firstly observe that $\binom{i}{i} - \binom{i-p^{\mathrm{l}(i)}}{i} = 1$ if $i \geq 1$, so $|e_i(i) - e_i(i-p^{\mathrm{l}(i)})| = |\lambda_i| = |p^{\mathrm{l}(i)}|^\rho$. Hence necessarily $o(e_i) > \mathrm{l}(i)$.

Let us prove $o(e_i) \leq \mathrm{l}(i)+1$. By [Sch84, Lemma 47.4], we have

$$\left|\binom{x}{i} - \binom{y}{i}\right| \leq |x-y|p^{\mathrm{l}(i)} = |x-y|^\rho p^{\mathrm{l}(i)\rho}(|x-y|p^{\mathrm{l}(i)})^{1-\rho}.$$

By definition of e_i thus $|e_i(x) - e_i(y)| \leq (|x-y|p^{\mathrm{l}(i)})^{1-\rho}|x-y|^\rho$. So if $|x-y| < p^{-\mathrm{l}(i)}$, then $|e_i(x) - e_i(y)| < |x-y|^\rho$. □

Corollary 2.42. *Assume that $v(\rho) \in v(\mathbf{K})$ and let $\{e_0, e_1, \dots\}$ be as in Lemma 2.41. Then $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathscr{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$ for all $n \geq 0$.*

Proof: We have $\mathrm{l}(i) < n$ for $i = 0, \dots, p^n - 1$. By Lemma 2.41 therefore $o(f) = \mathrm{l}(i) + 1 \leq n$. Now Corollary 2.40 applies. □

Proposition 2.43. *Let $\mathbf{K}$ be a complete non-Archimedeanly nontrivially discretely valued field with $\rho \in v(\mathbf{K})$. Let $e_0, e_1, \dots$ be the normalized Mahler polynomials $\binom{*}{i}$ as in Lemma 2.41. Then $\{e_0, e_1, \dots\}$ is an orthonormal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.*

Proof: By Corollary 2.42, this is a direct application of Corollary 2.39. □

Finally we show that for general $\mathbf{K}$, we still obtain that $\{\binom{*}{i}\}$ is an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ by reducing to the special case of a discretely valued $\mathbf{K}$ with $v(\mathbf{K}) \ni \rho$. For this, the following property of theirs is crucial:

Lemma 2.44. *Let $f\colon \mathbb{Z}_p \to \mathbf{K}$ be a continuous mapping and assume $f = \sum_{i\geq 0} a_i \binom{*}{i}$ with respect to $\|\cdot\|_{\sup}$ for coefficients $a_i \in \mathbf{K}$. Then $\operatorname{im} f \subseteq \mathbb{Q}_p$ if and only if $\{a_i\} \subseteq \mathbb{Q}_p$.*

Proof: Define the endomorphism Δ of the $\mathbf{K}$-vector space $\mathbf{K}^{\mathbb{Z}_p}$ by $g \mapsto g(\cdot + 1) - g$. Since $\Delta\binom{*}{0} = 0$ and $\Delta\binom{*}{i} = \binom{*}{i-1}$, we find $\Delta f(x) = \sum_{i\geq 0} a_{i+1}\binom{x}{i}$. Transitively, we obtain

$$\Delta^{\circ n} f(x) := \underbrace{\Delta \circ \cdots \circ \Delta}_{n\text{-times}} f(x) = \sum_{i\geq 0} a_{i+n}\binom{x}{i}.$$

In particular $a_n = \Delta^{\circ n} f(0)$ and hence the result. □

Lemma 2.45. *Let $\{b_i\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ be such that for any complete non-Archimedeanly nontrivially valued field $\mathbf{K} \supseteq \mathbb{Q}_p$ we have:*

(i) $\{b_i\}$ is an orthogonal system of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$,

(ii) for every continuous function $f\colon \mathbb{Z}_p \to \mathbf{K}$ with $f = \sum_{i\geq 0} a_i b_i$ with respect to $\|\cdot\|_{\sup}$ for coefficients $a_i \in \mathbf{K}$, we have $\operatorname{im} f \subseteq \mathbb{Q}_p$ if and only if $\{a_i\} \subseteq \mathbb{Q}_p$.

Then for any complete non-Archimedeanly nontrivially valued field $\mathbf{K} \supseteq \mathbb{Q}_p$ we have: $\{b_i\}$ is an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if it is one of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof: Let $\{b_i\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ be as above and $\mathbf{K} \supseteq \mathbb{Q}_p$ a complete non-Archimedeanly nontrivially valued field. By [Sch84, Exercise 50.F], an orthogonal system of a $\mathbf{K}$-Banach space $\mathbf{E}$ will be an orthogonal basis if its $\mathbf{K}$-linear span is dense in $\mathbf{E}$. Since $\{b_i\}$ is assumed to be orthogonal, it remains to prove that the $\mathbb{Q}_p$-linear span of $\{b_i\}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if its $\mathbf{K}$-linear one is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. We will denote by $< \{b_i\} >_{\mathbb{Q}_p}$ the $\mathbb{Q}_p$-linear span of the b_i and define $< \{b_i\} >_{\mathbf{K}}$ likewise.

Firstly, suppose $< \{b_i\} >_{\mathbf{K}}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Fix a function $f \in \mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$. As a convergent sum in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$, we have

$$f = \sum_{i\geq 0} a_i b_i \quad \text{for coefficients } a_i \in \mathbf{K}.$$

By assumption $\{a_i\} \subseteq \mathbb{Q}_p$. So $< \{b_i\} >_{\mathbb{Q}_p}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$.

Contrariwise, suppose $< \{b_i\} >_{\mathbb{Q}_p}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$. By Corollary 1.6, the locally constant functions $f\colon \mathbb{Z}_p \to \mathbf{K}$ are dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. It will thus suffice to prove that $< \{b_i\} >_{\mathbf{K}}$ is dense in $\mathscr{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. Fix $\varepsilon > 0$ and some locally constant $f\colon \mathbb{Z}_p \to \mathbf{K}$. Then f is constant on the $p^n\mathbb{Z}_p$ cosets for some n. Hence we may write

$$f = \sum_{i=0,\dots,p^n-1} a_i \mathbf{1}_{i+p^n\mathbb{Z}_p} \quad \text{for coefficients } a_0, \dots, a_{p^n-1} \in \mathbf{K}.$$

Let $\mathrm{C} := \max_i |a_i| \vee 1$ and $\varepsilon' = \varepsilon/\mathrm{C}$. By the density of $< \{b_i\} >_{\mathbb{Q}_p}$ in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$, we find $\{f_0, \dots, f_{p^n-1}\} \subseteq < \{b_i\} >_{\mathbb{Q}_p}$ such that $\|f_i - \mathbf{1}_{i+p^n\mathbb{Z}_p}\|_{\mathscr{C}^\rho} \leq \varepsilon'$ for $i = 0, \dots, p^n - 1$. Then

$$\|f - \sum_i a_i f_i\|_{\mathscr{C}^\rho} = \|\sum_i a_i(\mathbf{1}_{i+p^n\mathbb{Z}_p} - f_i)\|_{\mathscr{C}^\rho} \leq \max_i |a_i| \|\mathbf{1}_{i+p^n\mathbb{Z}_p} - f_i\|_{\mathscr{C}^\rho} \leq \mathrm{C}\varepsilon' = \varepsilon.$$

Therefore $< \{b_i\} >_{\mathbf{K}}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. □

Lemma (Lemma 2.45'). *Let $\{b_i\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ satisfy the assumptions of Lemma 2.45. If $\{b_i\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{F})$ is an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{F})$ for* one *complete non-Archimedean field $\mathbf{F} \supseteq \mathbb{Q}_p$, then $\{b_i\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ will be one of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ for* every *complete non-Archimedeanly valued field $\mathbf{K} \supseteq \mathbb{Q}_p$.*

Proof: This is a reformulation of the conclusion of Lemma 2.45. □

In particular we could choose in Lemma 2.45' our field $\mathbf{F} \supseteq \mathbb{Q}_p$ to be discretely valued with $\rho \in v(\mathbf{F})$ if such $\mathbf{F}$ existed. The next Lemma constructs such $\mathbf{F}$.

Lemma 2.46. *There is a complete non-Archimedeanly nontrivially discretely valued field $\mathbf{F} \supseteq \mathbb{Q}_p$ with $\rho \in v(\mathbf{F})$.*

Proof: Let $\mathrm{R} = \mathbb{Q}_p[t]$ endowed with the valuation $v_{\mathbf{F}}(\sum_i a_i t^i) := \inf_i v(a_i) + \rho i$. Then its induced norm, denoted $|\cdot|_{\mathbf{F}}$, is quickly checked to be multiplicative on R. It extends to the completed fraction field $\mathbf{F}$ of R, denoted likewise. □

Remark. As a set, $\mathbf{F}$ consists of all formal Laurent series $\sum_{i\in\mathbb{Z}} a_i t^i$ with coefficients in $\mathbb{Q}_p$ such that, putting $c := p^{-\rho}$, we have $|a_i|c^i \to 0$ as $i \to -\infty$ and $\{|a_i|c^i : i \geq 0\}$ bounded — with norm $|\sum_{i\in\mathbb{Z}} a_i t^i|_{\mathbf{F}} = \max_{i\in\mathbb{Z}} |a_i|c^i$.

Proposition 2.47. *The family $\{\binom{*}{0}, \binom{*}{1}, \dots\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ is an orthogonal basis with $\|\binom{*}{i}\| = p^{\mathrm{l}(i)\rho}$.*

Proof: The norms of the $\binom{*}{i}$ were calculated in Lemma 2.37. Then Lemma 2.37(ii) yields the first and Lemma 2.44 the second assumption of Lemma 2.45 regarding this family. By Lemma 2.45', it suffices to prove this theorem for some discretely valued field $\mathbf{F}$ with $\rho \in v(\mathbf{F})$, which exists by the preceding Lemma 2.46. By Proposition 2.43, we can rescale the $\binom{*}{i}$ such that they form an orthonormal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{F})$. But then $\{\binom{*}{i}\}$ is still an orthogonal basis of $\mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{F})$. □

Interlude: $\mathscr{C}^{\boldsymbol{\rho}}$-functions for $\boldsymbol{\rho} \in [0,1[^d$. Throughout this subsection, we will fix a tuple of real numbers $\rho \in [0,1[^d$.

We provide the necessary definitions to obtain a description of the coefficients of the Mahler basis of $\mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$, as carried out in the next subsection.

Definition 2.48. Let $f\colon X \to Y$ be a mapping on the metric spaces $X = X_1 \times \cdots \times X_d$ and Y. We put $d(x,y)^\rho := d_1(x_1,y_1)^{\rho_1} \vee \ldots \vee d_d(x_d,y_d)^{\rho_d}$ with the convention $0^0 = 0$ (See the following Remark).

(i) Let a be some point in X. We will say that f is $\mathscr{C}^\rho$ **at** a if for every $\varepsilon > 0$, there is a neighborhood $U \ni a$ such that

$$d(f(x), f(y)) \leq \varepsilon \cdot d(x,y)^\rho \quad \text{for all } x, y \in U.$$

We will say that f is a $\mathscr{C}^\rho$**-function** if f is $\mathscr{C}^\rho$ at all points $a \in X$. The set of all $\mathscr{C}^\rho$-functions $f\colon X \to Y$ will be denoted by $\mathscr{C}^\rho(X,Y)$.

(ii) We define $|f^{]\rho[}|\colon \nabla X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\rho[}|(x,y) := \frac{\|f(x) - f(y)\|}{d(x,y)^\rho}.$$

Remark.

(i) In case $\rho_k = 0$ for some $k \in \{1,\ldots,d\}$, the scurrilous convention $0^0 = 0$ ensures that in a neighborhood of the point $a \in X$, the condition $d(f(x), f(y)) \leq \varepsilon \cdot d(x,y)^\rho$ is still stronger than the mere continuity condition $d(f(x), f(y)) \leq \varepsilon$, which were in place if we had adopted the common convention $0^0 = 1$.

(ii) Keeping the above notations, we see that if $f\colon X \to Y$ is $\mathscr{C}^\rho$ at $a \in X$, then it will be $\mathscr{C}^{\tilde{\rho}}$ thereat for any $\tilde{\rho} \leq \rho$ componentwise. In particular if f is $\mathscr{C}^\rho$ at $a \in X$, then it will be $\mathscr{C}^\rho$ thereat with $\rho := \rho_1 \wedge \ldots \wedge \rho_d \in [0,1[$.

The mapping $f\colon X \to \mathbf{E}$ is $\mathscr{C}^\rho$ if and only if the function $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|\colon X \times X \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle X \times X$. We moreover saw above that every $\mathscr{C}^\rho$-function is in particular continuous. We can therefore establish:

Definition. For every compact $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathscr{C}^\rho, C}$ on $\mathscr{C}^\rho(X, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^\rho, C} = \|f_{|C}\|_{\sup} \vee \||(f_{|C})^{[\rho]}|\|_{\sup}.$$

We equip the $\mathbf{K}$-vector space $\mathscr{C}^\rho(X, \mathbf{E})$ with the locally convex topology given by the family of seminorms $\{\|\cdot\|_{\mathscr{C}^\rho, C} : C \subseteq X \text{ compact}\}$.

If X itself is compact, then we will turn $\mathscr{C}^\rho(X, \mathbf{E})$ into a normed $\mathbf{K}$-vector space by endowing it with the norm $\|\cdot\|_{\mathscr{C}^\rho} := \|\cdot\|_{\mathscr{C}^\rho, X}$.

Remark 2.49.

(i) We have an equality of locally convex $\mathbf{K}$-vector spaces $\mathscr{C}^{\vec{\rho}}(\mathrm{X}, \mathbf{E}) = \mathscr{C}^{\rho}(\mathrm{X}, \mathbf{E})$ with $\vec{\rho} = (\rho, \ldots, \rho)$. It holds $\|\cdot\|_{\mathscr{C}^{\vec{\rho}},\mathrm{C}} = \|\cdot\|_{\mathscr{C}^{\rho},\mathrm{C}}$ for any $\mathrm{C} \subseteq \mathrm{X}$ compact.

(ii) The locally convex $\mathbf{K}$-vector space $\mathscr{C}^{\rho}(\mathrm{X}, \mathbf{E})$ is the initial locally convex $\mathbf{K}$-vector space with respect to the inclusion mappings

$$\begin{array}{ccc} & & \mathscr{C}^{\rho_1 \cdot \boldsymbol{e}_1}(\mathrm{X}, \mathbf{E}) \\ & \overset{\text{incl.}}{\nearrow} & \\ \mathscr{C}^{\rho}(\mathrm{X}, \mathbf{E}) & & \vdots \\ & \underset{\text{incl.}}{\searrow} & \\ & & \mathscr{C}^{\rho_d \cdot \boldsymbol{e}_d}(\mathrm{X}, \mathbf{E}). \end{array}$$

It holds $\|\cdot\|_{\mathscr{C}^{\rho},\mathrm{C}} = \|\cdot\|_{\mathscr{C}^{\rho_1 \cdot \boldsymbol{e}_d},\mathrm{C}} \vee \ldots \vee \|\cdot\|_{\mathscr{C}^{\rho_d \cdot \boldsymbol{e}_d},\mathrm{C}}$ for any compact $\mathrm{C} \subseteq \mathrm{X}$.

(iii) The locally convex $\mathbf{K}$-vector space $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ is complete.

The Mahler Base of $\mathscr{C}^{\rho}(\mathbb{Z}_p^d, \mathbf{K})$. Recall that a function f is r-times differentiable if its ν-th difference quotient $f^{]\nu[}$, a function of many variables, is $\mathscr{C}^{\rho}$. Thence, as done in this subsection, we will also need to establish the criterion for a function $f : \mathbb{Z}_p^d \to \mathbf{K}$ to be a $\mathscr{C}^{\rho}$-function in terms of its Mahler coefficients.

We will use the following notational convention: Let $f_1 : \mathrm{X}_1 \to \mathbf{K}, \ldots, f_d : \mathrm{X}_d \to \mathbf{K}$ be functions. Then we denote by $f_1 \odot \cdots \odot f_d : \mathrm{X}_1 \times \cdots \times \mathrm{X}_d \to \mathbf{K}$ the function given by

$$f_1 \odot \cdots \odot f_d(x_1, \ldots, x_d) := f_1(x_1) \cdots f_d(x_d).$$

Definition. Cf. Definition 2.36, we define the $\boldsymbol{i}$-th **Mahler polynomial** $\binom{*}{\boldsymbol{i}} : \mathbb{Z}_p^d \to \mathbf{K}$ for $\boldsymbol{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d$ by $\binom{*}{\boldsymbol{i}} := \binom{*}{i_1} \odot \cdots \odot \binom{*}{i_d}$.

The following Lemma 2.50 and Lemma 2.51 hold for an arbitrary coordinate index $k \in \{1, \ldots, d\}$ but will for notational convenience only be stated and proved for $k = 1$.

Lemma 2.50. *Let $\mathrm{X}_1, \ldots, \mathrm{X}_d \subseteq \mathbf{K}$ be nonempty compact subsets without isolated points. Consider the mapping*

$$\mathscr{C}^{\rho}(\mathrm{X}_1, \mathbf{K}) \times \mathscr{C}^{0}(\mathrm{X}_2, \mathbf{K}) \times \cdots \times \mathscr{C}^{0}(\mathrm{X}_d, \mathbf{K}) \xrightarrow{\Psi} \mathscr{C}^{(\rho, 0, \ldots, 0)}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_d, \mathbf{K}),$$
$$(f_1, \ldots, f_d) \mapsto f := [(x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)].$$

If $\{e_{i_1}\} \subseteq \mathscr{C}^{r_1}(\mathrm{X}_1, \mathbf{K}), \ldots, \{e_{i_d}\} \subseteq \mathscr{C}^{r_d}(\mathrm{X}_d, \mathbf{K})$ are orthogonal families, so $\{e_{i_1} \odot \cdots \odot e_{i_d}\} \subseteq \mathscr{C}^{(\rho, 0, \ldots, 0)}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_d, \mathbf{K})$ will be an orthogonal family with $\|e_{i_1} \odot \cdots \odot e_{i_d}\|_{\mathscr{C}^{(\rho, 0, \ldots, 0)}} = \|e_{i_1}\|_{\mathscr{C}^{r_1}} \cdots \|e_{i_d}\|_{\mathscr{C}^{r_d}}$.

Proof: By direct computation. See [Nag11, Corollary 3.41]. □

Lemma 2.51. *The family $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}^{\rho \cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K})$ is an orthogonal family with $\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_1}} = p^{\rho\, \mathrm{l}(i_1)}$.*

Proof: By Proposition 2.47, the family $\{\binom{*}{i}\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ is in particular an orthogonal family for arbitrary $\rho \in [0, 1[$. Hence we find by Lemma 2.50

$$\{\binom{*}{i_1} \odot \cdots \odot \binom{*}{i_d}\} \subseteq \mathscr{C}^{\rho\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K})$$

to be an orthogonal family with

$$\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}_1}} = \|\binom{*}{i_1}\|_{\mathscr{C}^\rho} \cdot \|\binom{*}{i_d}\|_{\mathscr{C}^0} \cdots \|\binom{*}{i_d}\|_{\mathscr{C}^0} = p^{\rho\cdot \mathrm{l}(i_1)};$$

the last equality by Proposition 2.47. □

Lemma 2.52. *The closure of the set of all polynomial functions inside the* $\mathbf{K}$*-Banach space* $\mathscr{C}^\rho(\mathrm{X}, \mathbf{K})$ *with* $\mathrm{X} = \mathbb{Z}_p^d$ *contains all locally constant functions.*

Proof: The proof is divided into two steps.

(i) Fix an indicator function $\mathbf{1}_\mathrm{B} \colon \mathrm{X} \to \mathbf{K}$ of a closed ball $\mathrm{B} \subseteq \mathrm{X}$ of positive radius and $\varepsilon > 0$. Then there is a polynomial function $p \colon \mathrm{X} \to \mathbf{K}$ such that $\|\mathbf{1}_\mathrm{B} - p\|_{\mathscr{C}^\rho} \leq \varepsilon$.

(ii) The closure of the set of polynomial functions inside $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ contains all locally constant functions.

Ad (i): This is proved by induction on $d \geq 1$. If $d = 1$, then this in particular contained in the statement of Proposition 2.47

Let $d > 1$. Let $\mathrm{B} = \mathrm{B}' \times \mathrm{B}'' \subseteq \mathrm{X}$ with $\mathrm{B}' = \mathrm{B}_1 \times \cdots \times \mathrm{B}_{d-1} \subseteq \mathrm{X}_1 \times \cdots \times \mathrm{X}_{d-1} =: \mathrm{X}'$ and $\mathrm{B}'' := \mathrm{B}_d \subseteq \mathrm{X}_d =: \mathrm{X}''$. By induction, there is a polynomial function $p' \colon \mathrm{X}' \to \mathbf{K}$ with $\|\mathbf{1}_{\mathrm{B}'} - p'\|_{\mathscr{C}^\rho} \cdot \mathrm{M}'' \leq \varepsilon$ with $\mathrm{M}'' = \|\mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^\rho} \geq 0$. Then by the basis case $d = 1$, there is a polynomial function $p'' \colon \mathrm{X}'' \to \mathbf{K}$ with $\|\mathbf{1}_{\mathrm{B}''} - p''\|_{\mathscr{C}^\rho} \cdot \mathrm{M}' \leq \varepsilon$ with $\mathrm{M}' = \|p'\|_{\mathscr{C}^\rho} \geq 0$. We put $p := p' \odot p'' \colon \mathrm{X} \to \mathbf{K}$ and compute

$$\begin{aligned}
\|\mathbf{1}_\mathrm{B} - p\|_{\mathscr{C}^\rho} &= \|\mathbf{1}_{\mathrm{B}'} \odot \mathbf{1}_{\mathrm{B}''} - p' \odot p''\|_{\mathscr{C}^\rho} \\
&\leq \|\mathbf{1}_{\mathrm{B}'} \odot \mathbf{1}_{\mathrm{B}''} - p' \odot \mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^\rho} \vee \|p' \odot \mathbf{1}_{\mathrm{B}''} - p' \odot p''\|_{\mathscr{C}^\rho} \\
&= \|(\mathbf{1}_{\mathrm{B}'} - p') \odot \mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^\rho} \vee \|p' \odot (\mathbf{1}_{\mathrm{B}''} - p'')\|_{\mathscr{C}^\rho} \\
&= \quad (\|(\mathbf{1}_{\mathrm{B}'} - p') \odot \mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}'}} \vee \|(\mathbf{1}_{\mathrm{B}'} - p') \odot \mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}''}}) \\
&\quad \vee (\|p' \odot (\mathbf{1}_{\mathrm{B}''} - p'')\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}'}} \vee \|p' \odot (\mathbf{1}_{\mathrm{B}''} - p'')\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}''}}) \\
&= \quad (\|\mathbf{1}_{\mathrm{B}'} - p'\|_{\mathscr{C}^\rho} \cdot \|\mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^0} \vee \|\mathbf{1}_{\mathrm{B}'} - p'\|_{\mathscr{C}^0} \cdot \|\mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^\rho}) \\
&\quad \vee (\|p'\|_{\mathscr{C}^\rho} \cdot \|\mathbf{1}_{\mathrm{B}''} - p''\|_{\mathscr{C}^0} \vee \|p'\|_{\mathscr{C}^0} \cdot \|\mathbf{1}_{\mathrm{B}''} - p''\|_{\mathscr{C}^\rho}) \\
&\leq \|\mathbf{1}_{\mathrm{B}'} - p'\|_{\mathscr{C}^\rho} \cdot \|\mathbf{1}_{\mathrm{B}''}\|_{\mathscr{C}^\rho} \vee \|p'\|_{\mathscr{C}^\rho} \cdot \|\mathbf{1}_{\mathrm{B}''} - p''\|_{\mathscr{C}^\rho} \leq \varepsilon;
\end{aligned}$$

here $\boldsymbol{e}'$ respectively $\boldsymbol{e}''$ corresponding to the coordinates of X' respectively X'' in the cartesian product of metric spaces $\mathrm{X} = \mathrm{X}' \times \mathrm{X}''$.

Ad (ii): The closed balls $\mathrm{B} \subseteq \mathrm{X}$ constitute a basis of the topological space X. Hence by compactness of X, every locally constant function g is the finite sum $f = \sum_i \lambda_i \mathbf{1}_{\mathrm{B}_i}$ with

$\lambda_i \in \mathbf{K}$ and indicator functions $\mathbf{1}_{B_i}$ of closed balls $B_i \subseteq X$ for $i \in I$. By (i), for every $\varepsilon > 0$, there are polynomial functions $p_i : X \to \mathbf{K}$ such that $\|p_i - \mathbf{1}_{B_i}\|_{\mathscr{C}^\rho} M_i \leq \varepsilon$ with $M_i := |\lambda_i| \geq 0$. Then $p := \sum_i \lambda_i p_i : X \to \mathbf{K}$ satisfies $\|p - f\|_{\mathscr{C}^\rho} \leq \max_i |\lambda_i| \|p_i - \mathbf{1}_{B_i}\|_{\mathscr{C}^\rho} \leq \varepsilon$. □

Corollary 2.53. *The polynomial functions are dense in* $\mathscr{C}^\rho(X, \mathbf{K})$.

Proof: By the previous Lemma 2.52 and Corollary 1.6. □

Lemma 2.54. *Let* W *be the initial* $\mathbf{K}$*-Banach space with respect to finitely many inclusion mappings*

$$\begin{array}{ccccc} & & V_1 & & \\ & \overset{incl.}{\nearrow} & & \overset{incl.}{\searrow} & \\ W & & \vdots & & V \\ & \underset{incl.}{\searrow} & & \underset{incl.}{\nearrow} & \\ & & V_n & & \end{array}$$

for $\mathbf{K}$*-Banach spaces* $V_1, \ldots, V_n$ *and* V*. That is,* $W = V_1 \cap \ldots \cap V_n$ *as an abstract* $\mathbf{K}$*-vector space and its norm* $\|\cdot\|_W$ *on* W *is given by the pointwise maximum* $\|\cdot\|_W = \|\cdot\|_{V_1} \vee \ldots \vee \|\cdot\|_{V_n}$*. If* $\{e_i\} \subseteq W$ *is an orthogonal family of* $V_1, \ldots, V_n$ *and* V*, then* $\{e_i\}$ *will be an orthogonal family of* W*.*

Proof: Let $\{e_i\}$ be orthogonal in $V_1, \ldots, V_n$. We prove $\{e_i\} \subseteq W$ to be orthogonal by the following computation:

$$\begin{aligned} \|\sum_i \lambda_i e_i\|_W &= \|\sum_i \lambda_i e_i\|_{V_1} \vee \ldots \vee \|\sum_i \lambda_i e_i\|_{V_n} \\ &= \max_i |\lambda_i| \|e_i\|_{V_1} \vee \ldots \vee \max_i |\lambda_i| \|e_i\|_{V_n} \\ &= \max_i |\lambda_i| \|e_i\|_W. \end{aligned}$$

□

Proposition 2.55. *The family* $\{\binom{*}{i}\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal basis with* $\|\binom{*}{i}\|_{\mathscr{C}^\rho} = p^{\rho \cdot [l(i_1) \vee \ldots \vee l(i_d)]}$.

Proof: By Lemma 2.51 for $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$, we find $\{\binom{*}{i}\}$ to be an orthogonal family of the $\mathbf{K}$-Banach spaces $\mathscr{C}^{\rho \cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}), \ldots, \mathscr{C}^{\rho \cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K})$ and for $\rho = 0$ one of $\mathscr{C}^0(\mathbb{Z}_p^d, \mathbf{K})$. Consequently by Remark 2.49(i),(ii) and Lemma 2.54, we find $\{\binom{*}{i}\} \subseteq \mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with

$$\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^r} = \left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_1}} \vee \ldots \vee \left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_d}} = p^{\rho \cdot [l(i_1) \vee \ldots \vee l(i_d)]};$$

the last equality by Proposition 2.47. By [Sch84, Exercise 50.F], an orthogonal family whose $\mathbf{K}$-linear span is dense is an orthogonal base. It thus remains to show that the $\mathbf{K}$-linear span of $\{\binom{*}{i}\}$ is dense in $\mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$.

As this span consists of all polynomial functions on $\mathbb{Z}_p^d$ and these are by the previous Corollary 2.53 dense inside $\mathscr{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$. □

The Mahler Base of $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$. We now have all necessary tools at hand to finally compute the Mahler coefficients of a $\mathscr{C}^r$-function $f\colon \mathbb{Z}_p \to \mathbf{K}$.

Definition 2.56. Let $f\colon \mathbb{Z}_p \to \mathbf{K}$ be an arbitrary mapping. Then we define its ***n*-th Mahler coefficient** a_n for $n \in \mathbb{N}$ by

$$a_n = \Delta^{\circ n} f(0);$$

here we refer to Lemma 2.44 for the definition of the $\mathbf{K}$-linear endomorphism Δ on $\mathbf{K}^{\mathbb{Z}_p}$.

Lemma 2.57. *Let $f\colon \mathbb{Z}_p \to \mathbf{K}$ and $a_0, a_1, \ldots$ its Mahler Coefficients. For $x_1, \ldots, x_\nu \in \mathbb{Z}_{\geq 1}$ and $y \in \mathbb{Z}_{\geq 0}$, put $z = (x_1 + \cdots + x_\nu + y, \ldots, x_1 + y, y) \in \nabla\mathbb{Z}_{\geq 0}^{\nu+1}$. Then*

$$f^{]\nu[}(z) = \sum_{\substack{j\geq 0\\ m_1,\ldots,m_\nu \geq 1}} \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1})\cdots(m_\nu + \cdots + m_1)} \binom{x_1-1}{m_1-1}\cdots\binom{x_\nu-1}{m_\nu-1}\binom{y}{j}.$$

Proof: By induction on $\nu \geq 0$. □

Lemma 2.58. *Let $f\colon \mathbb{Z}_p \to \mathbf{K}$. Then*

(i) $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ *if and only if* $|b_{\boldsymbol{m}}| p^{[l(m_1-1)\vee\ldots\vee l(m_\nu-1)\vee l(j)]\cdot\rho} \to 0$ *as* $|\boldsymbol{m}| \to \infty$,

(ii) $\|f^{[\nu]}\|_{\mathscr{C}^\rho} = \max_{\boldsymbol{m}\geq 0} |b_{\boldsymbol{m}}| p^{[l(m_1-1)\vee\ldots\vee l(m_\nu-1)\vee l(j)]\cdot\rho}$;

here $b_{\boldsymbol{m}} \in \mathbf{K}$ for $\boldsymbol{m} = (m_1 - 1, \ldots, m_\nu - 1, j) \in \mathbb{N}^\nu \times \mathbb{N}$, is given by

$$b_{(m_1-1,\ldots,m_\nu-1,j)} = \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1})\cdots(m_\nu + \cdots + m_1)}.$$

Proof: Ad (i): On $\mathbb{Z}_p^{\nu+1}$, consider the bijection φ defined by

$$(x_1, \ldots, x_\nu, y) \mapsto ((x_1 + 1) + \cdots + (x_\nu + 1) + y, \cdots, (x_1 + 1) + y, y).$$

We denote likewise its restriction onto the preimage of $\nabla\mathbb{Z}_p^{\nu+1}$. By Lemma 2.57, for $z = (x_1 + \cdots + x_\nu + y, \ldots, x_1 + y, y) \in \nabla\mathbb{Z}_{\geq 0}^{\nu+1}$, we find

$$f^{]\nu[}(z) = \sum_{\substack{j\geq 0,\\ m_1,\ldots,m_\nu \geq 1}} \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1})\cdots(m_\nu + \cdots + m_1)} \binom{x_1-1}{m_1-1}\cdots\binom{x_\nu-1}{m_\nu-1}\binom{y}{j}.$$

Therefore the $\boldsymbol{m} = (m_1 - 1, \ldots, m_\nu - 1, j)$-th coefficient $b_{\boldsymbol{m}}$ of $\widetilde{f^{]\nu[}} := f^{]\nu[} \circ \varphi$ is given by

$$b_{(m_1-1,\ldots,m_\nu-1,j)} = \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1})\cdots(m_\nu + \cdots + m_1)}.$$

By Proposition 2.55, we find that

$$\widetilde{f^{]\nu[}} \text{ extends to } \widetilde{f^{[\nu]}} \in \mathscr{C}^\rho(\mathbb{Z}_p^{\nu+1}, \mathbf{K}) \quad \text{iff} \quad |b_{\boldsymbol{m}}| p^{[l(m_1)\vee\ldots\vee l(m_\nu)\vee l(j)]\cdot\rho} \to 0 \text{ as } |\boldsymbol{m}| \to \infty.$$

For this note that the above expansion determines $\widetilde{f^{[\nu]}}$ on the dense subset $\mathbb{Z}_{\geq 1}^{\nu} \times \mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}_p^{\nu+1}$ and hence everywhere by continuity. By Proposition 2.4, we find

$$f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) \quad \text{if and only if} \quad f^{]\nu[} \text{ extends to } f^{[\nu]} \in \mathscr{C}^{\rho}(\mathbb{Z}_p^{\nu+1}, \mathbf{K}).$$

Since φ is a locally Lipschitzian automorphism, $f^{]\nu[}$ extends to a $\mathscr{C}^{\rho}$-function $f^{[\nu]}$ on $\mathbb{Z}_p^{\nu+1}$ if and only if $\widetilde{f^{]\nu[}}$ extends to a $\mathscr{C}^{\rho}$-function $\widetilde{f^{[\nu]}} = f^{[\nu]} \circ \varphi$ on $\mathbb{Z}_p^{\nu+1}$. The proposition follows.

Ad (ii): As $\|\widetilde{f^{[\nu]}}\|_{\sup} = \|f^{[\nu]}\|_{\sup}$, it holds $\|f^{[\nu]}\|_{\mathscr{C}^{\rho}} = \max_{\boldsymbol{m} \geq \boldsymbol{0}} |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1) \vee \ldots \vee \mathrm{l}(m_{\nu+1})] \cdot \rho}$. ☐

Lemma 2.59. *Let $f \colon \mathbb{Z}_p \to \mathbf{K}$ and $b_{\boldsymbol{m}}$ with $\boldsymbol{m} = (m_1, \ldots, m_{\nu+1}) \in \mathbb{N}^{\nu+1}$ as in Lemma 2.58. Then*

$$\max_{|\boldsymbol{m}|=m} |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1) \vee \ldots \vee \mathrm{l}(m_{\nu+1})] \cdot \rho} = |a_{m+\nu}| p^{v_r(m+\nu)};$$

here $v_r(n)$ for $n \geq \nu$ is given by

$$\max_{0 \leq l_1 < \ldots < l_\nu \leq n} v(l_1) + \cdots + v(l_\nu) + \rho \max\{\mathrm{l}(l_1 - 1), \mathrm{l}(l_2 - l_1 - 1), \ldots, \mathrm{l}(l_\nu - l_{\nu-1} - 1), \mathrm{l}(n - l_\nu)\}.$$

Proof: Fix $m \geq 0$ and let $\boldsymbol{m} = (m_1 - 1, \ldots, m_\nu - 1, j) \in \mathbb{N}^{\nu+1}$ with $m_1, \ldots, m_\nu \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\geq 0}$ such that $|\boldsymbol{m}| = m$. Then $b_{\boldsymbol{m}}$ is given by

$$b_{\boldsymbol{m}} = \frac{a_{m+\nu}}{m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)}.$$

We find

$$\begin{aligned}
&\max_{|\boldsymbol{m}|=m} |b_{\boldsymbol{m}}| p^{\rho \cdot [\mathrm{l}(m_1 - 1) \vee \ldots \vee \mathrm{l}(m_\nu - 1) \vee \mathrm{l}(j)]} \\
&= \max_{\substack{m_1, \ldots, m_\nu \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\geq 0} \\ \text{with } m_1 + \cdots + m_\nu + j = m + \nu}} \frac{|a_{m+\nu}|}{|m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)|} p^{\rho \cdot [\mathrm{l}(m_1 - 1) \vee \ldots \vee \mathrm{l}(m_\nu - 1) \vee \mathrm{l}(j)]} \\
&= \max_{0 \leq l_1 < \ldots < l_\nu \leq m+\nu} \frac{|a_{m+\nu}|}{|l_1 \cdots l_\nu|} p^{\rho \cdot [\mathrm{l}(l_1) \vee \mathrm{l}(l_2 - l_1) \vee \ldots \vee \mathrm{l}(l_\nu - l_{\nu-1}) \vee \mathrm{l}(m - l_\nu)]};
\end{aligned}$$

the last equality by the bijection $l_n \mapsto m_1 + \cdots + m_n$ for $n = 1, \ldots, \nu$ and noting that given $i \geq 0$, there is $j \geq 0$ with $i + j = m$ if and only if $i \leq m$ and unique $j = m - i$. ☐

Corollary 2.60. *Let $f \colon \mathbb{Z}_p \to \mathbf{K}$. Then*

(i) $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ *if and only if* $|a_m| p^{v_r(m)} \to 0$ *as* $m \to \infty$,

(ii) $\|f\|_{\mathscr{C}^r} = |a_0| \vee |a_1/1!| \vee \ldots \vee |a_{\nu-1}/(\nu-1)!| \vee \max_{m \geq \nu} |a_m| p^{v_r(m)}$.

Proof: Ad (i): By definition,

$$|b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1) \vee \ldots \vee \mathrm{l}(m_{\nu+1})] \cdot \rho} \to 0 \quad \text{as } |\boldsymbol{m}| \to \infty$$

if and only if

$$\max_{|\boldsymbol{m}|=m} |b_{\boldsymbol{m}}|p^{[\mathrm{l}(m_1)\vee\ldots\vee\mathrm{l}(m_{\nu+1})]\cdot\rho} = |a_{m+\nu}|p^{v_r(m+\nu)} \to 0 \quad \text{as } m \to \infty.$$

Ad (ii): Let $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K}) \subseteq \mathscr{C}^n(\mathbb{Z}_p, \mathbf{K})$ for $n = 0, \ldots, \nu - 1$, the inclusion by Lemma 2.3. Then Lemma 2.58 for $r = n + \rho$ with $n = 0, \ldots, \nu - 1$ and $\rho = 0$ yields $\|f^{[\nu]}\|_{\sup} = \max_{m \geq n} |a_m| p^{v_n(m)}$ with $v_n(m)$ defined by $v_n(m) = \max_{0 \leq l_1 < \ldots < l_\nu \leq m} v(l_1) + \cdots + v(l_\nu)$, whereas for $n = \nu$ and ρ, Lemma 2.58 yields $\|f^{[\nu]}\|_{\mathscr{C}^\rho} = \max_{m \geq \nu} |a_m| p^{v_r(m)}$. We observe that if $n \leq n' \leq m$, then by definition $v_n(m) \leq v_{n'}(m)$. We thus obtain

$$\begin{aligned}
\|f\|_{\mathscr{C}^r} &= \|f\|_{\sup} \vee \|f^{[1]}\|_{\sup} \vee \ldots \vee \|f^{[\nu-1]}\|_{\sup} \vee \|f^{[\nu]}\|_{\mathscr{C}^\rho} \\
&= \max_{m \geq 0} |a_m| \vee \max_{m \geq 0} |a_{m+1}| p^{v_1(m)} \vee \ldots \vee \max_{m \geq 0} |a_{m+\nu-1}| p^{v_{\nu-1}(m)} \vee \max_{m \geq 0} |a_{m+\nu}| p^{v_r(m)} \\
&= |a_0| \vee |a_1| p^{v_1(1)} \vee \ldots \vee |a_{\nu-1}| p^{v_{\nu-1}(\nu-1)} \vee \max_{m \geq 0} |a_{m+\nu}| p^{v_r(m)}.
\end{aligned}$$

We can therefore conclude by

$$v_n(n) = \max_{0 \leq l_1 < \ldots < l_n \leq n} v(l_1) + \cdots + v(l_n) = v(1) + \ldots + v(n) = v(n!).$$

Theorem 2.61. *The family* $\{\binom{*}{0}, \binom{*}{1}, \ldots\} \subseteq \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ *constitutes an orthogonal basis of* $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ *with* $\|\binom{*}{m}\|_{\mathscr{C}^r} = p^{w_r(m)}$*; here*

$$w_r(m) = \begin{cases} v(m!), & \text{if } m < \nu, \\ v_r(m), & \text{otherwise.} \end{cases}$$

Proof: By Corollary 2.60(ii) applied to the mapping $e_m = \binom{*}{m}$, we find $\|e_m\|_{\mathscr{C}^r} = p^{w_r(m)}$. Moreover by the same token, if $f = \sum_{m \geq 0} a_m \binom{*}{m} \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$, then

$$\|f\|_{\mathscr{C}^r} = \max_{m \geq 0} |a_m| p^{w_r(m)} = \max_{m \geq 0} |a_m| \|e_m\|_{\mathscr{C}^r}.$$

In other words, $\{\binom{*}{m}\} \subseteq \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ is an orthogonal family with $\|\binom{*}{m}\|_{\mathscr{C}^r} = p^{w_r(m)}$. Since $w_r(m) = v_r(m)$ for $m \geq \nu$, we find

$$|a_m| p^{v_r(m)} \to 0 \text{ as } m \to \infty \quad \text{if and only if} \quad |a_m| \left\| \binom{*}{m} \right\|_{\mathscr{C}^r} \to 0 \text{ as } m \to \infty.$$

By Corollary 2.60(i), we see $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ if and only if $|a_m| \|\binom{*}{m}\|_{\mathscr{C}^r} \to 0$ as $m \to \infty$. That is, $\{\binom{*}{m}\}$ is an orthogonal basis of $\mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$. □

Lemma 2.62. *For* $m \geq \nu$ *with* $\nu \geq 1$ *holds*

$$v_\nu(m) := \max_{0 \leq l_1 < \ldots < l_\nu \leq m} v(l_1) + \cdots + v(l_\nu) = \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu).$$

Proof: Let $\mathrm{L} \subset \{1, \ldots, m\}$ with $\#\mathrm{L} = \nu$ and $b := \mathrm{l}(m)$, the maximal exponent e such that $p^e \leq m$. Then

$$v(l_1) + \cdots + v(l_\nu) = \#\{l \in \mathrm{L} : v(l) \geq 1\} + \cdots + \#\{l \in \mathrm{L} : v(l) \geq b\}.$$

Let $a = \mathrm{l}(m/\nu)$ be the maximal exponent e such that $p^a \nu \leq m$. Then the subsets $\mathrm{L} \subset \{1, \ldots, m\}$ with $\#\mathrm{L} = \nu$ for which the right-hand side above is maximal are precisely those with

$$\{x \leq m : v(x) > a\} \subseteq \mathrm{L} \subseteq \{x \leq m : v(x) \geq a\}.$$

For such $\mathrm{L} = \{l_1 < \ldots < l_\nu\}$, we find

$$\begin{aligned} & v(l_1) + \cdots + v(l_\nu) \\ = & a \cdot \nu + \#\{l = 1, \ldots, m : v(l) \geq a + 1\} + \cdots + \#\{l = 1, \ldots, m : v(l) \geq b\}. \end{aligned}$$

We likewise add together

$$\begin{aligned} & \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu) \\ = & \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq 1\} + \cdots + \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq b\} \\ = & \ \nu \cdot a + \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq a + 1\} \\ & + \cdots + \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq b\}. \end{aligned}$$

Observe that $\mathrm{l}(m/k) > a = \mathrm{l}(m/\nu)$ implies in particular $k \in \{1, \ldots, \nu\}$. Hence for $c \in \mathbb{Z}_{>a}$, we find $\#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq c\} = \#\{k = 1, \ldots, \nu : \mathrm{l}(m/k) \geq c\}$. To obtain the proposed equality, we are thus reduced to: For any $h \in \mathbb{Z}_{>a}$,

$$\#\{l = 1, \ldots, m : v(l) \geq h\} = \#\{k = 1, \ldots, m : \mathrm{l}(m/k) \geq h\}.$$

The left-hand side is the number of elements $l \leq m$ divisible by p^h. Since $\mathrm{l}(x) \geq h$ if and only if $x \geq p^h$, the right-hand side equals the number of elements $k \leq m$ with $m \geq p^h k$. This is also the number of elements below m divisible by p^h. □

Lemma 2.63. *For $m \geq \nu$, we find*

$$v_r(m) = \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu) + \rho \cdot \begin{cases} \mathrm{l}(m/\nu), & \text{if } q(\nu+1) \leq m, \\ \mathrm{l}(m/\nu) - 1, & \text{otherwise;} \end{cases}$$

here $q = \max\{x : x = p^h \text{ for some } h \in \mathbb{N} \text{ and } x\nu \leq m\}$.

Proof: For $0 \leq l_1 < \ldots \leq l_\nu \leq m$ with $m \geq \nu$, let $\breve{w} := v(l_1) + \cdots + v(l_\nu)$ and

$$w = \breve{w} + \rho \cdot [\mathrm{l}(k_1 - 1) + \cdots + \mathrm{l}(k_\nu - 1) + \mathrm{l}(m - l_\nu)]$$

with $k_1 := l_1, k_2 := l_2 - l_1, \ldots, k_\nu := l_\nu - l_{\nu-1}$.

Let $\mathrm{L} = \{0 \leq l_1 < \ldots < l_\nu \leq m\}$ be such that $\breve{w} = v_\nu(m)$ is maximal. Let $q = p^a$ be the maximal p-power such that $q\nu \leq m$. Then $\{x \leq m : v(x) > a\} \subseteq \mathrm{L} \subseteq \{x \leq m : v(x) \geq a\}$.

If and only if $q\nu \le m - q$, we can find an index $n \in \{1, \ldots, \nu\}$ with $k_n > q$. In this case, we can assume $l_\nu = q\nu$ and $m - l_\nu \ge q$. Therefore

$$\tilde{a} := \mathrm{l}(k_1 - 1) \vee \ldots \vee \mathrm{l}(k_\nu - 1) \vee \mathrm{l}(m - l_\nu) = \begin{cases} a, & \text{if } q(\nu + 1) \le m, \\ a - 1, & \text{otherwise.} \end{cases}$$

We prove that if $l_1, \ldots, l_\nu$ are such that $w(l_1, \ldots, l_\nu)$ is maximal for all possible $\{0 \le l_1 < \ldots < l_\nu \le m\}$, so will be $\breve{w}(l_1, \ldots, l_\nu)$. As $a = \mathrm{l}(m/\nu)$ this will by the above consideration prove the proposition.

Let $0 \le l_1 < \ldots < l_\nu \le m$. As $\rho < 1$, it suffices to prove that $\mathrm{l}(k_n - 1) = \tilde{a} + c$ for some $n \in \{1, \ldots, \nu\}$ or $\mathrm{l}(m - l_\nu) = \tilde{a} + c$ for some $c > 0$ implies $\breve{w}(l_1, \ldots, l_\nu) + c \le v_\nu(m)$.

Let us define $k_{\nu+1} := m - l_\nu$, so that $k_1 + \cdots + k_{\nu+1} = m$. Since $\nu \cdot q \le m < \nu \cdot qp$, there must by the pigeonhole principle exist $s := p^c - p + 1$ indices $n_1, \ldots, n_s$ with $k_{n_1}, \ldots, k_{n_s} < q$. Hence $v(k_{n_1}), \ldots, v(k_{n_s}) < a$. For $n = 1, \ldots, \nu$, if $v(k_n) < a$, either $v(l_n) < a$ or $v(l_{n+1}) = v(l_n + k_n) < a$. Hence there must exist $\lceil s/2 \rceil := \min\{i \in \mathbb{N} : i \ge s/2\} \ge c$ elements $l_n \in \{l_1, \ldots, l_\nu\}$ with $v(l_n) < a$. But if $\breve{w}(\mathrm{L}) = v_\nu(m)$ is maximal, then $\{x \le m : v(x) > a\} \subseteq \mathrm{L} \subseteq \{x \le m : v(x) \ge a\}$. Therefore $\breve{w}(l_1, \ldots, l_\nu) \le v_\nu(m) - c$. □

Lemma 2.64. *There are positive constants $c \le 1 \le \mathrm{C}$ with $c \cdot m^r \le p^{w_r(m)} \le \mathrm{C} \cdot m^r$.*

Proof: By Lemma 2.63, up to a possible deduction of the constant $\rho > 0$ holds

$$v_r(m) = \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu) + \rho\, \mathrm{l}(m/\nu).$$

As $\mathrm{l}(xy) \le \mathrm{l}(x) + \mathrm{l}(y) + 1$ implies $\mathrm{l}(x) - \mathrm{l}(y) - 1 \le \mathrm{l}(x/y)$, we find accordingly

$$r \cdot \mathrm{l}(m) - \tilde{c} \le v_r(m) \le r \cdot \mathrm{l}(m) \quad \text{with } \tilde{c} := \mathrm{l}(2) + \cdots + \mathrm{l}(\nu) + \rho\, \mathrm{l}(\nu) + r.$$

Since $p^{\mathrm{l}(m)} \le m \le p^{\mathrm{l}(m)+1}$, we find $c \cdot m^r \le p^{v_r(m)} \le m^r$ with $c := 1/p^{(\tilde{c}+\rho)\cdot r} > 0$. Recall that v_r differs from w_r only in the finitely many nonzero values $w_r(m)$ for $m = 0, \ldots, \nu - 1$. Hence we can decrease $c > 0$ and increase $\mathrm{C} := 1$ such that this inequality holds for $w_r(m)$ instead of $v_r(m)$. □

Corollary 2.65. *Let $f : \mathbb{Z}_p \to \mathbf{K}$ and $a_0, a_1, \ldots$ its Mahler coefficients.*

(i) We have $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ if and only if $|a_m| m^r \to 0$ as $m \to \infty$,

(ii) The norm $\|\cdot\|_{\mathscr{C}^r}$ is equivalent to the one given by $|a_0| \vee \max_{m \ge 1} |a_m| m^r \in \mathbb{R}_{\ge 0}$.

Proof: By Corollary 2.60, it suffices to see that there are positive constants $c \le 1 \le \mathrm{C}$ with $c \cdot m^r \le p^{w_r(m)} \le \mathrm{C} \cdot m^r$ and Lemma 2.64 yields the existence of these. □

References

[BB10] L. Berger and C. Breuil, *Sur quelques représentations potentiellement cristallines de* $\mathbf{GL}_2(\mathbf{Q}_p)$, Astérisque **330** (2010), 155–211.

[Col10] P. Colmez, *Fonctions d'une variable p-adique*, Astérisque (2010), no. 330, 13–59. MR 2642404.

[Nag11] E. Nagel, *Fractional non-Archimedean differentiability*, Univ. Münster, Mathematisch-Naturwissenschaftliche Fakultät (Diss.), 2011. zbMATH 1223.26011. Confer `http://nbn-resolving.de/urn:nbn:de:hbz:6-75409405856`.

[Sch78] W. H. Schikhof, *Non-Archimedean calculus*, Report, vol. 7812, Katholieke Universiteit, Mathematisch Instituut, Nijmegen, 1978, Lecture notes. MR 522166.

[Sch84] ______, *Ultrametric calculus*, Cambridge Studies in Advanced Mathematics, vol. 4, Cambridge University Press, Cambridge, 1984, An introduction to p-adic analysis. MR 791759.

[Sch95] ______, *A perfect duality between p-adic Banach spaces and compactoids*, Indag. Math. (N.S.) **6** (1995), no. 3, 325–339. MR 1351151. DOI 10.1016/0019-3577(95)93200-T.

Westfälische Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster

e-mail address: enno.nagel@math.uni-muenster.de