# Fractional non-Archimedean calculus in many variables

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Let $r \geq 0$ be a real number. We will introduce a notion of $r$-fold differentiability for functions of many variables over a non-Archimedeanly valued complete field $\mathbf{K}$ and then examine properties of theirs such as localness, completeness as a locally convex $\mathbf{K}$-algebra, density of (locally) polynomial functions, closure under composition and, for the dual, under convolution.

The definition of a $\mathscr{C}^r$-function will be given through partial difference quotients and build up on the one-variable case already studied in [Nag12]. In line with [BB10], we will also show a function on $\mathbb{Z}_p^d$ to be $r$-times differentiable if and only if its Mahler coefficients obey $|a_{\boldsymbol{n}}||\boldsymbol{n}|^r \to 0$ as $|\boldsymbol{n}| \to \infty$. As a corollary, a characterization of $\mathscr{C}^r$-functions $f \colon \mathrm{X} \to \mathbf{K}$ on open $\mathrm{X} \subseteq \mathbb{Q}_p^d$ by partial Taylor-polynomials is obtained.

## Contents

## Introduction

Let us fix a real number $r \geq 0$. Let $\mathbf{K}$ be a complete non-Archimedeanly nontrivially valued field and $X \subseteq \mathbf{K}$ a subset without isolated points. In [Nag12] we introduced the notion of an $r$-times differentiable function $f\colon X \to \mathbf{K}$, shown to generalize the previous differentiability notions given in [Sch84] (for $r = \nu \in \mathbb{N}$) and [BB10] respectively [Col10] (for $X = \mathbb{Z}_p$). In this article, we will extend this notion by introducing a concept of $r$-fold differentiability in many variables, in particular allowing for a reasonable notion of a finite dimensional $\mathscr{C}^r$-manifold.

To begin with, let us outline the conceptual viewpoint guiding us while establishing our final notion of fractional differentiability: Recall the common notion of differentiability over non-Archimedean vector spaces, in order to distinguish it from the usual one over the real numbers, which is significantly weaker, also often referred to as *strict differentiability*: Start with two normed finite-dimensional vector spaces V and W over a valued field $\mathbf{K}$. Let $f\colon U \to W$ be some mapping defined on an open subset $U \subseteq V$. Then $f$ is called *differentiable* or $\mathscr{C}^1$ in the point $a \in U$ if there is a linear map $D_a\colon V \to W$ such that for every $\varepsilon > 0$ there is a neighborhood $U_\varepsilon \ni a$ in U with

$$\|f(x+h) - f(x) - D_a \cdot h\| \leq \varepsilon \|h\| \quad \text{for all } x+h, x \in U_\varepsilon.$$

Now to iterate this differentiability notion, we opt for a choice of coordinates on the function's domain. We therefore assume $V = \mathbf{K}^d$ and let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ be its canonical basis vectors. Then given any two points $x+h, x \in U$ with $h \in \mathbf{K}^{*d}$, we can define $A := f^{]1[}(x+h, h) \in \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V, W)$ by the partial difference quotients

$$A(h_k \cdot \boldsymbol{e}_k) = f(x + h_1 \cdot \boldsymbol{e}_1 + \cdots + h_k \cdot \boldsymbol{e}_k) - f(x + h_1 \cdot \boldsymbol{e}_1 + \cdots + h_{k-1} \cdot \boldsymbol{e}_{k-1}) \quad \text{for } k = 1, \dots, d.$$

Then this map $f^{]1[}\colon U^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V, W)$ extends to a continuous function $f^{[1]}\colon U^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V, W)$ with $U^{[1]} = U \times U$ if and only if $f$ is $\mathscr{C}^1$ at every point of $a$. (See Remark 1.17.) This function's domain lies again in the $\mathbf{K}$-vector space $V \times V$ inheriting a natural choice of coordinates, its range is in a natural way again a $\mathbf{K}$-vector space, and so we can define $f$ to be *twice continuously differentiable* if

$$f^{]2[} = (f^{[1]})^{]1[}\colon (X^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V \times V, W), W)$$

extends to a continuous function $f^{[2]}$ on all of $X^{[2]} = X^{[1]} \times X^{[1]}$, and we can continue in this manner to arrive at our notion of $\nu$-fold differentiability for any $\nu \in \mathbb{N}$.

To account for the fractional part of the differentiability condition, we now introduce the notion of a $\mathscr{C}^\rho$-point for $\rho \in [0, 1[$ as follows: The mapping $f$ is $\mathscr{C}^\rho$ at the point $a \in U$ if for every $\varepsilon > 0$ there is a neighborhood $U_\varepsilon \ni a$ in U with

$$\|f(x+h) - f(x)\| \leq \varepsilon \|h\|^\rho \quad \text{for all } x+h, x \in U_\varepsilon.$$

(Note that this amounts to a tightened Hölder condition, where we demand the difference quotient to asymptotically vanish at the point $a$ instead of the usual boundedness condition around it.) Now write $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu \in \mathbb{N}$ and $\rho \in [0,1[$. Then for $f$ to be a $\mathscr{C}^r$-*function*, we demand its $\nu$-th iterated difference quotient not merely to extend continuously, but $\mathscr{C}^\rho$-wise at all critical limit points.

Then to arrive at our final Definition 2.2 of a $\mathscr{C}^r$-point, we notice that a mapping symmetric in two arguments is partially differentiable in both arguments if and if only if it is so in any one of them. For example if $\mathrm{V} = \mathbf{K}$ is one-dimensional, its first difference quotient is given by the symmetric mapping $f^{[1]}(x,y) = [f(x) - f(y)]/(x-y)$. If we now define a function $f$ to be twice differentiable if firstly $f^{[1]}$ exists on $\mathrm{U} \times \mathrm{U}$ and then is again differentiable, we are hence brought down to checking partial differentiability solely in $f^{[1]}$'s first coordinate, reducing an exponential growth of parameters to a linear one. This observation underlies the definition of the iterated difference quotients and subsequently iterated differentiability as introduced in [Sch84], which we likewise employ here for our iterated partial difference quotients as defined at the beginning of Section 2.

This conceptual viewpoint of our notion of fractional differentiability will render a lot of the attained results plausible, in particular those concerning the verification of natural properties, albeit their formal proofs can be elaborate. We will now proceed to explain each section in detail:

We firstly introduce in Section 1 the notion of a $\mathscr{C}^\rho$-function for $\rho \in [0,1[$. Since the definition of fractional differentiability will be given through a fixed choice of coordinates, it will also be technically important to have a coordinatewise variant of this notion at hand, the one of a $\mathscr{C}^{\boldsymbol{\rho}}$-function for $\boldsymbol{\rho} \in [0,1[^d$. It will be introduced and studied in the following Section 1

In Section 1 we make precise the more conceptual definition of $1+\rho$-fold differentiability as already introduced at the beginning of this introduction. This point of view will shed light on natural questions such as the one about the composition of $\mathscr{C}^r$-functions, whereas answering these by the initial definitions becomes overly technical and rests unenlightening.

In Section 2 we will then make precise the definition of $r$-fold differentiability. Following the approach by Schikhof and de Smedt, we firstly introduce for a function $f$ its iterated difference partial quotients $f^{]\boldsymbol{n}[}$ for $\boldsymbol{n} \in \mathbb{N}^d$. Their definitions incorporate already the symmetry of the previously introduced differentials $f^{]1[}, f^{]2[}, \ldots$, reducing their number of variables and yielding more precisely that $f^{]\boldsymbol{n}[}$ takes $|\boldsymbol{n}| = n_1 + \cdots + n_d$ variables. (At the same time we observe though that these partial iterated difference quotients remain nevertheless much more complicated than the partial derivatives in the Archimedean case, which at any degree of differentiability only take $d$ arguments. This becomes necessary through the lack of an adequate counterpart to the mean value theorem.) The definition of $r$-fold differentiability for $r = \nu + \rho$ with $\nu \in \mathrm{N}$ and $\rho \in [0,1[$ is then given point-wise by demanding the difference quotients $f^{]\boldsymbol{n}[}$ with $|\boldsymbol{n}| = \nu$ to be $\mathscr{C}^\rho$ at the critical limit points. We will show how to endow the space of $\mathscr{C}^r$-functions with a natural locally convex topology.Thereby and since a $\mathscr{C}^r$-function is defined pointwise, in particular a local notion, we arrive at a reasonable notion of a $\mathscr{C}^r$-manifold.

To explain the main result of Section 2 most succinctly, let us view $\mathscr{D}(\mathrm{X},\mathbf{K}) = \varinjlim \mathscr{D}^r(\mathrm{X},\mathbf{K})$ as the filtered $\mathbf{K}$-vector space of all $\mathbf{K}$-linear forms defined on all arbitrarily often differentiable functions $\mathscr{C}^\infty(\mathrm{X},\mathbf{E})$ which extend continuously onto $\mathscr{C}^r(\mathrm{X},\mathbf{E})$ for some $r \geq 0$. When X is moreover a group with $\mathscr{C}^\infty$-multiplication, we can endow $\mathscr{D}(\mathrm{X}, \mathbf{K})$ with a convolution product. The main result is then that this is indeed a filtered $\mathbf{K}$-algebra with respect to the given multiplication and filtration.

In its most naive way presented above, the condition to be a $\mathscr{C}^r$-function is hardly handy, and can in some instances be simplified: Let $\mathbf{K} \supseteq \mathbb{Q}_p$ as a valued field. Just as mentioned at the beginning for the one-variable case, there is a distinguished orthogonal basis of the continuous $\mathbf{K}$-valued functions $\mathscr{C}^0(\mathbb{Z}_p^d, \mathbf{K})$ relating to the domain's cyclic topological group structure, given by the called *Mahler polynomials* $\binom{*}{\boldsymbol{i}}$ for $\boldsymbol{i} \in \mathbb{N}^d$. In Section 2 we want to establish a sufficient and necessary condition of $r$-fold differentiability for a function $f\colon \mathbb{Z}_p^d \to \mathbf{K}$ regarding its *Mahler coefficients* $(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$ with respect to this basis. To this end, we will describe the space of $\mathscr{C}^r$-functions of several variables as intersection of $\mathscr{C}^{\boldsymbol{r}}$-function spaces for $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$, resembling tensor products of $\mathscr{C}^r$-functions of one variable. By the computation of the Mahler coefficients of functions of one variable in for example, [Nag12, Section 2.3], directly yields the orthogonality of the multivariate Mahler polynomials and their $\mathscr{C}^r$-norms. It rests to prove that the $\mathbf{K}$-linear span of the Mahler-polynomials — which is just the $\mathbf{K}$-vector space of all polynomial functions — is dense in $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$.

For this, we show generally at first all locally polynomial functions of total degree at most $\nu$ and consequently all polynomial functions to constitute dense subspaces of the $\mathbf{K}$-Banach space $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ on any compact domain $\mathrm{X} \subseteq \mathbf{K}^d$.

This finally yields the Mahler polynomials to indeed form an orthogonal basis of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. By their computed $\mathscr{C}^r$-norms, we can conclude the $\mathscr{C}^r$-functions $f\colon \mathbb{Z}_p^d \to \mathbf{K}$ to be describable by its Mahler coefficients obeying $|a_{\boldsymbol{n}}||\boldsymbol{n}|^r \to 0$ as $|\boldsymbol{n}| \to \infty$.

This last condition is equivalent to $|a_{\boldsymbol{n}}|n_k^r$ as $|\boldsymbol{n}| \to \infty$ for $k = 1, \dots, d$, each of these conditions describing the completed tensor product of the space of $\mathscr{C}^r$-functions $f\colon \mathbb{Z}_p \to \mathbf{K}$ in the $k$-th variable with the one of continuous functions in the other ones. By the equivalent description of $r$-fold differentiability in one variable through Taylor polynomials obtained in [Nag12, Section 2.2], we show in Section 2 how to equivalently characterize this tensor product through a function's Taylor polynomial where only its first $\nu$ partial derivatives in the $k$-th variable are considered. As a consequence of these two observations, we conclude to have an equivalent description of $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathbf{K}$ on open domains $\mathrm{X} \subseteq \mathbb{Q}_p^d$ through partial Taylor polynomials.

## Notations and Conventions

Throughout this paper $\mathbf{K}$ will denote a complete non-Archimedeanly valued field whose valuation $v$ is nontrivial. If we fix a positive real constant $c_v < 1$, we obtain a norm $|x| := c_v^{v(x)}$. Define $\mathbf{o}_{<\lambda} = \{x \in \mathbf{K} : |x| < \lambda\}$ respectively $\mathbf{o}_{\leq\lambda} = \{x \in \mathbf{K} : |x| \leq \lambda\}$ for $\lambda \in \mathbb{R}_{\geq 0}$; put $\mathbf{o} = \mathbf{o}_{\leq 1}$ and $\mathbf{k} = \mathbf{o}/\mathbf{o}_{<1}$. If the residue field $\mathbf{k}$ of $\mathbf{K}$ has positive characteristic $p$, we will always put $c_v = p^{-1}$. Then $v(p) > 0$ and if this value is finite, we will assume $v(p) = 1$.

The letter $\mathbf{E}$ will in the following denote a $\mathbf{K}$-Banach space.

## Cartesian products

Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be a finite cartesian product of sets. Then we will call a subset $\mathrm{A} \subseteq \mathrm{X}$ **cartesian** if $\mathrm{A} = \mathrm{A}_1 \times \cdots \times \mathrm{A}_d$ with $\mathrm{A}_1 \subseteq \mathrm{X}_1, \ldots, \mathrm{A}_d \subseteq \mathrm{X}_d$. If $\mathrm{X}_1, \ldots, \mathrm{X}_d \ni \{0, 1\}$, we will denote by $\boldsymbol{e}_k$ the tuple whose sole nonzero entry is 1 at the $k$-th place.

Let $\mathrm{A} \subseteq \mathrm{X}^{\mathrm{I}}$ for a set X and an index set I. We will denote by $\triangle\mathrm{A}$ the diagonal subset

$$\triangle\mathrm{A} = \{(x, \ldots, x) \in \mathrm{A} : x \in \mathrm{X}\}$$

and by $\triangledown\mathrm{A}$ its subset of tuples with pairwise distinct coordinates

$$\triangledown\mathrm{A} = \{(x_i)_{i \in \mathrm{I}} \in \mathrm{A} : x_{i'} \neq x_{i''} \text{ if } i', i'' \in \mathrm{I} \text{ distinct }\}.$$

If $d = 1$, then $\triangle\mathrm{A} = \triangledown\mathrm{A} = \mathrm{A}$.

## Metric and normed spaces

We will throughout assume all seminorms to be non-Archimedean. All normed respectively metric spaces are implicitly assumed to be endowed with a norm $\|\cdot\|$ respectively metric d, through whose arguments it will be clear whereon it is defined. Every normed space gives rise to a metric $\mathrm{d}(x, y) := \|x - y\|$.

Let the set $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be the cartesian product of normed respectively metric spaces $\mathrm{X}_1, \ldots, \mathrm{X}_d$ with correspondingly indexed norms respectively metrics. Then we endow X with the structure of a normed respectively metric space through the norm

$$\|x\| = \max\{\|x_1\|_1, \ldots, \|x_d\|_d\}$$

respectively metric

$$\mathrm{d}(x, y) = \max\{\mathrm{d}_1(x_1, y_1), \ldots, \mathrm{d}_d(x_d, y_d)\}.$$

We will then call X a **cartesian** normed respectively metric space.

If X is an arbitrary set and Y a normed space, we define a *quasinorm* $\|\cdot\|_{\sup}$ (a map with image in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying all axioms of a norm) on the mappings $f : \mathrm{X} \to \mathrm{Y}$ by

$$\|f\|_{\sup} = \begin{cases} \sup_{x \in \mathrm{X}} \|f(x)\|, & \text{if this supremum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

For a subset $\mathrm{A} \subseteq \mathrm{X}$, we define $\|f\|_{\mathrm{A}} := \|f_{|\mathrm{A}}\|_{\sup}$.

Let X be a metric space. Then for a subset $\mathrm{A} \subset \mathrm{X}$, we define its *diameter* by $\operatorname{dia}\mathrm{A} := \sup\{\mathrm{d}(x, y) : x, y \in \mathrm{A}\}$. If $\varepsilon \geq 0$ and $x_0 \in \mathrm{X}$, we define the *ball of radius* $\varepsilon$ *around* $x_0$ by $\mathrm{B}_{\leq \varepsilon}(x_0) := \{x \in \mathrm{X} : \mathrm{d}(x_0, x) \leq \varepsilon\}$.

## Notational conventions

- We denote by $\mathbb{N} = \{0, 1, 2, \cdots\}$ the set of nonnegative integers.

- For a multi-index $\boldsymbol{n} \in \mathbb{N}^d$, we put $|\boldsymbol{n}| = n_1 + \ldots + n_d$. We define for $\nu \in \mathbb{N}$ finite sets of multi-indices
$$\mathbb{N}^d_{=\nu} = \{\boldsymbol{n} \in \mathbb{N}^d : |\boldsymbol{n}| = \nu\}$$
and accordingly $\mathbb{N}^d_{<\nu}$ or $\mathbb{N}^d_{\leq\nu}$ by replacing $=$ with $\leq$ or $<$. For multi-indices $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^d$, we define their natural partial ordering by
$$\boldsymbol{i} \leq \boldsymbol{j} \quad \text{if} \quad i_1 \leq j_1, \ldots, i_d \leq j_d.$$

- We might abbreviate $\min\{a, b\}$ respectively $\max\{a, b\}$ for two real numbers $a$ and $b$ by the associative logical conjunction respectively disjunction operator $a \wedge b$ respectively $a \vee b$.

# 1 Basic definitions and observations

## $\mathscr{C}^\rho$-functions for $\rho \in [0, 1[$

*Assumption.* Throughout this subsection, we will fix a real number $\rho \in [0, 1[$.

In this section, we briefly introduce the space of $\mathscr{C}^\rho$-functions for $\rho \in [0, 1[$ and collect their most basic properties.

*Definition of $\mathscr{C}^\rho$-functions.*

**Definition.** Let X be a metric space, Y a complete metric space, $f : \mathrm{A} \to \mathrm{Y}$ a mapping defined on a subset $\mathrm{A} \subseteq \mathrm{X}$ and $a$ some point in X; we will say that $f$ is $\mathscr{C}^\rho$ **at** $a$, if for every $\varepsilon > 0$ there is a neighborhood $\mathrm{U} \ni a$ in X such that
$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x, y)^\rho \quad \text{for all } x, y \in \mathrm{U} \cap \mathrm{A}.$$
Then $f$ will be a $\mathscr{C}^\rho$**-function** if $f$ is $\mathscr{C}^\rho$ at all points $a \in \mathrm{A}$, where we note that this notion is independent of the ambient space X. We will denote the set of all $\mathscr{C}^\rho$-functions $f : \mathrm{A} \to \mathrm{Y}$ by $\mathscr{C}^\rho(\mathrm{A}, \mathrm{Y})$.

*Remark* 1.1. Keeping the notations above, let us assume that $a \in \mathrm{X}$ is a boundary point in $\partial\mathrm{A} = \bar{\mathrm{A}} - \mathrm{A} \subseteq \mathrm{X}$. Then by completeness of Y, a function $f$ is $\mathscr{C}^0$ at $a$ if and only if there is a *unique* limit $f(a) \in \mathrm{Y}$ such that for every $\varepsilon > 0$, there is a neighborhood $\mathrm{U} \ni a$ in X such that
$$\mathrm{d}(f(x), f(a)) \leq \varepsilon \quad \text{for all } x \in \mathrm{U} \cap \mathrm{A}.$$
If even $a \in \mathrm{A}$, then a function $f : \mathrm{A} \to \mathrm{Y}$ will be $\mathscr{C}^0$ at $a$ if and only if it will be continuous at $a$.

**Proposition 1.2.** *Let* X *be a metric space,* Y *a complete metric space and* $f : \mathrm{A} \to \mathrm{Y}$ *a* $\mathscr{C}^\rho$*-function defined on subset* $\mathrm{A} \subseteq \mathrm{X}$. *Let* $\mathrm{A} \subseteq \mathrm{B} \subseteq \bar{\mathrm{A}} \subseteq \mathrm{X}$ *denote the* $\mathscr{C}^\rho$*-points of* $f$. *Then* $f$ *extends uniquely to a* $\mathscr{C}^\rho$*-function* $\mathrm{F} : \mathrm{B} \to \mathrm{Y}$.

*Proof:* See [Nag12, Proposition 1.3] □

*The locally convex topology on $\mathscr{C}^\rho$-functions.*

**Definition.** Let X be a metric space and $f\colon X \to \mathbf{E}$ a mapping thereon. We define $|f^{]\rho[}|\colon \nabla X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\rho[}|(x, y) = \frac{\|f(x) - f(y)\|}{d(x, y)^\rho}.$$

Then the mapping $f\colon X \to E$ is $\mathscr{C}^\rho$ if and only if the function $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|\colon X \times X \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle X \times X$. Therefore the following definition is meaningful.

**Definition.** For every compact $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathscr{C}^\rho, C}$ on $\mathscr{C}^\rho(X, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^\rho, C} = \|f_{|C}\|_{\sup} \vee \||(f_{|C})^{[\rho]}|\|_{\sup}.$$

We equip the $\mathbf{K}$-vector space $\mathscr{C}^\rho(X, \mathbf{E})$ with the locally convex topology given by the set of seminorms $\{\|\cdot\|_{\mathscr{C}^\rho, C} : C \subseteq X \text{ compact}\}$.

If X itself is compact, then we will turn $\mathscr{C}^\rho(X, \mathbf{E})$ into a normed $\mathbf{K}$-vector space by endowing it with the norm $\|\cdot\|_{\mathscr{C}^\rho} := \|\cdot\|_{\mathscr{C}^\rho, X}$.

*Remark.* The locally convex $\mathbf{K}$-vector space $\mathscr{C}^\rho(X, \mathbf{E})$ is complete.

*Properties of the space of $\mathscr{C}^\rho$-functions.*

**Definition.** Let X and Y be metric spaces, $f\colon X \to Y$ a mapping on X and $a$ some point in X; we will say that $f$ **is $\mathscr{C}^{\text{lip}}$** or is **locally Lipschitzian** at $a$ if there is a constant $C > 0$ and a neighborhood $U \ni a$ such that

$$d(f(x), f(y)) \leq C \cdot d(x, y) \text{ for all } x, y \in U.$$

Then $f$ will be a **$\mathscr{C}^{\text{lip}}$-function** or **a locally Lipschitzian function** if $f$ is $\mathscr{C}^{\text{lip}}$ at all points $a \in X$. We will denote the set of all $\mathscr{C}^{\text{lip}}$-functions $f\colon X \to Y$ by $\mathscr{C}^{\text{lip}}(X, Y)$.

**Proposition 1.3.** *Let X, Y and Z be metric spaces. Then the $\mathscr{C}^\rho$-functions are closed under composition with locally Lipschitzian functions, that is, if $g\colon X \to Y$ and $f\colon Y \to Z$, then if one of these functions will be $\mathscr{C}^\rho$ and the other one $\mathscr{C}^{\text{lip}}$, then $f \circ g \in \mathscr{C}^\rho(X, Z)$.*

*Proof:* By definition. See [Nag11, Propositon 1.7]. □

**Definition.** Let X be a metric spaces and Y a set; a mapping $g\colon X \to Y$ will be called **$\delta$-constant** if $d(x, y) \leq \delta$ implies $g(x) = g(y)$.

**Lemma 1.4.** *Let X be a metric space and $f\colon X \to \mathbf{E}$ a mapping such that for fixed $\varepsilon > 0$, there is $0 < \delta \leq 1$ such that $d(x, y) \leq \delta$ implies $\|f(x) - f(y)\| \leq \varepsilon \cdot d(x, y)^\rho$ for all $x, y \in X$. Then there is a $\delta$-constant function $g\colon X \to \mathbf{E}$ with $\|f - g\|_{\mathscr{C}^\rho, C} \leq \varepsilon$ for all $C \subseteq X$ compact.*

*Proof:* This is proved more generally later on, see Lemma 1.9. For a direct proof, see [Nag11, Lemma 1.11]. □

## $\mathscr{C}^{\boldsymbol{\rho}}$-functions for $\boldsymbol{\rho} \in [0,1[^d$

In this section, we briefly introduce the space of $\mathscr{C}^{\boldsymbol{\rho}}$-functions for $\boldsymbol{\rho} \in [0,1[^d$ and collect their most basic properties. This coordinate-wise generalization of the previously introduced space of $\mathscr{C}^{\rho}$-functions will come into play when we examine later on iteratively differentiable functions of many variables whose definition depends on a fixed choice of coordinates.

*Assumption.* Throughout this subsection, we will fix a tuple of real numbers $\boldsymbol{\rho} \in [0,1[^d$.

*Definition of $\mathscr{C}^{\boldsymbol{\rho}}$-functions.*

**Definition 1.5.** Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a mapping on the metric spaces $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ and Y. We put $\mathrm{d}(x,y)^{\boldsymbol{\rho}} := \mathrm{d}_1(x_1,y_1)^{\rho_1} \vee \ldots \vee \mathrm{d}_d(x_d,y_d)^{\rho_d}$ with the convention $0^0 = 0$ (See the following Remark).

(i) Let $a$ be some point in X. We will say that $f$ is $\mathscr{C}^{\boldsymbol{\rho}}$ **at** $a$ if for every $\varepsilon > 0$, there is a neighborhood $\mathrm{U} \ni a$ such that

$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x,y)^{\boldsymbol{\rho}} \quad \text{for all } x, y \in \mathrm{U}.$$

We will say that $f$ is a $\mathscr{C}^{\boldsymbol{\rho}}$**-function** if $f$ is $\mathscr{C}^{\boldsymbol{\rho}}$ at all points $a \in \mathrm{X}$. The set of all $\mathscr{C}^{\boldsymbol{\rho}}$-functions $f\colon \mathrm{X} \to \mathrm{Y}$ will be denoted by $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X},\mathrm{Y})$.

(ii) We define $|f^{]\boldsymbol{\rho}[}|\colon \triangledown\mathrm{X} \times \mathrm{X} \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\boldsymbol{\rho}[}|(x,y) := \frac{\|f(x) - f(y)\|}{\mathrm{d}(x,y)^{\boldsymbol{\rho}}}.$$

*Remark.*

(i) In case $\rho_k = 0$ for some $k \in \{1, \ldots, d\}$, the scurrilous convention $0^0 = 0$ ensures that in a neighborhood of the point $a \in \mathrm{X}$, the condition $\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x,y)^{\boldsymbol{\rho}}$ is still stronger than the mere continuity condition $\mathrm{d}(f(x), f(y)) \leq \varepsilon$, which were in place if we had adopted the common convention $0^0 = 1$.

(ii) Keeping the above notations, we see that if $f\colon \mathrm{X} \to \mathrm{Y}$ is $\mathscr{C}^{\boldsymbol{\rho}}$ at $a \in \mathrm{X}$, then it will be $\mathscr{C}^{\tilde{\boldsymbol{\rho}}}$ thereat for any $\tilde{\boldsymbol{\rho}} \leq \boldsymbol{\rho}$ componentwise. In particular if $f$ is $\mathscr{C}^{\boldsymbol{\rho}}$ at $a \in \mathrm{X}$, then it will be $\mathscr{C}^{\rho}$ thereat with $\rho := \rho_1 \wedge \ldots \wedge \rho_d \in [0,1[$.

*The locally convex topology on $\mathscr{C}^{\boldsymbol{\rho}}$-functions.* The mapping $f\colon \mathrm{X} \to \mathbf{E}$ is $\mathscr{C}^{\boldsymbol{\rho}}$ if and only if the function $|f^{]\boldsymbol{\rho}[}|$ extends to a continuous function $|f^{[\boldsymbol{\rho}]}|\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle\mathrm{X} \times \mathrm{X}$. We moreover saw above that every $\mathscr{C}^{\boldsymbol{\rho}}$-function is in particular continuous. We can therefore establish:

**Definition.** For every compact $\mathrm{C} \subseteq \mathrm{X}$, we define the seminorm $\|\cdot\|_{\mathscr{C}^{\boldsymbol{\rho}},\mathrm{C}}$ on $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^{\boldsymbol{\rho}},\mathrm{C}} = \|f_{|\mathrm{C}}\|_{\sup} \vee \||(f_{|\mathrm{C}})^{[\boldsymbol{\rho}]}|\|_{\sup}.$$

We equip the **K**-vector space $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ with the locally convex topology given by the family of seminorms $\{\|\cdot\|_{\mathscr{C}^{\boldsymbol{\rho}},\mathrm{C}} : \mathrm{C} \subseteq \mathrm{X} \text{ compact}\}$.

If X itself is compact, then we will turn $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ into a normed **K**-vector space by endowing it with the norm $\|\cdot\|_{\mathscr{C}^{\boldsymbol{\rho}}} := \|\cdot\|_{\mathscr{C}^{\boldsymbol{\rho}},\mathrm{X}}$.

*Remark* 1.6.

(i) We have an equality of locally convex **K**-vector spaces $\mathscr{C}^{\vec{\rho}}(\mathrm{X}, \mathbf{E}) = \mathscr{C}^{\rho}(\mathrm{X}, \mathbf{E})$ with $\vec{\rho} = (\rho, \dots, \rho)$. It holds $\|\cdot\|_{\mathscr{C}^{\vec{\rho}},\mathrm{C}} = \|\cdot\|_{\mathscr{C}^{\rho},\mathrm{C}}$ for any $\mathrm{C} \subseteq \mathrm{X}$ compact.

(ii) The locally convex **K**-vector space $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ is the initial locally convex **K**-vector space with respect to the inclusion mappings

$$\begin{array}{ccc} & & \mathscr{C}^{\rho_1 \cdot \boldsymbol{e}_1}(\mathrm{X}, \mathbf{E}) \\ & \overset{\text{incl.}}{\nearrow} & \\ \mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E}) & & \vdots \\ & \underset{\text{incl.}}{\searrow} & \\ & & \mathscr{C}^{\rho_d \cdot \boldsymbol{e}_d}(\mathrm{X}, \mathbf{E}). \end{array}$$

It holds $\|\cdot\|_{\mathscr{C}^{\boldsymbol{\rho}},\mathrm{C}} = \|\cdot\|_{\mathscr{C}^{\rho_1 \cdot \boldsymbol{e}_d},\mathrm{C}} \vee \dots \vee \|\cdot\|_{\mathscr{C}^{\rho_d \cdot \boldsymbol{e}_d},\mathrm{C}}$ for any compact $\mathrm{C} \subseteq \mathrm{X}$.

(iii) The locally convex **K**-vector space of $\mathscr{C}^{\boldsymbol{\rho}}$-functions $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ is the initial locally convex **K**-vector space with respect to all restriction mappings

$$\begin{aligned} \mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E}) &\to \mathscr{C}^{\boldsymbol{\rho}}(\mathrm{C}, \mathbf{E}), \\ f &\mapsto f_{|\mathrm{C}} \end{aligned}$$

with C running through the family of all compact subsets $\mathrm{C} \subseteq \mathrm{X}$.

(iv) The locally convex **K**-vector space $\mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ is complete.

*Properties of the space of $\mathscr{C}^{\boldsymbol{\rho}}$-functions.*

**Definition 1.7.** Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be a cartesian metric space.

(i) Put

$$\mathbf{d}(x, y) := (\mathrm{d}_1(x_1, y_1), \dots, \mathrm{d}_d(x_d, y_d)) \quad \text{for } x, y \in \mathrm{X}.$$

Let $\boldsymbol{\delta} \in \mathbb{R}^d_{\geq 0}$. Then we write

$$\mathbf{d}(x, y) \leq \boldsymbol{\delta} \quad \text{if} \quad \mathrm{d}_1(x_1, y_1) \leq \delta_1, \dots, \mathrm{d}_d(x_d, y_d) \leq \delta_d.$$

Given $a \in \mathrm{X}$, we denote $\mathrm{B}_{\leq \boldsymbol{\delta}}(a) := \{x \in \mathrm{X} : \mathbf{d}(x, a) \leq \boldsymbol{\delta}\}$.

(ii) Let Y be a metric space. A mapping $g \colon \mathrm{X} \to \mathrm{Y}$ will be called **locally $\boldsymbol{\delta}$-constant** for $\boldsymbol{\delta} \in \mathbb{R}^d_{\geq 0}$ if it is locally constant and $\mathbf{d}(x, y) \leq \boldsymbol{\delta}$ implies $g(x) = g(y)$.

In case $\boldsymbol{\delta} \in \mathbb{R}^d_{>0}$, we find a locally $\boldsymbol{\delta}$-constant function to be $\delta$-constant with $\delta = \delta_1 \wedge \ldots \wedge \delta_d > 0$. But we will also be interested in the case where there is only one positive $\delta_k > 0$ and where we do not know particular positive lower bounds for the other entries of $\boldsymbol{\delta}$ - even though they might exist, for example, by compactness of X.

**Lemma 1.8.** *Let $f \in \mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ with $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ a compact metric space. Then*

$$\|f\|_{\mathscr{C}^{\boldsymbol{\rho}}} = \|f\|_{\sup} \vee |||f^{[\boldsymbol{\rho}]}|||_{\mathrm{X}^{[\boldsymbol{\rho}]}}$$

*with $\mathrm{X}^{[\boldsymbol{\rho}]} := \{(x, y) \in \mathrm{X} \times \mathrm{X} : x_k = y_k \text{ if } \rho_k = 0\} \subseteq \mathrm{X} \times \mathrm{X}$.*

*Proof:* This reduces by definition of $\|\cdot\|_{\mathscr{C}^{\boldsymbol{\rho}}}$ to verification of $|||f^{[\boldsymbol{\rho}]}|||_{\mathrm{X}^{[\boldsymbol{\rho}]}} \geq |||f^{[\boldsymbol{\rho}]}|||_{\sup}$. We have by definition $|f^{[\boldsymbol{\rho}]}|(x, y) \geq \|f\|_{\sup}$ only if $\mathrm{d}(x, y) \leq 1$. But if $\rho_k = 0$ and $x_k \neq y_k$, then $\mathrm{d}^{\boldsymbol{\rho}}(x, y) \geq \mathrm{d}_k(x_k, y_k)^0 = 1$. The assertion follows. □

**Lemma 1.9.** *Let $\boldsymbol{\rho} \in [0, 1[^d$. Let $\boldsymbol{\delta} \in [0, 1]^d$ such that:*

(i) *For any $k = 1, \ldots, d$, we have $\delta_k = 0$ only if $\rho_k = 0$.*

(ii) *Put $\mathrm{D} = \{\delta_k^{\rho_k} : k = 1, \ldots, d \text{ and } \delta_k > 0\}$. Then we have $\max \mathrm{D} = \min \mathrm{D}$ and $\gamma := \max \mathrm{D} > 0$.*

*Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be a compact cartesian metric space and $f \in \mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$ such that for fixed $\varepsilon > 0$, we find $\mathbf{d}(x, y) \leq \boldsymbol{\delta}$ to imply $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x, y)^{\boldsymbol{\rho}}$ for all $x, y \in \mathrm{X}$. Then there is locally $\boldsymbol{\delta}$-constant $g\colon \mathrm{X} \to \mathbf{E}$ with $\|f - g\|_{\sup} \leq \varepsilon\gamma$ and $\|f - g\|_{\mathscr{C}^{\boldsymbol{\rho}}} \leq \varepsilon$.*

*Proof:* Fix such $f\colon \mathrm{X} \to \mathbf{E}$ and $\varepsilon > 0$. Because $f \in \mathscr{C}^{\boldsymbol{\rho}}(\mathrm{X}, \mathbf{E})$, there is a tuple $\tilde{\boldsymbol{\delta}} \in ]0, 1]^d$ - and for which we may by premiss on $f$ assume $\tilde{\delta}_k = \delta_k$ for all $k = 1, \ldots, d$ with $\delta_k > 0$ - such that $\mathbf{d}(x, y) \leq \tilde{\boldsymbol{\delta}}$ implies $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}^{\boldsymbol{\rho}}(x, y)$. Because $\mathbf{E}$ is non-Archimedean, we can partition X into equivalence classes $\mathrm{U}_i \subseteq \mathrm{X}$ by declaring

$$x \sim y \quad \text{if} \quad \|f(x) - f(y)\| \leq \varepsilon\gamma.$$

Since $f$ is in particular continuous, every $\mathrm{U}_i$ is open. We now choose an element $a_i$ from each $\mathrm{U}_i$ and define locally constant $g\colon \mathrm{X} \to \mathbf{E}$ by

$$g(x) := f(a_i) \quad \text{if} \quad x \in \mathrm{U}_i.$$

We note that two points $x$ and $y$ will be equivalent if $\mathbf{d}(x, y) \leq \tilde{\boldsymbol{\delta}}$. Because $\boldsymbol{\delta} \leq \tilde{\boldsymbol{\delta}}$, we find thus $g$ to be in particular locally $\boldsymbol{\delta}$-constant.

By construction $\|f - g\|_{\sup} \leq \varepsilon\gamma \leq \varepsilon$ and

$$\begin{aligned}
|||(f - g)^{[\boldsymbol{\rho}]}|||_{\sup} &= |||(f - g)^{[\boldsymbol{\rho}]}|||_{\mathrm{X}^{[\boldsymbol{\rho}]}} \\
&= |||(f - g)^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in \mathrm{X}^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\leq\tilde{\boldsymbol{\delta}}\}} \vee |||(f - g)^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in \mathrm{X}^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\not\leq\tilde{\boldsymbol{\delta}}\}} \\
&\leq |||f^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in \mathrm{X}^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\leq\tilde{\boldsymbol{\delta}}\}} \vee |||g^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in \mathrm{X}^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\leq\tilde{\boldsymbol{\delta}}\}} \\
&\quad \vee \max_{(x,y)\in \mathrm{X}^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\not\leq\tilde{\boldsymbol{\delta}}} \left(\frac{\|f(x) - g(x)\|}{\mathrm{d}(x, y)^{\boldsymbol{\rho}}} \vee \frac{\|f(y) - g(y)\|}{\mathrm{d}(x, y)^{\boldsymbol{\rho}}}\right) \\
&\leq \varepsilon \vee 0 \vee \varepsilon\gamma/\gamma = \varepsilon;
\end{aligned}$$

the first equality by the preceding Lemma 1.8. Regarding the last inequality, we note that for $(x, y) \in X^{[\boldsymbol{\rho}]}$, we have $\mathbf{d}(x, y) \nleq \tilde{\boldsymbol{\delta}}$ if and only if there is $k \in \{1, \ldots, d\}$ with $\rho_k > 0$ such that $d_k^{\rho_k}(x_k, y_k) > \delta_k^{\rho_k} = \gamma$, and so $d(x, y)^{\boldsymbol{\rho}} > \gamma$. □

**Corollary 1.10.** *Let $X = X_1 \times \cdots \times X_d$ be a compact cartesian metric space. For $\boldsymbol{\rho} \in [0, 1[^d$, the locally constant functions $g \colon X \to \mathbf{E}$ are dense in $\mathscr{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$.*

*Proof:* Fix $\varepsilon > 0$ and $f \in \mathscr{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$. Then $|f^{[\boldsymbol{\rho}]}| \colon X \times X \to \mathbb{R}_{\geq 0}$ is by compactness of $X \times X$ a uniformly continuous function vanishing on $\triangle(X \times X)$. Hence we find $\boldsymbol{\delta} \in ]0, 1]^d$ such that in particular for all $\vec{a} = (a, a) \in X \times X$,

$$||f^{[\boldsymbol{\rho}]}|(x, y) - |f^{[\boldsymbol{\rho}]}|(\vec{a})| = |f^{[\boldsymbol{\rho}]}|(x, y) \leq \varepsilon \quad \text{for all } (x, y) \in X \times X \cap B_{\leq (\boldsymbol{\delta}, \boldsymbol{\delta})}(\vec{a}).$$

By possibly shrinking $\boldsymbol{\delta} \in ]0, 1]^d$ coordinatewise, we can moreover assume $\delta_k^{\rho_k} = \gamma$ for all $k \in L := \{l \in \{1, \ldots, d\} : \rho_l > 0\}$ with $\gamma := \min\{\delta_l^{\rho_l} : l \in L\} > 0$. Then $\boldsymbol{\delta}$ fulfills the conditions of the preceding Lemma 1.9

By the triangle inequality, if $\mathbf{d}(x, y) \leq \boldsymbol{\delta}$ for $(x, y) \in X \times X$, then $\mathbf{d}((x, y), (a, a)) \leq (\boldsymbol{\delta}, \boldsymbol{\delta})$ for some $a \in X$. Thus for all $(x, y) \in X \times X$ holds

$$\|f(x) - f(y)\| \leq \varepsilon \cdot d(x, y)^{\boldsymbol{\rho}} \quad \text{if } \mathbf{d}(x, y) \leq \boldsymbol{\delta}.$$

By Lemma 1.9 we find therefore locally $\boldsymbol{\delta}$-constant $g \colon X \to \mathbf{E}$ with $\|f - g\|_{\mathscr{C}^{\boldsymbol{\rho}}} \leq \varepsilon$. In particular the locally constant functions are dense in $\mathscr{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$. □

*Symmetry properties of the space of $\mathscr{C}^{\rho}$-functions.*

**Definition 1.11.** Let $A_1, \ldots, A_d$ be sets and put $A = A_1 \times \cdots \times A_d$. Denote by $\sigma \colon A \to A$ the mapping swapping the $k$-th and $l$-th coordinate. Then we will call:

(i) A point $a \in A$ **symmetric in its $k$-th and $l$-th coordinate** if $\sigma a = a$.

(ii) A subset $U \subseteq A$ **symmetric in its $k$-th and $l$-th coordinate** if $\sigma U = U$.

(iii) A function $f \colon U \to \mathbf{E}$ on a subset $U \subseteq A$ **symmetric in its $k$-th and $l$-th coordinate** if U is symmetric in its $k$-th and $l$-th coordinate and $f \circ \sigma = f$.

**Lemma 1.12.** *We assume $\varrho \in [0, 1]$. Let $U \subseteq \mathbf{K}^d$ be a subset and $f \colon U \to \mathbf{E}$ a mapping symmetric in its k-th and l-th coordinate. Fix $\varepsilon > 0$. Then*

$$\|f(x + t \cdot \boldsymbol{e}_k) - f(x)\| \leq \varepsilon \cdot |t|^{\varrho} \quad \textit{for all } x + t \cdot \boldsymbol{e}_k, x \in U$$

*if and only if*

$$\|f(x + t \cdot \boldsymbol{e}_l) - f(x)\| \leq \varepsilon \cdot |t|^{\varrho} \quad \textit{for all } x + t \cdot \boldsymbol{e}_l, x \in U.$$

*Proof:* By symmetry it suffices to prove one direction, for example, if

$$\|f(x + t \cdot \boldsymbol{e}_l) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_l, x \in \mathrm{U},$$

then

$$\|f(x + t \cdot \boldsymbol{e}_k) - f(x)\| \leq \varepsilon |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_k, x \in \mathrm{U}.$$

Denote by $\sigma \colon \mathrm{K}^d \to \mathrm{K}^d$ the map swapping the $k$-th and $l$-th coordinate. By assumption, U is left stable under $\sigma$, that is, if $y, x \in \mathrm{U}$, then $y^\sigma, x^\sigma \in \mathrm{U}$. Now let $y = x + t \cdot \boldsymbol{e}_k, x \in \mathrm{U}$. By symmetry of $f$ in its $k$-th and $l$-th coordinate, we find

$$\|f(x+t\cdot\boldsymbol{e}_k)-f(x)\| = \|f(y)-f(x)\| = \|f(y^\sigma)-f(x^\sigma)\| = \|f(x^\sigma+t\cdot\boldsymbol{e}_l)-f(x^\sigma)\| \leq \varepsilon|t|^\varrho,$$

the last inequality as $y^\sigma, x^\sigma \in \mathrm{U}$.

□

**Lemma 1.13.** *Let* $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ *be a compact cartesian metric space and* $f \colon \mathrm{X} \to \mathbf{E}$ *be a mapping symmetric in its k-th and l-th coordinate. Then* $f \in \mathscr{C}^{\rho\cdot\boldsymbol{e}_k}(\mathrm{C}, \mathbf{E})$ *if and only if* $f \in \mathscr{C}^{\rho\cdot\boldsymbol{e}_l}(\mathrm{C}, \mathbf{E})$ *and it holds* $\|f\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}_k}} = \|f\|_{\mathscr{C}^{\rho\cdot\boldsymbol{e}_l}}$.

*Proof:* Since X is compact, we find $f \in \mathscr{C}^{\rho\cdot\boldsymbol{e}_k}(\mathrm{X}, \mathbf{E})$ for some $k \in \{1, \ldots, d\}$ if and only if for every $\varepsilon > 0$ exists $\delta > 0$ such that $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x, y)^{\rho\cdot\boldsymbol{e}_k}$ for all $x, y \in \mathrm{X}$ with $\mathrm{d}(x, y) \leq \delta$. Since $f$ is symmetric in its $k$-th and $l$-th coordinate, we see by Lemma 1.12 that $f \in \mathscr{C}^{\rho\cdot\boldsymbol{e}_k}(\mathrm{X}, \mathbf{E})$ if and only if $f \in \mathscr{C}^{\rho\cdot\boldsymbol{e}_l}(\mathrm{X}, \mathbf{E})$.

Moreover, denote by $\sigma$ the permutation map on X swapping the $k$-th and $l$-th coordinate. Then $(\sigma, \sigma)$ acts on $\mathrm{X} \times \mathrm{X} \supseteq \mathrm{X}^{[\rho\cdot\boldsymbol{e}_k]}$ and

$$|||f^{[\rho\cdot\boldsymbol{e}_k]}|||_{\mathrm{X}^{[\rho\cdot\boldsymbol{e}_k]}} = |||f^{[\rho\cdot\boldsymbol{e}_l]}| \circ (\sigma, \sigma)\|_{\mathrm{X}^{[\rho\cdot\boldsymbol{e}_k]}} = |||f^{[\rho\cdot\boldsymbol{e}_l]}|||_{(\sigma,\sigma)\mathrm{X}^{[\rho\cdot\boldsymbol{e}_k]}} = |||f^{[\rho\cdot\boldsymbol{e}_l]}|||_{\mathrm{X}^{[\rho\cdot\boldsymbol{e}_l]}}.$$

**Definition (1.11').** Let $\mathrm{A}_1, \ldots, \mathrm{A}_d$ be sets and put $\mathrm{A} = \mathrm{A}_1 \times \cdots \times \mathrm{A}_d$. Let $\mathrm{I} \subseteq \{1, \ldots, d\}$ be a subset of indices. Then we will call:

(i) A point $a \in \mathrm{A}$ **symmetric in its coordinates indexed by** I if $a$ is symmetric in its $k$-th and $l$-th coordinates for all $k, l \in \mathrm{I}$.

(ii) A subset $\mathrm{U} \subseteq \mathrm{A}$ **symmetric in its coordinates indexed by** I if U is symmetric in its $k$-th and $l$-th coordinate for all $k, l \in \mathrm{I}$.

(iii) A function $f \colon \mathrm{U} \to \mathbf{E}$ on a subset $\mathrm{U} \subseteq \mathrm{A}$ **symmetric in its coordinates indexed by** I if $f$ is symmetric in its $k$-th and $l$-th coordinate for all $k, l \in \mathrm{I}$.

**Corollary 1.14.** *We assume* $\varrho \in [0, 1]$. *Let* $\mathrm{U} \subseteq \mathbf{K}^d$ *be a cartesian subset and* $f \colon \mathrm{U} \to \mathbf{E}$ *a mapping symmetric in its coordinates indexed by* $\mathrm{I}_1, \ldots, \mathrm{I}_e$ *for a partition* $\{1, \ldots, d\} = \mathrm{I}_1 \dot\cup \ldots \dot\cup \mathrm{I}_e$. *Let* $i_1, \ldots, i_e$ *be representatives of these subsets. Fix* $\varepsilon > 0$. *Then*

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\varrho \quad \textit{for all } x, y \in \mathrm{U}$$

*if and only if for* $j = 1, \ldots, e$ *holds*

$$\|f(x + t \cdot \boldsymbol{e}_{i_j}) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \textit{for all } x + t \cdot \boldsymbol{e}_{i_j}, x \in \mathrm{U}.$$

*Proof:* By Lemma 1.12. □

**Corollary 1.15.** *Let* $X = X_1 \times \cdots \times X_d$ *be a compact cartesian metric space. Let* $\{1, \ldots, d\} = I_1 \dot\cup \ldots \dot\cup I_e$ *with representatives* $i_1, \ldots, i_d$ *and* $f \colon X \to \mathbf{E}$ *a mapping symmetric in its coordinates indexed by* $I_1, \ldots, I_e$*. Then*

$$f \in \mathscr{C}^{\rho}(X, E) \quad \textit{if and only if} \quad f \in \mathscr{C}^{\rho \cdot \boldsymbol{e}_{i_1}}(X, \mathbf{E}) \cap \ldots \cap \mathscr{C}^{\rho \cdot \boldsymbol{e}_{i_e}}(X, \mathbf{E})$$

*and*

$$\|f\|_{\mathscr{C}^{\rho}} = \|f\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_{i_1}}} \vee \ldots \vee \|f\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_{i_e}}}.$$

*Proof:* We have

$$\mathscr{C}^{\rho}(X, \mathbf{E}) = \mathscr{C}^{\vec{\rho}}(X, \mathbf{E}) = \mathscr{C}^{\rho \cdot \boldsymbol{e}_1}(X, \mathbf{E}) \cap \ldots \cap \mathscr{C}^{\rho \cdot \boldsymbol{e}_d}(X, \mathbf{E})$$

and

$$\|\cdot\|_{\mathscr{C}^{\rho}} = \|\cdot\|_{\mathscr{C}^{\vec{\rho}}} = \|\cdot\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_1}} \vee \ldots \vee \|\cdot\|_{\mathscr{C}^{\rho \cdot \boldsymbol{e}_d}},$$

the first respectively second equality by Remark 1.6 (i) respectively (ii). We can then conclude by applying Lemma 1.13. □

## $\mathscr{C}^{1+\rho}$-functions

For latter use we introduce here a more conceptual approach to the definition of $1 + \rho$-fold differentiability for $\rho \in [0, 1[$ by the function's approachability through linear maps, generalizing the common notion of strict differentiability given in the introduction, see Remark 1.17.

*Assumption.* Throughout this subsection, we will make the following assumptions:

- We will fix a real number $\rho \in [0, 1[$.
- We let $X = X_1 \times \cdots \times X_d \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point.

A quick remark on the latter assumption's origin: Define the — for example, first — partial difference quotient of a function $f \colon X \to \mathbf{E}$ by

$$f^{]1,0,\ldots,0[}(x, t) = \frac{f(x + t \cdot \boldsymbol{e}_1) - f(x)}{t} \quad \text{for } x \in X, t \in \mathbf{K}^* \text{ with } x + t \cdot \boldsymbol{e}_1 \in X.$$

Then $f$ is defined to be once partially differentiable in its first coordinate at $a \in X$ if and only if this function is $\mathscr{C}^0$ at $(a, 0)$. But this function $f^{]1,0,\ldots,0[}$ has a *unique* extension onto $(a, 0)$ with value $D_{1,0,\ldots,0} f(a) := \lim_{t \to 0} f^{]1,0,\ldots,0[}(a, t)$ if and only if $a_1$ is an accumulation point of $X_1$.

**Definition.** Let $f \colon X \to \mathbf{E}$ be a mapping.

(i) Put $X^{]e_k[} = X_1 \times \cdots \times X_{k-1} \times \triangledown X_k^2 \times X_{k+1} \times \cdots \times X_d$. We define $f^{]e_k[} \colon X^{]e_k[} \to \mathbf{E}$ by

$$f^{]e_k[}(-; y_k, x_k; -) = \frac{f(x + t \cdot e_k) - f(x)}{t};$$

here $x := (x_1, \ldots, x_d)$ and $t := y_k - x_k \neq 0$ - the hyphenations to the left and right of the semicolons representing the omitted coordinate entries $x_1, \ldots, x_{k-1}$ and $x_{k+1}, \ldots, x_d$.

(ii) Put $X^{[e_k]} = X_1 \times \cdots \times X_{k-1} \times X_k^2 \times X_{k+1} \times \cdots \times X_d$. Then $f$ will be a $\mathscr{C}^{(1+\rho)\cdot e_k}$**-function** if $f$ is continuous and $f^{]e_k[} \colon X^{]e_k[} \to \mathbf{E}$ extends (uniquely) to a $\mathscr{C}^\rho$-function $f^{[e_k]} \colon X^{[e_k]} \to \mathbf{E}$. Let us denote the set of all $\mathscr{C}^{(1+\rho)\cdot e_k}$-functions $f \colon X \to \mathbf{E}$ by $\mathscr{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E})$. For compact cartesian $C \subseteq X$ we define the seminorm $\|\cdot\|_{\mathscr{C}^{(1+\rho)\cdot e_k}, C}$ on $\mathscr{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^{(1+\rho)\cdot e_k}, C} = \|f\|_C \vee \|f^{[e_k]}\|_{\mathscr{C}^\rho, C^{[e_k]}}.$$

(iii) We define $\mathscr{C}^{1+\rho}(X, \mathbf{E})$ as the initial locally convex $\mathbf{K}$-vector space with respect to the inclusion mappings $\mathscr{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E}) \hookrightarrow \mathscr{C}^0(X, \mathbf{E})$ for $k = 1, \ldots, d$, that is we put $\mathscr{C}^{1+\rho}(X, \mathbf{E}) = \mathscr{C}^{(1+\rho)\cdot e_1}(X, \mathbf{E}) \cap \ldots \cap \mathscr{C}^{(1+\rho)\cdot e_d}(X, \mathbf{E}) \subseteq \mathscr{C}^0(X, \mathbf{E})$ and for compact cartesian $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathscr{C}^{1+\rho}, C}$ on $\mathscr{C}^{1+\rho}(X, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^{1+\rho}, C} = \|f\|_{\mathscr{C}^{(1+\rho)\cdot e_1}, C} \vee \ldots \vee \|f\|_{\mathscr{C}^{(1+\rho)\cdot e_d}, C}.$$

**Proposition 1.16.** *Let $f \colon X \to \mathbf{E}$ be a mapping. Consider the following*

**Definition.**

(i) For all $y, x \in X$ with $y = x + t_1 e_1 + \cdots + t_d e_d$ and $t_1, \ldots, t_d \in \mathbf{K}^*$, there is a unique $\mathbf{K}$-linear map $A =: f^{]1[}(y, x) \colon \mathbf{K}^d \to \mathbf{E}$ defined through

$$A \cdot t_k e_k := f(x + t_1 \cdot e_1 + \cdots + t_k \cdot e_k) - f(x + t_1 \cdot e_1 + \cdots + t_{k-1} \cdot e_{k-1}) \quad \text{for } k = 1, \ldots, d.$$

(ii) Define $X^{]1[} := \{(y, x) \in X \times X \colon y = x + t_1 e_1 + \cdots + t_d e_d \text{ with } t_1, \ldots, t_d \in \mathbf{K}^*\}$ and $X^{[1]} := X \times X$. We will say that $f \colon X \to \mathbf{E}$ is a $\tilde{\mathscr{C}}^{1+\rho}$**-function** if $f^{]1[} \colon X^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E})$ extends (uniquely) to a $\mathscr{C}^\rho$-function $f^{[1]} \colon X^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E})$ with respect to the operator norm on its range. We will denote the set of all $\tilde{\mathscr{C}}^{1+\rho}$-functions $f \colon X \to \mathbf{E}$ by $\tilde{\mathscr{C}}^{1+\rho}(X, \mathbf{E})$. For every compact cartesian $C \subseteq X$ we define the seminorm $\|\cdot\|_{\tilde{\mathscr{C}}^{1+\rho}, C}$ on $\tilde{\mathscr{C}}^{1+\rho}(X, \mathbf{E})$ by

$$\|f\|_{\tilde{\mathscr{C}}^{1+\rho}, C} = \|f\|_C \vee \|(f^{[1]}\|_{\mathscr{C}^\rho, C^{[1]}}.$$

*Then $\tilde{\mathscr{C}}^{1+\rho}(X, \mathbf{E}) = \mathscr{C}^{1+\rho}(X, \mathbf{E})$ and $\|\cdot\|_{\tilde{\mathscr{C}}^{1+\rho}, C} = \|\cdot\|_{\mathscr{C}^{1+\rho}, C}$ for $C \subseteq X$ compact cartesian.*

*Proof:* It holds $A \cdot \boldsymbol{e}_1 = f^{]\boldsymbol{e}_1[}(y_1, x_1; x_2; \dots; x_d), \dots, A \cdot \boldsymbol{e}_d = f^{]\boldsymbol{e}_d[}(y_1; \dots; y_{d-1}; y_d, x_d)$ for $y, x \in X$ with $y = x + t_1 \cdot \boldsymbol{e}_1 + \dots + t_d \cdot \boldsymbol{e}_d$ and $t_1, \dots, t_d \in \mathbf{K}^*$. Hence under the isometric isomorphism of $\mathbf{K}$-Banach spaces

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E}) &\to \mathbf{E}^d\\ A &\mapsto (A \cdot \boldsymbol{e}_1, \dots, A \cdot \boldsymbol{e}_d),\end{aligned}$$

we obtain $f^{]1[}(y, x) = (f^{]\boldsymbol{e}_1[}(y_1, x_1; x_2; \dots; x_d), \dots, f^{]\boldsymbol{e}_d[}(y_1; \dots; y_{d-1}; y_d, x_d))$. Therefore $f^{]1[} \colon X^{]1[} \to \mathbf{E}^d$ extends to a $\mathscr{C}^\rho$-function on all $x, y \in X$ if and only if $f^{]\boldsymbol{e}_k[} \colon X^{]\boldsymbol{e}_k[} \to \mathbf{E}$ extends to a $\mathscr{C}^\rho$-function $f^{[\boldsymbol{e}_k]} \colon X^{]\boldsymbol{e}_k[} \to \mathbf{E}$ for $k = 1, \dots, d$. That is,

$$\mathscr{C}^{(1+\rho)\cdot \boldsymbol{e}_1}(X, \mathbf{E}) \cap \dots \cap \mathscr{C}^{(1+\rho)\cdot \boldsymbol{e}_d}(X, \mathbf{E}) = \mathscr{C}^{1+\rho}(X, \mathbf{E}).$$

For $y, x \in X$ with $y = x + t_1 \boldsymbol{e}_1 + \dots + t_d \boldsymbol{e}_d$ and $t_1, \dots, t_d \in \mathbf{K}$, we find

$$\begin{aligned}\|f^{[1]}(y, x)\| =& \|A\| = \|A \cdot \boldsymbol{e}_1\| \vee \dots \vee \|A \cdot \boldsymbol{e}_d\|\\ =& \|f^{[\boldsymbol{e}_1]}(y_1, x_1; x_2; \dots; x_d)\| \vee \dots \vee \|f^{[\boldsymbol{e}_d]}(y_1; \dots; y_{d-1}; y_d, x_d)\|.\end{aligned}$$

If we let $x, y$ run through $C \subseteq X$ compact cartesian, we find

$$\|f^{[1]}\|_{\mathscr{C}^\rho, C^{[1]}} = \|f^{[\boldsymbol{e}_1]}\|_{\mathscr{C}^\rho, C^{[\boldsymbol{e}_1]}} \vee \dots \vee \|f^{[\boldsymbol{e}_d]}\|_{\mathscr{C}^\rho, C^{[\boldsymbol{e}_d]}}.$$

Therefore $\|f\|_{\tilde{\mathscr{C}}^{1+\rho}, C} = \|f\|_{\mathscr{C}^{1+\rho}, C}$. □

*Remark* 1.17. Let $f \in \mathscr{C}^1(X, \mathbf{E})$ and $a \in X$. Consider the $\mathbf{K}$-linear mapping $D_a f := f^{[1]}(a, a) \in \operatorname{Hom}_{\mathbf{K}\text{-vctsp}}(\mathbf{K}^d, \mathbf{E})$. Then for every $\varepsilon > 0$, there is a neighborhood $U \ni a$ in X such that $\|f^{[1]}(x + h, x) - D_a f\| \le \varepsilon$ for all $x + h, x \in U$. In particular

$$\begin{aligned}\|f(x + h) - f(x) - D_a f \cdot h\| =& \|f^{[1]}(x + h, x) \cdot h - D_a f \cdot h\|\\ \le& \|f^{[1]}(x + h, x) - D_a f\|\|h\| \le \varepsilon\|h\|\end{aligned}$$

for all $x + h, x \in U$. This is usually called *strict differentiability*. Therefore if a function is $\mathscr{C}^1$ at a point $a \in X$, then it is strictly differentiable at $a$. In the other direction, given $\varepsilon > 0$ we find a neighborhood $U \ni a$ in X such that in particular for all $y = x + t \cdot \boldsymbol{e}_k, x \in U$ with $k = 1, \dots, d$ holds

$$\|f^{]\boldsymbol{e}_k[}(-; x_k + t, x_k; -) - D_a f \cdot \boldsymbol{e}_k\| = \|1/t \cdot (f(x + t \cdot \boldsymbol{e}_k) - f(x) - D_a f)\| \le \varepsilon \cdot |t||t\boldsymbol{e}_k| = \varepsilon.$$

Thus $f^{]\boldsymbol{e}_k[}$ is $\mathscr{C}^0$ at $\vec{a} = (a_1; \dots; a_{k-1}; a_k, a_k; a_{k+1}; \dots; a_d) \in X^{[\boldsymbol{e}_k]}$ for $k = 1, \dots, d$ or, by the preceding Proposition 1.16, equivalently the function $f^{]1[}$ is $\mathscr{C}^0$ at $(a, a)$.

**Lemma 1.18.** *We have norm-nonincreasing inclusions*

$$\mathscr{C}^1(X, \mathbf{E}) \subseteq \mathscr{C}^{\mathrm{lip}}(X, \mathbf{E}) \subseteq \mathscr{C}^\rho(X, \mathbf{E}).$$

*Proof:* By definition, see [Nag11, Lemma 1.37]. □

## 2 Fractional differentiability in many variables

*Assumption.* Throughout this section, we will fix a real number $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with integral part $\nu = \lfloor r \rfloor := \max\{n \in \mathbb{N} : n \leq r\} \in \mathbb{N}$ and fractional one $\rho = \{r\} := r - \lfloor r \rfloor \in [0, 1[$.

### $\mathscr{C}^r$-functions for $r \in \mathbb{R}_{\geq 0}$

In this section, we define the locally convex $\mathbf{K}$-vector space of $\mathscr{C}^r$-functions and collect its most basic properties.

*Definition of $\mathscr{C}^r$-functions.* Let $\mathbf{X} \subseteq \mathbf{K}^d$ be a subset. We recall that $\mathbf{X}$ is called *cartesian* if $\mathbf{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ with $\mathrm{X}_1, \ldots, \mathrm{X}_d \subset \mathbf{K}$.

**Definition.**

(i) For a subset $\mathrm{X} \subseteq \mathbf{K}$ and $n \in \mathbb{N}$, we define

$$\mathrm{A}^{[n]} = \mathrm{X}^{\{0,\ldots,n\}} \quad \text{and} \quad \mathrm{X}^{]n[} = \nabla \mathrm{X}^{[n]} = \{(x_0, \ldots, x_n) : x_i = x_j \text{ only if } i = j\}.$$

Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d \subset \mathbf{K}^d$ with $\mathrm{X}_1, \ldots, \mathrm{X}_d \subset \mathbf{K}$ be a cartesian subset. We put

$$\mathrm{X}^{]\boldsymbol{n}[} := \mathrm{X}_1^{]n_1[} \times \cdots \times \mathrm{X}_d^{]n_d[} \quad \text{and} \quad \mathrm{X}^{[\boldsymbol{n}]} := \mathrm{X}_1^{[n_1]} \times \cdots \times \mathrm{X}_d^{[n_d]}.$$

We will write elements $x \in \mathrm{X}^{[\boldsymbol{n}]}$ as $x = ({}^1x; -; {}^dx)$ with ${}^1x \in \mathrm{X}_1^{[n_1]}, \ldots, {}^dx \in \mathrm{X}_d^{[n_d]}$.

(ii) Let $f : \mathrm{X} \to \mathbf{E}$ be a function on a cartesian subset $\mathrm{X} \subseteq \mathbf{K}^d$. Let $\boldsymbol{n} \in \mathbb{N}^d$. Through recursion on $n = |\boldsymbol{n}|$, we define functions $f^{]\boldsymbol{n}[} : \mathrm{X}^{]\boldsymbol{n}[} \to \mathbf{E}$ by

$$f^{]\boldsymbol{0}[} = f,$$

and if $\boldsymbol{n}^+ = \boldsymbol{n} + \boldsymbol{e}_k$ for $k \in \{1, \ldots, d\}$, then

$$f^{]\boldsymbol{n}^+[}(-; {}^kx_0, {}^kx_1, {}^kx_2, \ldots, {}^kx_{n_k+1}; -) = \frac{f^{]\boldsymbol{n}[}(-; {}^kx_0, {}^kx_2, \ldots, {}^kx_{n_k+1}; -) - f^{]\boldsymbol{n}[}(-; {}^kx_1, {}^kx_2, \ldots, {}^kx_{n_k+1}; -)}{{}^kx_0 - {}^kx_1}.$$

Here and in the following the hyphenations to the left and right of the semicolons representing the same omitted arguments ${}^1x; -; {}^{k-1}x$ and ${}^{k+1}x; -; {}^dx$.

We remark that this definition does not depend on the order of summation of $\boldsymbol{n} = n_1\boldsymbol{e}_1 + \cdots + n_d\boldsymbol{e}_d \in \mathbb{N}^d$ by $\mathbf{K} \subseteq \mathbf{E}$ being central.

**Example.** For notational convenience, we consider the case of two variables and a function $f : \mathrm{X} \times \mathrm{Y} \to \mathbf{E}$ for $\mathrm{X}, \mathrm{Y} \subseteq \mathbf{K}$.

(i) We have $X^{]1,0[} = \{(x, x'; y) : x, x' \in X, y \in Y \text{ with } x \neq x'\}$ and

$$f^{]1,0[}(x, x'; y) = \frac{f(x, y) - f(x', y)}{x - x'}.$$

In other words the $f^{]1,0[}, f^{]0,1[}$ are the first partial difference quotients of $f$.

(ii) We have $X^{]1,1[} = \{(x + u, x; y + v, y) : x + u, x \in X, y + v, y \in Y \text{ with } u, v \neq 0\}$ and

$$f^{]1,1[}(x + u, x; y + v, y) \\ = \frac{[f(x + u, y + v) - f(x, y + v)] - [f(x + u, y) - f(x, y)]}{u \cdot v}.$$

In other words $f^{]1,1[}$ is the first mixed partial difference quotient of $f$.

**Lemma 2.1.** *Let $X \subseteq \mathbf{K}^d$ be a cartesian subset.*

*(i) Let $f : X \to \mathbf{E}$ be a mapping on X. Then the mapping $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ is symmetric in its $X_k$-coordinates for each $k = 1, \dots, d$.*

*(ii) Let $f, g : X \to \mathbf{E}$ be two mappings on X. Then for all $({}^1x; -; {}^dx) \in X^{]\boldsymbol{n}[}$, we find*

$$(f \cdot g)^{]\boldsymbol{n}[}({}^1x; -; {}^dx) \\ = \sum_{\boldsymbol{j} \leq \boldsymbol{n}} f^{]\boldsymbol{j}[}({}^1x_0, \dots, {}^1x_{j_1}; -; {}^dx_0, \dots, {}^dx_{j_d}) \cdot g^{]\boldsymbol{n}-\boldsymbol{j}[}({}^1x_{j_1}, \dots, {}^1x_{n_1}; -; {}^dx_{j_d}, \dots, {}^dx_{n_d}).$$

*Proof:* Ad (i): This is a multivariable version of [Sch84, Lemma 29.2(ii)] and proved likewise.
Ad (ii): In case $d = 1$, this is proved in [Sch84, Lemma 29.2(v)]. In the general case $d > 1$, the result follows by induction, see [Nag11, Lemma 3.4(iii)]. □

To account for more general domains of $\mathscr{C}^r$-functions such as open subsets $X \subseteq \mathbf{K}^d$ we introduce the following notion of a locally cartesian subset.

**Definition.** Let $V = V_1 \times \dots \times V_d$ be a topological space. Then we will call a subset $X \subseteq V$ **locally cartesian** if every point $x \in X$ has a cartesian neighborhood with respect to the relative topology in X.

**Definition 2.2.**

(i) Let $X \subseteq \mathbf{K}^d$ be a cartesian subset; we will say that a mapping $f : X \to \mathbf{E}$ is $\mathscr{C}^r$ or ***r*-times continuously differentiable** at some point $a \in X$ if $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ is $\mathscr{C}^\rho$ at $\vec{a} := (\vec{a}_1; -; \vec{a}_d) \in X^{[\boldsymbol{n}]}$ for all $\boldsymbol{n} \in \mathbb{N}^d$ with $n_1 + \dots + n_d = \nu$. (Here $\vec{a}_k := (a_k, \dots, a_k) \in X_k^{[n_k]}$ for $k = 1, \dots, d$.)

(ii) Let $X \subseteq \mathbf{K}^d$ be locally cartesian and $f : X \to \mathbf{E}$ a map thereon. We will say that $f$ is $\mathscr{C}^r$ at $a$ if $f_{|U}$ is $\mathscr{C}^r$ at $a$ for some cartesian neighborhood $U \subseteq X$.

(iii) Let $\mathrm{X} \subseteq \mathbf{K}^d$ be a locally cartesian subset. Then $f$ will be a **$\mathscr{C}^r$-function** or an **$r$-times continuously differentiable function** if $f$ is $\mathscr{C}^r$ at all points $a \in \mathrm{X}$. The set of all $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathbf{E}$ is denoted by $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

**Lemma 2.3.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ be cartesian. Then a mapping $f\colon \mathrm{X} \to \mathbf{E}$ is $\mathscr{C}^r$ at a point $a \in \mathrm{X}$ if and only if for every $\varepsilon > 0$, $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $k = 1, \ldots, d$, there is a cartesian neighborhood $\mathrm{U} \ni a$ in $\mathrm{X}$ such that*

$$\|f^{]\boldsymbol{n}[}(-; {}^k x_0, {}^k x_1, \ldots, {}^k x_{n_k}; -) - f^{]\boldsymbol{n}[}(-; {}^k \tilde{x}_0, {}^k x_1, \ldots, {}^k x_{n_k}; -)\| \leq \varepsilon \cdot |{}^k x_0 - {}^k \tilde{x}_0|^\rho \text{ on } \mathrm{U}^{]\boldsymbol{n}[}. \tag{$*$}$$

*Proof:* By symmetry of $f^{]\boldsymbol{n}[}$ in its $\mathrm{X}_k$-coordinates, see [Nag11, Lemma 3.2]. □

*Remark* 2.4.

(i) By definition, being $\mathscr{C}^r$ is a local property. In the following we will therefore formulate local results on $\mathscr{C}^r$-functions solely for cartesian subsets $\mathrm{X} \subseteq \mathbf{K}^d$.

(ii) We observe that the differentiability at some point a may vanish if the function's domain expands in $\mathbf{K}^d$ - as long as there is no neighborhood $\mathrm{U} \ni a$ in $\mathbf{K}^d$ lying in the domain.

(iii) If $\mathbf{E} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_d$ with $\mathbf{K}$-Banach spaces $\mathbf{E}_k$ for $k = 1, \ldots, d$, then $f\colon \mathrm{X} \to \mathbf{E}$ will be $\mathscr{C}^\rho$ at $a \in \mathrm{X}$ if and only if its $k$-th component $f_k\colon \mathrm{X} \to \mathbf{E}_k$ will be $\mathscr{C}^\rho$ at $a$ for $k = 1, \ldots, d$. Hence $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) = \mathscr{C}^r(\mathrm{X}, \mathbf{E}_1) \times \cdots \times \mathscr{C}^r(\mathrm{X}, \mathbf{E}_d)$.

(iv) Let $f\colon \mathrm{X} \to \mathbf{K}$ be some mapping, $\boldsymbol{n} \in \mathbb{N}^d$ and $a_1, \ldots, a_d$ accumulation points in $\mathrm{X}_1, \ldots, \mathrm{X}_d$. Then $\vec{a}$ is an accumulation point of $\mathrm{X}^{]\boldsymbol{n}[}$. As $\mathbf{E}$ is complete, we find by Remark 1.1 that $f^{]\boldsymbol{n}[}$ is $\mathscr{C}^0$ at $\vec{a}$ if and only if there is a limit $\mathrm{D}_{\boldsymbol{n}} f(a) \in \mathbf{E}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|f^{]\boldsymbol{n}[}(x) - \mathrm{D}_{\boldsymbol{n}} f(a)\| \leq \varepsilon \quad \text{for all } x \in \mathrm{X}^{]\boldsymbol{n}[} \text{ with } \|x - \vec{a}\| \leq \delta.$$

*The locally convex topology on $\mathscr{C}^r$-functions and first properties.*

**Definition.**

(i) Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d \subseteq \mathbf{K}^d$ be a cartesian subset. Then we say that **its factors contain no isolated points** if each subset $\mathrm{X}_1, \ldots, \mathrm{X}_d \subseteq \mathbf{K}$ contains no isolated points.

(ii) If $\mathrm{X} \subseteq \mathbf{K}^d$ is a locally cartesian subset, we say that **its local factors contain no isolated points** if it can be covered by cartesian neighborhoods $\mathrm{U} = \mathrm{U}_1 \times \cdots \times \mathrm{U}_d$ in $\mathrm{X}$ whose factors contain no isolated points.

The following Proposition will be crucial for the definition of the locally convex topology of $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ to be introduced.

**Proposition 2.5.** *Let* $X \subseteq \mathbf{K}^d$ *be a nonempty cartesian subset whose factors contain no isolated points and* $f : X \to \mathbf{E}$ *a mapping thereon. Then* $f \in \mathscr{C}^r(X, \mathbf{E})$ *if and only if for all* $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$, *the function* $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ *extends to a unique* $\mathscr{C}^\rho$*-function* $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{E}$.

*Proof:* By induction on $\nu \in \mathbb{N}$ and Lemma 2.3 one finds $f^{]\boldsymbol{n}[}$ to be a $\mathscr{C}^\rho$-function. Since $X^{]\boldsymbol{n}[} \subseteq X^{[\boldsymbol{n}]}$ is a dense subset, we find $f^{]\boldsymbol{n}[}$ by Proposition 1.2 to extend to a $\mathscr{C}^\rho$-function on all of $X^{[\boldsymbol{n}]}$. □

**Lemma 2.6.** *Let* $X \subseteq \mathbf{K}^d$ *be a cartesian subset and* $f : X \to \mathbf{E}$ *a mapping thereon. Let* $a \in X$. *If* $f$ *is* $\mathscr{C}^r$ *at* $a$, *then* $f$ *will be* $\mathscr{C}^s$ *at* $a$ *for every* $s \leq r$.

*Proof:* By definition, using Lemma 2.3. See [Nag11, Lemma 3.5] for more details. □

**Definition.** Let $X \subseteq \mathbf{K}^d$ be a locally cartesian subset with local factors free of isolated points. Let $f \in \mathscr{C}^r(X, \mathbf{E})$. Let $C \subseteq X$ be a compact cartesian subset contained in some nonempty cartesian neighborhood $U \supseteq C$ in X with factors free of isolated points. By Lemma 2.6 and Proposition 2.5, for all $\boldsymbol{n} \in \mathbb{N}^d$ with $|\boldsymbol{n}| < \nu$ respectively $|\boldsymbol{n}| = \nu$, the mapping $f_{|U}^{]\boldsymbol{n}[}$ extends to a continuous respectively $\mathscr{C}^\rho$-function $f_{|U}^{[\boldsymbol{n}]}$. Hence we can define for each compact cartesian $C \subseteq X$ the seminorm $\|\cdot\|_{\mathscr{C}^r, C}$ by

$$\|f\|_{\mathscr{C}^r, C} := \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| < \nu} \|f_{|U}^{[\boldsymbol{n}]}\|_{C^{[\boldsymbol{n}]}} \vee \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| = \nu} \|f_{|U}^{[\boldsymbol{n}]}\|_{\mathscr{C}^\rho, C^{[\boldsymbol{n}]}}.$$

We provide $\mathscr{C}^r(X, \mathbf{E})$ with the locally convex topology induced through this family of seminorms $\{\|\cdot\|_{\mathscr{C}^r, C}\}$ with $C \subseteq X$ compact cartesian — the **topology of compact (cartesian) convergence**.

*Remark* 2.7. A prior version of this article the definition of the locally convex topology contained an inaccuracy, residing on an invalid and dispensable lemma as kindly pointed out by Helge Glöckner in [Glö13] and which has now been skipped. Despite the omission the numbering of the article has been left intact.

We note that if $\mathbf{K}$ is locally compact, then $\mathscr{C}^r(X, \mathbf{E})$ can be more concisely described as the initial locally convex $\mathbf{K}$-vector space with respect to all restriction mappings

$$\mathscr{C}^r(X, \mathbf{E}) \to \mathscr{C}^r(C, \mathbf{E}),$$
$$f \mapsto f_{|C};$$

here C running through the family of all balls $C \subseteq X$.

*Remark.* We have for $s \leq r$ a norm-nonincreasing inclusion of locally convex $\mathbf{K}$-vector spaces $\mathscr{C}^r(X, \mathbf{E}) \subseteq \mathscr{C}^s(X, \mathbf{E})$.

*Proof:* The statement is natural, but once more the symmetry properties of the iterated difference quotients have to be invoked to give a formal proof. See [Nag11, Lemma 3.11]. □

**Proposition 2.8.** *The locally convex* $\mathbf{K}$*-vector space* $\mathscr{C}^r(X, \mathbf{E})$ *is complete.*

*Proof:* This is proved in [Nag11, Proposition 3.13]. □

*Remark* 2.9. We list some natural properties of the locally convex $\mathbf{K}$-vector space $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$. For proofs, we confer to [Nag11, Section 3].

(i) Let $\mathbf{E}$ be a $\mathbf{K}$-Banach algebra. Then the pointwise multiplication in $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ is continuous, that is, $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ is a locally convex $\mathbf{K}$-algebra.

(ii) Let us call a locally convex $\mathbf{K}$-vector space **Fréchet** if its topology can be induced by a countable family of seminorms and a topological space **σ-compact** if it is the (ascending) countable union of compact subsets. Then more or less by definition, $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ is Fréchet if and only if X is σ-compact.

However, we note that a closed subset of a complete normed group is σ-compact if and only if it is separable and locally compact. For example $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ is not Fréchet for a ball $\mathrm{X} \subseteq \mathbb{C}_p^d$.

*Locally analytic functions in $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ on an open domain.* We will show locally analytic functions to be in particular arbitrarily often differentiable, yielding a convenient sufficient criterion for a function to of class $\mathscr{C}^r$ for some real number $r \geq 0$.

**Definition.** Let $\mathrm{X} \subseteq \mathbf{K}^d$ be an open subset. A function $f : \mathrm{X} \to \mathbf{K}$ will be called **locally analytic** if for each point $\boldsymbol{a} \in \mathrm{X}$, there is a closed ball $\mathrm{B} \ni a$ in X such that

$$f(\boldsymbol{x} - \boldsymbol{a}) = \sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} (\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{i}} \quad \text{for all } \boldsymbol{x} \in \mathrm{B};$$

here $a_{\boldsymbol{i}} \in \mathbf{K}$ and $(\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{i}} := (x_1 - a_1)^{i_1} \cdots (x_d - a_d)^{i_d}$ for $\boldsymbol{i} \in \mathbb{N}^d$.

We will use the following notational convention: Let $\mathrm{X}_1 \subseteq \mathbf{K}^{e_1}, \ldots, \mathrm{X}_d \subseteq \mathbf{K}^{e_d}$ be subsets and $f_1 : \mathrm{X}_1 \to \mathbf{E}, \ldots, f_d : \mathrm{X}_d \to \mathbf{E}$ functions on these. Then we denote by $f_1 \odot \cdots \odot f_d : \mathrm{X}_1 \times \cdots \times \mathrm{X}_d \to \mathbf{E}$ the function given by

$$f_1 \odot \cdots \odot f_d(x_1, \ldots, x_d) := f_1(x_1) \cdots f_d(x_d).$$

**Lemma 2.10.** *Let $\mathrm{X}_1, \ldots, \mathrm{X}_d \subseteq \mathbf{K}$ and $f_1 : \mathrm{X}_1 \to \mathbf{E}, \ldots, f_d : \mathrm{X}_d \to \mathbf{E}$ be $d$ mappings. If $\boldsymbol{n} \in \mathbb{N}^d$, then*

$$(f_1 \odot \cdots \odot f_d)^{]\boldsymbol{n}[} = f_1^{]n_1[} \odot \cdots \odot f_d^{]n_d[}.$$

*Proof:* By definition. See [Nag11, Lemma 3.14(ii)]. □

**Lemma 2.11.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ be a cartesian subset and $\boldsymbol{i}, \boldsymbol{n} \in \mathbb{N}^d$.*

*(i) Let $f(x) = *^{\boldsymbol{i}} : \mathrm{X} \to \mathbf{K}$ be defined as the monomial function $\boldsymbol{x} \mapsto \boldsymbol{x}^{\boldsymbol{i}}$. Then*

$$f^{]\boldsymbol{n}[} = \begin{cases} 1, & \text{if } \boldsymbol{i} = \boldsymbol{n}, \\ 0, & \text{if } \boldsymbol{i} \ngeq \boldsymbol{n}. \end{cases}$$

*(ii) Let $f, g\colon \mathrm{X} \to \mathbf{E}$ be two maps with $g$ being δ-constant. Then $(g \cdot f)^{]\boldsymbol{n}[}({}^1x; -; {}^dx) = g(p) \cdot f^{]\boldsymbol{n}[}({}^1x; -; {}^dx)$ on $\mathrm{P}^{]\boldsymbol{n}[}$ with $\mathrm{P} = \mathrm{B}_{\leq\delta}(p)$ independent of the representative $p \in \mathrm{P}$. In particular on $\mathrm{P}^{]\boldsymbol{n}[}$ holds $(g \cdot *^{\boldsymbol{i}})^{]\boldsymbol{n}[} = \begin{cases} g(p), & \text{if } \boldsymbol{i} = \boldsymbol{n}, \\ 0, & \text{if } \boldsymbol{i} \ngeq \boldsymbol{n}. \end{cases}$*

*Proof:* Ad (i): By Lemma 2.10 and [Nag12, Lemma 2.19], we have

$$f^{]\boldsymbol{n}[} = *^{i_1]n_1[} \odot \cdots \odot *^{i_d]n_d[} = \begin{cases} 1, & \text{if } i_1 = n_1, \ldots, i_d = n_d, \\ 0, & \text{if } i_k < n_k \text{ for some } k \in \{1, \ldots, d\}. \end{cases}$$

Ad (ii): We have $g_{|\mathrm{P}} = g(p)$ for some representative $p \in \mathrm{P}$. By linearity of $f \mapsto f^{]\boldsymbol{n}[}$, we obtain $(g \cdot f)^{]\boldsymbol{n}[}({}^1x; -; {}^dx) = g(p) \cdot f^{]\boldsymbol{n}[}(x)$. Now apply (i). □

**Proposition 2.12.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ be an open subset. A locally analytic function $f\colon \mathrm{X} \to \mathbf{K}$ is a $\mathscr{C}^r$-function for any $r \in \mathbb{R}_{\geq 0}$.*

*Proof:* By Lemma 2.6, we find $\mathscr{C}^r(\mathrm{X}, \mathbf{K}) \supseteq \mathscr{C}^{\nu+1}(\mathrm{X}, \mathbf{K})$. Hence we may assume $r = \nu \in \mathbb{N}$. Since being $\mathscr{C}^r$ is a local property, it suffices to prove this for an analytic function $f\colon \mathrm{X} \to \mathbf{K}$ on a closed ball $\mathrm{X} \subseteq \mathbf{K}^d$ whose radius we may assume to lie in $|\mathbf{K}^*|_{\leq 1}$. Let $\mathrm{X} = \mathrm{B}_{\leq\varepsilon}(\boldsymbol{a})$ with $\boldsymbol{a} \in \mathrm{X}$ and $\varepsilon > 0$. For notational convenience, let us assume $\boldsymbol{a} = 0$. Our function $f$ is now defined as

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \quad \text{for all } \boldsymbol{x} \in \mathbf{K}^d \text{ with } \|\boldsymbol{x}\| \leq \varepsilon.$$

Since this power series converges for all $\boldsymbol{x}$ with $\|x\| = \varepsilon \in |\mathbf{K}^*|$, we find $|a_{\boldsymbol{i}}|\varepsilon^{|\boldsymbol{i}|} \to 0$ as $|\boldsymbol{i}| \to \infty$. It suffices to prove $|*^{\boldsymbol{i}[\boldsymbol{n}]}(\boldsymbol{x})| \leq \varepsilon^{|\boldsymbol{i}|-\nu}$ for all $\boldsymbol{x} \in \mathrm{X}^{[\boldsymbol{n}]}$ with $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$: Then uniformly in all compact cartesian $\mathrm{C} \subseteq \mathrm{X}$ holds $\|a_{\boldsymbol{i}} *^{\boldsymbol{i}}\|_{\mathscr{C}^\nu, \mathrm{C}} \leq |a_{\boldsymbol{i}}|\varepsilon^{|\boldsymbol{i}|}/\varepsilon^\nu \to 0$ as $|\boldsymbol{i}| \to \infty$, hence $f = \sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$ as a convergent sum in $\mathscr{C}^\nu(\mathrm{X}, \mathbf{K})$ by completeness.

Let $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$. By Lemma 2.11(i), we find $*^{\boldsymbol{i}[\boldsymbol{n}]} = 0$ if $\boldsymbol{i} \ngeq \boldsymbol{n}$. Otherwise, by Lemma 2.10 holds

$$|*^{\boldsymbol{i}[\boldsymbol{n}]}(\boldsymbol{x})| = |*^{i_1[n_1]}({}^1\boldsymbol{x})| \cdots |*^{i_d[n_d]}({}^d\boldsymbol{x})| \quad \text{for all } \boldsymbol{x} = ({}^1\boldsymbol{x}; -; {}^d\boldsymbol{x}) \in \mathrm{X}^{[\boldsymbol{n}]}.$$

We are hence reduced to proving $|*^{i[n]}(x)| \leq \varepsilon^{i-n}$ for $i \geq n \in \mathbb{N}$ and $x \in \mathrm{X}^{[n]}$ with $\mathrm{X} := \mathrm{B}_{\leq\varepsilon}(0)$, as then

$$|*^{\boldsymbol{i}[\boldsymbol{n}]}(\boldsymbol{x})| \leq \varepsilon^{i_1-n_1} \cdots \varepsilon^{i_d-n_d} = \varepsilon^{|\boldsymbol{i}|-|\boldsymbol{n}|} \leq \varepsilon^{|\boldsymbol{i}|-\nu} \quad \text{if } \boldsymbol{i} \geq \boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}.$$

Let $*^i = g \cdot h$ with $g = *^{i-1}$ and $h = *$. By the product rule [Sch84, Lemma 29.2(v)], we find

$$*^{i]n[}(x_0, \ldots, x_n) = g^{]n-1[}(x_0, \ldots, x_{n-1}) + g^{]n[}(x_0, \ldots, x_n)x_0.$$

As $|x| \leq \varepsilon$, it follows by induction on $i \geq 0$ that

$$|*^{i[n]}(x)| \leq \varepsilon^{(i-1)-(n-1)} \vee \varepsilon^{i-1-n} \cdot \varepsilon = \varepsilon^{i-n} \quad \text{for all } x \in \mathrm{X}^{[n]}.$$

## $\mathscr{C}^r$-manifolds

In this section, we want to introduce a well-defined notion of a $\mathscr{C}^r$-manifold. Crucial will be the local nature of the property of $r$-fold differentiability as well as the closure under composition of the space of $\mathscr{C}^r$-functions, which is also of independent interest.

*Composition properties of $\mathscr{C}^r$-functions.* This subsection is devoted to the proof of the closure under composition of the space of $\mathscr{C}^r$-functions (At least if $r \geq 1$). This essentially uses the viewpoint of $\mathscr{C}^r$-fold differentiability as iterated approachability of their difference quotients by linear maps, where their composition becomes a matrix product. As $\mathscr{C}^\rho$-functions are closed under *multiplication*, the closure of $\mathscr{C}^r$-functions under composition for $r \geq 1$ is inferred.

**Lemma 2.13.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point and $f \colon \mathrm{X} \to \mathbf{E}$ a mapping thereon. Let $n \leq r$ be a nonnegative integer. Then $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ if and only if $f \in \mathscr{C}^n(\mathrm{X}, \mathbf{E})$ and $f^{[\boldsymbol{n}]} \in \mathscr{C}^{r-n}(\mathrm{X}^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$. Moreover $\|f\|_{\mathscr{C}^r,\mathrm{C}} = \|f\|_{\mathscr{C}^n,\mathrm{C}} \vee \max_{\boldsymbol{n}\in\mathbb{N}^d_{=n}} \|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^{r-n},\mathrm{C}^{[\boldsymbol{n}]}}$.*

*Proof:* For a proof, see [Glö07, Remark 2.5] or [Nag11, Lemma 3.19]. □

*Remark.* This is informed by the viewpoint of $f^{[\boldsymbol{n}]}$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ as the $\boldsymbol{n}$-th column-vector of the $\nu$-th iterated difference quotient $f^{[\nu]}$ of $f \colon \mathrm{X} \to \mathbf{K}$, up to reduction by symmetry in the coordinates of $f^{[\nu]}$. Here for example, $f^{[0]} = f$, then $f^{[1]}$ is to be understood in the sense of Proposition 1.16, and if $f^{[1]} \colon \mathrm{X}^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E}) = \mathbf{E}^d$ exists, we let

$$f^{]2[} = (f^{[1]})^{]1[} \colon (\mathrm{X}^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d \times \mathbf{K}^d, \mathbf{E}), \mathbf{E}) = \mathbf{E}^{d^2}.$$

If existent, the unique continuous extension of $f^{]2[} \colon (\mathrm{X}^{[1]})^{]1[} \to \mathbf{E}$ to all of $\mathrm{X}^{[2]} = \mathrm{X}^{[1]} \times \mathrm{X}^{[1]}$ is then denoted $f^{[2]}$. This was also discussed in the introduction.

In the following, we will tacitly use that projection functions are $\mathscr{C}^r$-functions for any $r \geq 0$, a convenient criterion for this given by Proposition 2.12. Recall that by Remark 2.4(iii) a cartesian product of $\mathscr{C}^r$-functions is again a $\mathscr{C}^r$-function.

Let us henceforth abbreviate $(x;t)$ by $(x_1; \ldots; x_{k-1}; x_k + t, x_k; x_{k+1}; \ldots; x_d) \in \mathrm{X}^{[\boldsymbol{e}_k]}$.

**Lemma 2.14.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ and $\mathrm{Y} \subseteq \mathbf{K}^e$ be nonempty cartesian subsets whose factors contain no isolated point. Let $f \in \mathscr{C}^1(\mathrm{X}, \mathbf{K}^e)$ and $g \in \mathscr{C}^1(\mathrm{Y}, \mathbf{E})$ be two functions with $\operatorname{im} f \subseteq \mathrm{Y}$. Then $g \circ f \in \mathscr{C}^1(\mathrm{X}, \mathbf{E})$, seen by the matrix product*

$$g \circ f^{[\boldsymbol{e}_k]}(x;t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[\boldsymbol{e}_k]}(x;t) \quad \textit{for } k = 1, \ldots, d \textit{ and } (x;t) \in \mathrm{X}^{[\boldsymbol{e}_k]}.$$

*Proof:* Let $k \in \{1, \ldots, d\}$. We have to prove that the above equation's right-hand side composed linear map $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[\boldsymbol{e}_k]}(x;t) \colon \mathbf{K} \to \mathbf{E}$ sends any $t \in \mathbf{K}^*$ such that $x + t \cdot \boldsymbol{e}_k, x \in \mathrm{X}$ to $g \circ f(x + t\boldsymbol{e}_k) - g \circ f(x)$. Then by continuity this equality extends to all of $\mathrm{X}^{[\boldsymbol{e}_k]}$.

By definition $f^{[\boldsymbol{e}_k]}(x;t) \cdot t = f(x + t \cdot \boldsymbol{e}_k) - f(x)$ and

$$g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot (f(x + t \cdot \boldsymbol{e}_k) - f(x)) = g(f(x + t \cdot \boldsymbol{e}_k)) - g(f(x)),$$

where we recall Proposition 1.16 for the definition of $g^{]1[}\colon \mathrm{Y}^{[1]} \to \mathbf{E}$ (and use continuous extension). Together, we find

$$\begin{aligned} g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) f^{[\boldsymbol{e}_k]}(x;t) \cdot t &= g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot (f(x + t \cdot \boldsymbol{e}_k) - f(x)) \\ &= g(f(x + t \cdot \boldsymbol{e}_k)) - g(f(x)). \end{aligned} \qquad \square$$

**Lemma 2.15.** *Let* X *and* Y *be compact metric spaces,* $g \in \mathscr{C}^\rho(\mathrm{Y}, \mathbf{E})$ *and* $f \in \mathscr{C}^{\mathrm{lip}}(\mathrm{X}, \mathrm{Y})$. *Then* $\|g \circ f\|_{\mathscr{C}^\rho} \leq (1 \vee |||f^{]1[}|||_{\sup}^\rho) \cdot \|g\|_{\mathscr{C}^\rho} \leq (1 \vee \|f\|_{\mathscr{C}^{\mathrm{lip}}}^\rho) \cdot \|g\|_{\mathscr{C}^\rho}$.

*Proof:* We find $g \circ f \in \mathscr{C}^\rho(\mathrm{X}, \mathbf{E})$ by Proposition 1.3(i). For $x, y \in \mathrm{X}$ with $f(x), f(y) \in \mathrm{Y}$ distinct holds

$$\begin{aligned} |g \circ f^{]\rho[}|(x,y) =& \|g(f(x)) - g(f(y))\| / \|x - y\|^\rho \\ =& \|g(f(x)) - g(f(y))\| / \|f(x) - f(y)\|^\rho \cdot \|f(x) - f(y)\|^\rho / \|x - y\|^\rho \\ =& |g^{]\rho[}|(f(x), f(y)) \cdot |f^{]1[}|(x,y)^\rho. \end{aligned}$$

This equality extends to all other distinct $x, y \in \mathrm{X}$, with zeroes on both sides of the equation. Therefore

$$\|g \circ f\|_{\mathscr{C}^\rho} = \|g \circ f\|_{\sup} \vee |||g \circ f^{]\rho[}|||_{\sup} \leq \|g\|_{\sup} \vee \|g\|_{\mathscr{C}^\rho} |||f^{]1[}|||_{\sup}^\rho \leq (1 \vee \|f\|_{\mathscr{C}^{\mathrm{lip}}}^\rho) \cdot \|g\|_{\mathscr{C}^\rho}.$$

**Lemma 2.16.** *Let* $\mathrm{X} \subseteq \mathbf{K}^d$ *and* $\mathrm{Y} \subseteq \mathbf{K}^e$ *be nonempty cartesian subsets whose factors contain no isolated point. For* $r \geq 1$, *let* $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K}^e)$ *and* $g \in \mathscr{C}^r(\mathrm{Y}, \mathbf{E})$ *be two functions with* $\operatorname{im} f \subseteq \mathrm{Y}$. *Then* $g \circ f^{]\boldsymbol{e}_k[}(x,y)$ *extends to* $g \circ f^{[\boldsymbol{e}_k]} \in \mathscr{C}^{r-1}(\mathrm{X}^{[\boldsymbol{e}_k]}, \mathbf{E})$ *for* $k = 1, \ldots, d$.

*Proof:* We proceed by induction on $\nu \geq 1$. If $\nu = 1$, that is, $r = 1 + \rho$, then by Lemma 2.14 will hold

$$g \circ f^{[\boldsymbol{e}_k]}(x;t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[\boldsymbol{e}_k]}(x;t) \quad \text{for } k = 1, \ldots, d;$$

here the right-hand side meaning the matrix product. By Lemma 1.18, we firstly find $f \in \mathscr{C}^1(\mathrm{X}, \mathbf{K}^e) \subseteq \mathscr{C}^{\mathrm{lip}}(\mathrm{X}, \mathbf{K}^e)$ and by Proposition 1.16, it also holds $g^{[1]} \in \mathscr{C}^\rho(\mathrm{Y}^{[1]}, \mathbf{E})$. The function $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x))$ is therefore again a $\mathscr{C}^\rho$-function by Proposition 1.3(i). Also $f^{[\boldsymbol{e}_k]}$ is a $\mathscr{C}^\rho$-function as $f \in \mathscr{C}^{1+\rho}(\mathrm{X}, \mathbf{K}^e) \subseteq \mathscr{C}^\rho(\mathrm{X}, \mathbf{K}^e)$ by Lemma 2.6

If $\mathrm{B} = (b_j) \in \mathrm{M}_{1 \times e}(\mathbf{E})$ and $\mathrm{A} = (a_i) \in \mathrm{M}_{e \times 1}(\mathbf{K})$ are matrices whose coordinate entries are $\mathscr{C}^\rho$-functions on $\mathrm{X}^{[\boldsymbol{e}_k]}$ into $\mathbf{E} \supseteq \mathbf{K}$, then their matrix product $\mathrm{C} = \mathrm{B} \cdot \mathrm{A}\colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{E}$ will be again a $\mathscr{C}^\rho$-function: For this, note that $\mathrm{C} = a_1 b_1 + \cdots + a_e b_e$. By Proposition 1.3(ii), this sum of products is again a $\mathscr{C}^\rho$-function. Therefore with $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x))$ and $f^{[\boldsymbol{e}_k]}(x;t)$, so is their matrix-product $g \circ f^{[\boldsymbol{e}_k]}(x;t)$ a $\mathscr{C}^\rho$-function.

If $\nu > 1$, then we just saw $g \circ f \in \mathscr{C}^1(\mathrm{X}, \mathbf{E})$ and by Lemma 2.13, we must prove $g \circ f^{[\boldsymbol{e}_k]}$ to be a $\mathscr{C}^{r-1}$-function for $k = 1, \ldots, d$. By Lemma 2.14 holds

$$g \circ f^{[\boldsymbol{e}_k]}(x;t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[\boldsymbol{e}_k]}(x;t). \qquad (*)$$

We have $g^{[1]} = (g^{[\boldsymbol{e}_1]} \circ p_1, \dots, g^{[\boldsymbol{e}_e]} \circ p_e)$ with projection functions $p_l \colon \mathrm{Y}^{[1]} \to \mathrm{Y}^{[\boldsymbol{e}_l]}$ for $l = 1, \dots, e$ and $g^{[\boldsymbol{e}_l]} \in \mathscr{C}^{r-1}(\mathrm{X}^{[\boldsymbol{e}_l]}, \mathbf{E})$ by Lemma 2.13. By the induction hypothesis, $g^{[\boldsymbol{e}_l]} \circ p_l \in \mathscr{C}^{r-1}(\mathrm{Y}^{[1]}, \mathbf{E})$ and hence $g^{[1]} \in \mathscr{C}^{r-1}(\mathrm{Y}^{[1]}, \mathbf{E}^e)$. Moreover $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K}^e) \subseteq \mathscr{C}^{r-1}(\mathrm{X}, \mathbf{K}^e)$. Again by the induction hypothesis, we find $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{E}^e$ to be a $\mathscr{C}^{r-1}$-function. By Lemma 2.13 holds $f^{[\boldsymbol{e}_k]}(x; t) \in \mathscr{C}^{r-1}(\mathrm{X}^{[\boldsymbol{e}_k]}, \mathbf{K}^e)$. By Remark 2.9(i), we find their matrix product $(*)$ to be a $\mathscr{C}^{r-1}$-function. Hence $g \circ f^{[\boldsymbol{e}_k]}$ is a $\mathscr{C}^{r-1}$-function. □

**Corollary 2.17.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ and $\mathrm{Y} \subseteq \mathbf{K}^e$ be nonempty cartesian subsets whose factors contain no isolated point. For $r \geq 1$, let $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K}^e)$ and $g \in \mathscr{C}^r(\mathrm{Y}, \mathbf{E})$ be two functions with $\operatorname{im} f \subseteq \mathrm{Y}$. Then $g \circ f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$.*

*Proof:* By Lemma 2.13, we find that $f^{]\boldsymbol{e}_k[}$ and $g^{]\boldsymbol{e}_k[}$ extend to $f^{[\boldsymbol{e}_k]} \in \mathscr{C}^{r-1}(\mathrm{X}^{[\boldsymbol{e}_k]}, \mathbf{K})$ and $g^{[\boldsymbol{e}_k]} \in \mathscr{C}^{r-1}(\mathrm{Y}^{[\boldsymbol{e}_k]}, \mathbf{E})$ for $k = 1, \dots, d$. By the same token, $f \circ g \in \mathscr{C}^{r-1}(\mathrm{X}, \mathbf{E})$ if and only if $g \circ f^{]\boldsymbol{e}_k[} \colon \mathrm{X}^{]\boldsymbol{e}_k[} \to \mathbf{E}$ extends to a $\mathscr{C}^{r-1}$-function $g \circ f^{[\boldsymbol{e}_k]} \colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{E}$ for $k = 1, \dots, d$. We can conclude by Lemma 2.16. □

**Proposition 2.18.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ and $\mathrm{Y} \subseteq \mathbf{K}^e$ be nonempty locally cartesian subsets whose local factors contain no isolated point. Let $f \colon \mathrm{X} \to \mathbf{K}^e$ and $g \colon \mathrm{Y} \to \mathbf{E}$ be two functions with $\operatorname{im} f \subseteq \mathrm{Y}$. Let $r$ be a nonnegative real number. If $r \geq 1$ and $f$ and $g$ are both $\mathscr{C}^r$-functions, so will be their composition $g \circ f \colon \mathrm{X} \to \mathbf{E}$. If $r < 1$, then the same will hold true provided either $f$ or $g$ is locally Lipschitzian.*

*Proof:* Foremost if $r < 1$, this will hold by Proposition 1.3(i). In case $r \geq 1$, we can by assumption cover Y by nonempty open cartesian subsets $\mathrm{V} \subseteq \mathrm{Y}$ whose factors are free of isolated points. Since $g$ is in particular continuous, their preimages $\tilde{\mathrm{U}} \subseteq \mathrm{X}$ are again open. By assumption on X, we can find a nonempty open cartesian $\mathrm{U} \subseteq \tilde{\mathrm{U}}$ whose factors contain no isolated point. Such U covering X and since being $\mathscr{C}^r$ is a local property, we can restrict to the case X and Y being nonempty cartesian with factors free of isolated points. In this case, Corollary 2.17 yields the result. □

**Proposition 2.19.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ and $\mathrm{Y} \subseteq \mathbf{K}^e$ be nonempty locally cartesian subsets whose local factors contain no isolated point. Let $r$ be a nonnegative real number. Let $f \colon \mathrm{X} \to \mathrm{Y}$ be either of class $\mathscr{C}^r$ if $r \geq 1$ or locally Lipschitzian if $r < 1$. Then the precomposition operator $\mathscr{C}^r(\mathrm{Y}, \mathbf{E}) \ni g \mapsto g \circ f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ is continuous.*

*Proof:* The mapping is well defined by Proposition 2.18. Since the norms are defined on compact cartesian subsets inside nonempty open cartesian subsets whose factors contain no isolated point, we can reduce to the case that $\mathrm{X} \subseteq \mathbf{K}^d$ and $\mathrm{Y} \subseteq \mathbf{K}^e$ are nonempty cartesian subsets whose factors contain no isolated point. Let $\mathrm{C} \subseteq \mathrm{X}$ be compact cartesian. Then $f(\mathrm{C}) \subseteq \mathrm{Y}$ is again compact and we let $\mathrm{D} \supseteq f(\mathrm{C})$ be compact cartesian in Y, which exists as Y is cartesian. Then in any case $\|g \circ f\|_{\mathrm{C}} \leq \|g\|_{\mathrm{D}}$.

Foremost if $r = \rho < 1$, then $f \in \mathscr{C}^{\mathrm{lip}}(\mathrm{X}, \mathrm{Y})$ and by Lemma 2.15 will hold

$$\|g \circ f\|_{\mathscr{C}^\rho, \mathrm{C}} \leq (\|f\|^{\rho}_{\mathscr{C}^{\mathrm{lip}}, \mathrm{C}} \vee 1) \cdot \|g\|_{\mathscr{C}^\rho, \mathrm{D}},$$

proving continuity in case $r < 1$. If $r \geq 1$, we will prove by induction on $\nu \geq 1$ that $\|g \circ f\|_{\mathscr{C}^r, \mathrm{C}} \leq \mathrm{M} \cdot \|g\|_{\mathscr{C}^r, \mathrm{D}}$ for a constant $\mathrm{M} = \mathrm{M}(f, \mathrm{C}, r) \geq 1$ depending solely on $r \geq 0$, $\mathrm{C} \subseteq \mathrm{X}$ and $f \in \mathscr{C}^r(\mathrm{X}, \mathrm{Y})$. First off, we find by Lemma 2.14 that

$$g \circ f^{[\boldsymbol{e}_k]}(x; t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[\boldsymbol{e}_k]}(x; t) \quad \text{for } k = 1, \ldots, d.$$

We assume for convenience the operator norm of the multiplication mapping in $\mathbf{E}$ to be equal to 1. Defining $\mathrm{F}^{[\boldsymbol{e}_k]} \in \mathscr{C}^r(\mathrm{X}^{[\boldsymbol{e}_k]}, \mathrm{Y}^{[1]})$ by $(x; t) \mapsto (f(x + t \cdot \boldsymbol{e}_k), f(x))$, it therefore holds by the continuity of multiplication, cf. Remark 2.9(i), that

$$\max_{k=1,\ldots,d} \|[g \circ f]^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \leq \max_{k=1,\ldots,d} \|g^{[1]} \circ \mathrm{F}^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \|f^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \tag{$*$}$$

We also have by Lemma 2.13, for a general cartesian subset $\mathrm{X} \subseteq \mathbf{K}^d$ with factors free of isolated points and $\mathrm{C} \subseteq \mathrm{X}$ compact cartesian

$$\|h\|_{\mathscr{C}^r, \mathrm{C}} = \|h\|_{\mathrm{C}} \vee \max_{k=1,\ldots,d} \|h^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \quad \text{for any } h \in \mathscr{C}^r(\mathrm{X}, \mathbf{E}). \tag{$**$}$$

We can now turn to the starting case $\nu = 1$, that is, $r = 1 + \rho$. We compute

$$\begin{aligned}
\|g \circ f\|_{\mathscr{C}^r, \mathrm{C}} =& \|g \circ f\|_{\mathrm{C}} \vee \max_{k=1,\ldots,d} \|[g \circ f]^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^\rho, \mathrm{C}^{[\boldsymbol{e}_k]}} \\
\leq& \|g\|_{\mathrm{D}} \vee \max_{k=1,\ldots,d} \|g^{[1]} \circ \mathrm{F}^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^\rho, \mathrm{C}^{[\boldsymbol{e}_k]}} \|f^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^\rho, \mathrm{C}^{[\boldsymbol{e}_k]}} \\
\leq& \|g\|_{\mathrm{D}} \vee [\max_{k=1,\ldots,d} (\|\mathrm{F}^{[\boldsymbol{e}_k]}\|^\rho_{\mathscr{C}^{\mathrm{lip}}, \mathrm{C}^{[\boldsymbol{e}_k]}} \vee 1)] \cdot \|g^{[1]}\|_{\mathscr{C}^\rho, \mathrm{D}^{[1]}} \|f\|_{\mathscr{C}^r, \mathrm{C}} \\
=& \|g\|_{\mathrm{D}} \vee [\max_{k=1,\ldots,d} \mathrm{M}(\mathrm{F}^{[\boldsymbol{e}_k]}, \mathrm{C}^{[\boldsymbol{e}_k]}, \rho)] \|f\|_{\mathscr{C}^r, \mathrm{C}} \cdot \|g^{[1]}\|_{\mathscr{C}^\rho, \mathrm{D}^{[1]}} \\
\leq& \mathrm{M} \cdot \|g\|_{\mathscr{C}^r, \mathrm{D}};
\end{aligned}$$

where we put $\mathrm{M}(\mathrm{F}^{[\boldsymbol{e}_k]}, \mathrm{C}^{[\boldsymbol{e}_k]}, \rho) := \|\mathrm{F}^{[\boldsymbol{e}_k]}\|^\rho_{\mathscr{C}^{\mathrm{lip}}, \mathrm{C}^{[\boldsymbol{e}_k]}} \vee 1 \geq 1$ for $k = 1, \ldots, d$ and accordingly $\mathrm{M} := 1 \vee [\max_{k=1,\ldots,d} \mathrm{M}(\mathrm{F}^{[\boldsymbol{e}_k]}, \mathrm{C}^{[\boldsymbol{e}_k]}, \rho)] \cdot \|f\|_{\mathscr{C}^r, \mathrm{C}} \geq 1$.

Here the first equality by definition, the following inequality by Inequality $(*)$ and the next one by the case $r = \rho < 1$ just observed (as well as $\|f^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^\rho, \mathrm{C}^{[\boldsymbol{e}_k]}} \leq \|f\|_{\mathscr{C}^r, \mathrm{C}}$ for $k = 1, \ldots, d$ by definition) Finally the last inequality follows through Proposition 1.16 by

$$\|g^{[1]}\|_{\mathscr{C}^\rho, \mathrm{D}^{[1]}} = \|g^{[\boldsymbol{e}_1]}\|_{\mathscr{C}^\rho, \mathrm{D}^{[\boldsymbol{e}_1]}} \vee \ldots \vee \|g^{[\boldsymbol{e}_e]}\|_{\mathscr{C}^\rho, \mathrm{D}^{[\boldsymbol{e}_e]}} \leq \|g\|_{\mathscr{C}^r, \mathrm{D}}.$$

This settles the case $\nu = 1$. Let $\nu > 1$. Then we compute similarly

$$\begin{aligned}
\|g \circ f\|_{\mathscr{C}^r, \mathrm{C}} =& \|g \circ f\|_{\mathrm{C}} \vee \max_{k=1,\ldots,d} \|[g \circ f]^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \\
\leq& \|g\|_{\mathrm{D}} \vee \max_{k=1,\ldots,d} \|g^{[1]} \circ \mathrm{F}^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \|f^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}} \\
\leq& \|g\|_{\mathrm{D}} \vee (\max_{k=1,\ldots,d} \|g^{[1]} \circ \mathrm{F}^{[\boldsymbol{e}_k]}\|_{\mathscr{C}^{r-1}, \mathrm{C}^{[\boldsymbol{e}_k]}}) \|f\|_{\mathscr{C}^r, \mathrm{C}} \\
\leq& \|g\|_{\mathrm{D}} \vee ([\max_{k=1,\ldots,d} \mathrm{M}(\mathrm{F}^{[\boldsymbol{e}_k]}, \mathrm{C}^{[\boldsymbol{e}_k]}, r-1)] \cdot \|f\|_{\mathscr{C}^r, \mathrm{C}}) \|g^{[1]}\|_{\mathscr{C}^{r-1}, \mathrm{D}^{[1]}} \\
\leq& \mathrm{M} \cdot \|g\|_{\mathscr{C}^r, \mathrm{D}},
\end{aligned}$$

with $M := 1 \vee ([\max_{k=1,\dots,d} M(F^{[e_k]}, C^{[e_k]}, r-1)] \cdot \|f\|_{\mathscr{C}^r,C}) \cdot \tilde{M} \geq 1$ and the constant $\tilde{M} \geq 1$ defined below. Here the first equality by Equality (∗∗), the following inequality by Inequality (∗) and the one thereafter by Equality (∗∗).

The penultimate inequality is obtained by the induction hypothesis for $\nu - 1$.

The last inequality follows by

$$\|g^{[1]}\|_{\mathscr{C}^{r-1},D^{[1]}} = \max_{l=1,\dots,e} \|g^{[e_l]} \circ p_l\|_{\mathscr{C}^{r-1},D^{[1]}}$$
$$\leq \max_{l=1,\dots,e} \tilde{M}(p_l, Y^{[1]}, r-1) \cdot \|g^{[e_l]}\|_{\mathscr{C}^{r-1},D^{[e_l]}} \leq \tilde{M} \cdot \|g\|_{\mathscr{C}^r,D}$$

for projection functions $p_l \colon Y^{[1]} \to Y^{[e_l]}$ for $l = 1, \dots, e$ and $\tilde{M}(p_l, Y^{[1]}, r-1) \geq 1$ given the induction hypothesis, and putting $\tilde{M} := \max_{l=1,\dots,d} \tilde{M}(p_l, Y^{[1]}, r-1)$. The last inequality by Equality (∗∗). □

*$\mathscr{C}^r$-manifolds.* In this subsubsection we want to introduce the notion of a $\mathscr{C}^r$-function on a $\mathscr{C}^r$-manifold. The independence of the notion of a $\mathscr{C}^r$-function on a $\mathscr{C}^r$-manifold rests on the local nature of the concept of differentiability employed as well as the just established closure under composition of the space of $\mathscr{C}^r$-functions.

**Definition.** Let M be a topological Hausdorff space. We say that M is a **topological manifold** of dimension $d$ or a topological $d$-manifold if for every point $x \in M$, we can find a **chart** $(U, \phi)$ consisting of:

(i) an open set $U \subseteq M$ containing $x$, and

(ii) a map $\phi \colon U \to \mathbf{K}^d$ such that $\phi(U)$ is open in $\mathbf{K}^d$ and $\phi \colon U \to \phi(U)$ is a homeomorphism.

Assume that we have a notion of $\mathscr{C}^*$-function on open subsets in $\mathbf{K}^d$, that is, for all open subsets $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ we have the subset $\mathscr{C}^*(X, Y) \subseteq Y^X$ of all $\mathscr{C}^*$-functions $f \colon X \to Y$.

**Definition.** We will say that two charts $(U, \phi)$ and $(V, \psi)$ are **$\mathscr{C}^*$-compatible** if both maps

$$\mathbf{K}^d \supseteq \phi(U \cap V) \underset{\phi\circ\psi^{-1}}{\overset{\psi\circ\phi^{-1}}{\rightleftarrows}} \psi(V \cap U) \subseteq \mathbf{K}^d$$

are $\mathscr{C}^*$-functions.

**Definition.** A **$\mathscr{C}^*$-atlas** for M is a set $\mathscr{A} = \{(U_i, \phi_i)\}$ of charts on M such that any two of these are compatible and which covers M.

**Definition.** Let M be a topological $d$-manifold and $\mathscr{A} = \{(U_i, \phi_i) \colon i \in I\}$ a $\mathscr{C}^*$-atlas of M. Then we will say that a function $f \colon M \to \mathbf{E}$ is a $\mathscr{C}^*$-function with respect to $\mathscr{A}$, if

$$f \circ \phi_i^{-1} \in \mathscr{C}^*(\phi_i(U_i), \mathbf{E}) \quad \text{for all } i \in I.$$

**Definition.** We will call an atlas **maximal** if it is not contained in any strictly larger atlas.

*Remark.* Equivalently an atlas $\mathscr{A}_0$ is maximal, if each chart on M compatible with every chart in $\mathscr{A}_0$ will be already in $\mathscr{A}_0$.

**Proposition 2.20.** *Let* M *be a topological d-manifold endowed with a $\mathscr{C}^*$-atlas. We assume that:*

- *The class of $\mathscr{C}^*$-functions is closed under composition: That is, if* $U \subseteq \mathbf{K}^d, V \subseteq \mathbf{K}^e$ *and* $W \subseteq \mathbf{K}^f$ *are open subsets, and* $f \in \mathscr{C}^*(U, V), g \in \mathscr{C}^*(V, W)$ *then* $g \circ f \in \mathscr{C}^*(U, W)$.
- *The $\mathscr{C}^*$-property is local: That is, if* $\{U_i : i \in I\}$ *is a cover by open sets of* $U \subseteq \mathbf{K}^d$ *open and* $f : U \to \mathbf{E}$ *is such that* $f_{|U_i}$ *is a $\mathscr{C}^*$-function for all* $i \in I$*, then* $f$ *is a $\mathscr{C}^*$-function.*

*Then we find that:*

(i) *The manifold* M *has a maximal atlas* $\mathscr{A}_0$.

(ii) *The domains* $\{U\}$ *of all charts* $(U, \phi)$ *in* $\mathscr{A}_0$ *form a topological basis of* M.

(iii) *A function* $f : M \to \mathbf{E}$ *is a $\mathscr{C}^*$-function with respect to* any *atlas* $\mathscr{A} \subseteq \mathscr{A}_0$ *if and only if it is a $\mathscr{C}^*$-function with respect to* the maximal *atlas* $\mathscr{A}_0$.

*Proof:* Ad 1.: We firstly show the existence of the maximal atlas $\mathscr{A}_0$. Let $\mathscr{A}$ be an atlas on M whose existence we assume. We put

$$\mathscr{A}_0 = \{ \text{ all charts } (U, \phi) \text{ compatible with every chart in } \mathscr{A} \ \}.$$

We will show that $\mathscr{A}_0$ is an atlas on M. Then by definition, it will be maximal. For this, it remains to prove that every two charts $(U, \phi)$ and $(V, \psi)$ on M which are compatible with every chart in $\mathscr{A}$ are itself compatible: Let $x \in U \cap V$. By localness, we have to show that there is a neighborhood $W \subseteq U \cap V$ of $x$ such that

$$\mathbf{K}^d \supseteq \phi(W) \underset{\phi\circ\psi^{-1}}{\overset{\psi\circ\phi^{-1}}{\rightleftarrows}} \psi(W) \subseteq \mathbf{K}^d$$

are $\mathscr{C}^*$-functions. Let $(\tilde{W}, \tilde{\theta})$ be a chart in $\mathscr{A}$ with $\tilde{W} \ni x$. Put $W = \tilde{W} \cap (U \cap V)$. Then the maps $\phi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are by the assumed compatibility $\mathscr{C}^*$-functions. Therefore $\phi \circ \psi^{-1} = (\phi \circ \tilde{\theta}^{-1}) \circ (\theta^{-1} \circ \psi)$ is by closure under composition a $\mathscr{C}^*$-function on $\psi(W)$. By symmetry, we also have that $\psi \circ \phi^{-1}$ is a $\mathscr{C}^*$-function on $\phi(W)$.

Ad 2.: We now show that the domains of the charts in $\mathscr{A}_0$ form a topological basis of M. Let $U \subseteq M$ be an open subset. We have to show that for any point $x \in U$ we find a chart $(U_x, \phi_x)$ in $\mathscr{A}_0$ such that $x \in U_x \subseteq U$. Let $(\tilde{U}_x, \tilde{\phi}_x) \in \mathscr{A}_0$ be a chart with $\tilde{U}_x \ni x$. Then we put $U_x := \tilde{U}_x \cap U$ and $\phi_x := \tilde{\phi}_{x|U_x}$. Then clearly $(U_x, \phi_x)$ is a chart such that $x \in U_x \subseteq U$. Because $(U_x, \phi_x)$ is the restriction of a chart $(\tilde{U}_x, \tilde{\phi}_x)$ in the atlas $\mathscr{A}_0$ and hence being compatible with every chart in $\mathscr{A}_0$, we find it to be compatible with any chart in $\mathscr{A}_0$. Since $\mathscr{A}_0$ is maximal we just observed above that $(U_x, \phi_x) \in \mathscr{A}_0$.

Ad 3.: We assume that $f : M \to \mathbf{E}$ is a $\mathscr{C}^*$-function with respect to an atlas $\mathscr{A} \subseteq \mathscr{A}_0$. We have to show that $f \circ \phi$ is a $\mathscr{C}^*$-function with respect to any chart $(U, \phi) \in \mathscr{A}_0$ on M. By assumption, we find a cover by charts $\{(U_i, \psi_i) : i \in I\} \subseteq \mathscr{A}$ of U. We may assume $U_i \subseteq U$. By localness, it suffices to check that $f \circ \phi_{|U_i}$ is a $\mathscr{C}^*$-function for every $i \in I$. We have $\phi_{|U_i} = \psi_i \circ (\psi^{-1} \circ \phi_{|U_i})$. Because $(\psi_i, U_i)$ and $(\phi, U)$ in $\mathscr{A}_0$ are compatible, we find the (left|right) hand-map $(\psi^{-1} \circ \phi_{|U_i})$ to be a $\mathscr{C}^*$-function. By assumption $f \circ \psi_i$ is a $\mathscr{C}^*$-function. By closure under composition, therefore $f \circ \phi_{|U_i}$ is a $\mathscr{C}^*$-function. □

**Example 2.21.** Let $r \in \mathbb{R}_{\geq 0}$ and let $\mathscr{C}^r$ be the notion of $r$-fold differentiability, that is, for all open subsets $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$, we let $\mathscr{C}^r(X, Y) \subseteq Y^X$ be the subset of $\mathscr{C}^r$-functions as given in Definition 2.2. Then in the sense of Proposition 2.20, we find this notion for $r \geq 1$ to be local and closed under composition.

*Proof:* We check that:

(i) By Definition 2.2, the $\mathscr{C}^r$-property is defined pointwise, in particular it is local.

(ii) If $r \geq 1$, the $\mathscr{C}^r$-functions are by Proposition 2.18 closed under composition. □

**Theorem 2.22.** *Let $r \geq 1$ and let M be a topological $d$-manifold endowed with a $\mathscr{C}^r$-atlas. Then we find that:*

(i) *The manifold M has a maximal atlas $\mathscr{A}_0$.*

(ii) *The domains $\{U\}$ of all charts $(U, \phi)$ in $\mathscr{A}_0$ form a topological basis of M.*

(iii) *A function $f : M \to \mathbf{E}$ is a $\mathscr{C}^r$-function with respect to* any *atlas $\mathscr{A} \subseteq \mathscr{A}_0$ if and only if it is a $\mathscr{C}^r$-function with respect to* the maximal *atlas $\mathscr{A}_0$.*

*Proof:* By the preceding Example 2.21, we can apply Proposition 2.20. □

*Remark.*

(i) Let us call a pair $(M, \mathscr{A}_0)$ of a topological manifold M and a maximal atlas $\mathscr{A}_0$ on M a **$\mathscr{C}^*$-manifold**. Then the preceding Theorem 2.22 says that any $\mathscr{C}^r$-atlas on a topological manifold M gives rise to a reasonable notion of $\mathscr{C}^r$-manifold in the sense that the property of any mapping on M to be a $\mathscr{C}^r$-function does not depend on the particular choice of atlas whereon it is tested.

(ii) If $r < 1$, then we still have a good notion of $\mathscr{C}^r$-functions on $\mathscr{C}^{\text{lip}}$-manifolds: That is, let $(M, \mathscr{A}_0)$ be a $\mathscr{C}^{\text{lip}}$-manifold. Then as in Proposition 2.20, by the same arguments one can characterize a function $f : M \to \mathbf{E}$ to be a $\mathscr{C}^r$-function if it is a $\mathscr{C}^r$-function with respect to any atlas $\mathscr{A} \subseteq \mathscr{A}_0$.

## The space of distributions on $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ for a compact group X

Let $\mathscr{D}(\mathrm{X}, \mathbf{K}) = \varinjlim_{r \geq 0} \mathscr{D}^r(\mathrm{X}, \mathbf{K})$ with $\mathscr{D}^r(\mathrm{X}, \mathbf{K})$ defined as the continuous dual of $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$. Then by density of $\mathscr{C}^s(\mathrm{X}, \mathbf{K}) \subseteq \mathscr{C}^r(\mathrm{X}, \mathbf{K})$ for $s \geq r$ (a consequence of the later Proposition 2.40) these transitions maps are injective and we obtain a natural filtration of $\mathscr{D}(\mathrm{X}, \mathbf{K})$ by the degree of differentiability.

From this viewpoint, this subsubsection is devoted to proving that the convolution product comports well with respect to this filtration, so that $\mathscr{D}(\mathrm{X}, \mathbf{K})$ is a filtered $\mathbf{K}$-algebra.

**Definition.** For a compact locally cartesian subset $\mathrm{X} \subseteq \mathbf{K}^d$ with local factors free of isolated points, we define the $\mathbf{K}$-vector space $\mathscr{D}^r(\mathrm{X}, \mathbf{K})$ of **distributions** by

$$\mathscr{D}^r(\mathrm{X}, \mathbf{K}) = \{\text{all continuous } \mathbf{K}\text{-linear mappings } \mu\colon \mathscr{C}^r(\mathrm{X}, \mathbf{E}) \to \mathbf{K}\}.$$

We endow $\mathscr{D}^r(\mathrm{X}, \mathbf{K})$ with the structure of a complete topological $\mathbf{K}$-vector space by the **operator norm** $\|\cdot\|_{\mathscr{D}^r}$ defined on $\mathscr{D}^r(\mathrm{X}, \mathbf{K})$ by

$$\|\mu\|_{\mathscr{D}^r} = \inf\{\mathrm{C} \in \mathbb{R}_{\geq 0} : |\mu(f)| \leq \mathrm{C} \cdot \|f\|_{\mathscr{C}^r} \quad \text{for all } f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})\}.$$

We remark firstly that by [vR78, Chapter III, Section "Linear Operators"], the operator norm $\|\cdot\|_{\mathscr{D}^r}$ is well defined, as a $\mathbf{K}$-linear operator is continuous if and only if it is *bounded* - meaning the existence of such a largest lower bound C. Secondly, by [vR78, Exercise 3.M(i)], this normed $\mathbf{K}$-vector space is complete with respect to $\|\cdot\|_{\mathscr{D}^r}$ as $\mathbf{K}$ is.

We want to define a convolution product for $\mathscr{C}^r$-distributions on compact groups. For this a couple of technical preparations are in order.

**Definition.** Let $\mathrm{X} = \mathrm{X}' \times \mathrm{X}'' \subseteq \mathbf{K}^{d'} \times \mathbf{K}^{d''}$ be a nonempty compact cartesian subset whose factors have no isolated points and $f\colon \mathrm{X}' \times \mathrm{X}'' \to \mathbf{E}$ a mapping thereon. We consider $\mathbf{K}^{d'} \times \mathbf{K}^{d''}$ as direct sum $\mathrm{V}' \oplus \mathrm{V}''$ with $\mathrm{V}' = \mathbf{K}^{d'}$ and $\mathrm{V}'' = \mathbf{K}^{d''}$. Then we define $f$ to lie in $\mathscr{C}^{\rho' \otimes \rho''}(\mathrm{X}' \times \mathrm{X}'', \mathbf{E})$ for $\rho', \rho'' \in [0, 1[$, if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in \mathrm{X}$ and $h' \in \mathrm{V}', h'' \in \mathrm{V}''$ of norm at most $\delta$ holds, where defined,

$$\|[f(x + h' + h'') - f(x + h'')] - [f(x + h') - f(x)]\| \leq \varepsilon \|h'\|^{\rho'} \|h''\|^{\rho''}.$$

**Lemma 2.23.** *Let $\mathrm{X} = \mathrm{X}' \times \mathrm{X}'' \subseteq \mathbf{K}^{d'} \times \mathbf{K}^{d''} = \mathrm{V}' \oplus \mathrm{V}''$ be a compact cartesian subset whose factors have no isolated points and $f \in \mathscr{C}^{\rho}(\mathrm{X}, \mathbf{E})$ for $\rho \in [0, 2[$. Then $f \in \mathscr{C}^{\rho' \otimes \rho''}(\mathrm{X}' \times \mathrm{X}'', \mathbf{E})$ for all $\rho', \rho'' \in [0, 1[$ with $\rho' + \rho'' \leq \rho$.*

*Proof:* We distinguish two cases.

Case 1: $\rho < 1$. By compactness, there is a $\delta > 0$ such that $\|f(x + h) - f(x)\| \leq \varepsilon \|h\|^{\rho}$ for all $x + h, x \in \mathrm{X}$ with $\|h\| \leq \delta$. Applying this to $h = h'$ in $\mathrm{V}'$ respectively $h = h'' \in \mathrm{V}''$, the non-Archimedean triangle inequality yields

$$\|[f(x + h' + h'') - f(x + h'')] - [f(x + h') - f(x)]\| \leq \varepsilon \|h'\|^{\rho} \wedge \varepsilon \|h''\|^{\rho} \leq \varepsilon \|h'\|^{\rho'} \|h''\|^{\rho''}$$

for all $x \in \mathrm{X}$ and $h' \in \mathrm{V}', h'' \in \mathrm{V}''$ of norm at most $\delta$ such that the above term is defined.

Case 2: $\rho \geq 1$. By Proposition 1.16, we found $f \in \mathscr{C}^{1+\rho}(\mathrm{X}, \mathbf{E})$ if and only if $f^{]1[}: \mathrm{X}^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathrm{V}' \oplus \mathrm{V}'', \mathbf{E})$ extends to a $\mathscr{C}^{\rho}$-function $f^{[1]}: \mathrm{X}^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathrm{V}' \oplus \mathrm{V}'', \mathbf{E})$. By definition, we find by continuous extension $f^{[1]}(x+h, x) \cdot h = f(x+h) - f(x)$ for all $x+h, x \in \mathrm{X}$. In particular for $h' \in \mathrm{V}', h'' \in \mathrm{V}''$ holds, where defined,

$$\begin{aligned} &[f(x+h'+h'') - f(x+h'')] - [f(x+h') - f(x)] \\ =&f^{[1]}(x+h'+h'', x+h'') \cdot h' - f^{[1]}(x+h', x) \cdot h' \\ =&[f^{[1]}(x+h'+h'', x+h'') - f^{[1]}(x+h', x)] \cdot h'. \end{aligned}$$

By compactness, there is $\delta > 0$ such that $\|f^{[1]}(\tilde{x}, x) - f^{[1]}(\tilde{y}, y)\| \leq \varepsilon \|(\tilde{x}, x) - (\tilde{y}, y)\|^{\rho}$ for all $(\tilde{x}, x), (\tilde{y}, y) \in \mathrm{X}^{[1]} = \mathrm{X} \times \mathrm{X}$ with $\|(\tilde{x}, x) - (\tilde{y}, y)\| \leq \delta$. In particular

$$\begin{aligned} &\|[f(x+h'+h'') - f(x+h'')] - [f(x+h') - f(x)]\| \\ \leq&\|f^{[1]}(x+h'+h'', x+h'') - f^{[1]}(x+h', x)\| \|h'\| \\ \leq&\varepsilon \cdot \|h''\|^{\rho} \|h'\| && \text{if } \|h''\| \leq \delta. \end{aligned}$$

By symmetry of $h'$ and $h''$, we even have

$$\begin{aligned} &\|[f(x+h'+h'') - f(x+h'')] - [f(x+h') - f(x)]\| \\ \leq&\varepsilon \cdot \|h'\|^{\rho} \|h''\| \wedge \varepsilon \cdot \|h'\| \|h''\|^{\rho} \leq \varepsilon \cdot \|h'\|^{\rho'} \|h''\|^{\rho''} && \text{if } \|h'\|, \|h''\| \leq \delta. \end{aligned}$$

□

**Lemma 2.24.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point and $f \in \mathscr{C}^{\nu-1}(\mathrm{X}, \mathbf{E})$. Then $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ if and only if $f^{[\boldsymbol{n}]} \in \mathscr{C}^{1+\rho}(\mathrm{X}^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu-1}$.*

*Remark* 2.25. Let $\mathrm{X} \subseteq \mathbf{K}^d$ be a compact locally cartesian subset with local factors free of isolated points. For a finite covering $\mathfrak{U} = \{\mathrm{U}_1, \ldots, \mathrm{U}_n\}$ by balls of X (which are closed and hence compact), define the norm $\|\cdot\|_{\mathscr{C}^r, \mathfrak{U}} := \|\cdot\|_{\mathscr{C}^r, \mathrm{U}_1} \vee \ldots \vee \|\cdot\|_{\mathscr{C}^r, \mathrm{U}_n}$ on $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$. Then the locally convex topology on $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ is induced by any such norm.

**Lemma 2.26.** *Let $r', r''$ and $r = r' + r''$ be nonnegative real numbers and $\mathrm{X} \subseteq \mathbf{K}^d$ a compact locally cartesian group with $\mathscr{C}^r$-multiplication (respectively $\mathscr{C}^{\mathrm{lip}}$-multiplication if $r < 1$) whose local factors contain no isolated point. Then for $\mu \in \mathscr{D}^{r'}(\mathrm{X}, \mathbf{K})$ and $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$, their* ***convolution*** *$\mu \star f: \mathrm{X} \to \mathbf{K}$, defined by $y \mapsto \mu \cdot f(\_ \cdot y)$, is a $\mathscr{C}^{r''}$-function.*

*Proof:* Assume $r' = \nu' + \rho'$ and $r'' = \nu'' + \rho''$ with $\nu', \nu'' \in \mathbb{N}$ and $\rho', \rho'' \in [0, 1[$ so that $r = \nu + \rho$ with $\nu = \nu' + \nu'' \in \mathbb{N}$ and $\rho = \rho' + \rho'' \in [0, 2[$. Let $\mathrm{X}'' \subseteq \mathrm{X}$ be an open (and without loss of generality closed, hence compact) cartesian subset whose factors contain no isolated point. For every $\varepsilon > 0$ and $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$, we want to find a $\delta'' > 0$ such that

$$\|\mu \star f^{]\boldsymbol{n}''[}(y+h'') - \mu \star f^{]\boldsymbol{n}''[}(y)\| \leq \varepsilon \|h''\|^{\rho''} \quad \text{for all } y+h'', y \in \mathrm{X}^{]\boldsymbol{n}''[} \text{ with } \|h''\| \leq \delta''.$$

Then for every $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$, the function $\mu \star f^{]\boldsymbol{n}''[} \colon X''^{]\boldsymbol{n}''[} \to \mathbf{K}$ extends by Proposition 1.2 to a $\mathscr{C}^{\rho''}$-function $\mu \star f^{[\boldsymbol{n}'']} \colon X''^{[\boldsymbol{n}'']} \to \mathbf{K}$. That is, $\mu \star f_{|X''} \in \mathscr{C}^{r''}(X'', \mathbf{K})$ and so $\mu \star f \in \mathscr{C}^{r''}(X, \mathbf{K})$.

Consider the composed mapping $F \colon X \times X \to \mathbf{E}$ given by

$$\begin{aligned} X \times X &\to X &&\to \mathbf{E} \\ (x, y) &\mapsto x \cdot y &&\mapsto f(x \cdot y). \end{aligned}$$

By Proposition 2.18, this is again a $\mathscr{C}^r$-function on $X \times X$. Let $X' \subseteq X$ be an open (and closed, hence compact) cartesian subset whose factors contain no isolated point. By Lemma 2.24, we find $F^{[\boldsymbol{n}]} \in \mathscr{C}^{\rho}((X' \times X'')^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} = (\boldsymbol{n}', \boldsymbol{n}'') \in (\mathbb{N}^d \times \mathbb{N}^d)_{\leq \nu}$. Then $(X' \times X'')^{[\boldsymbol{n}]} = X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}$ and by Lemma 2.23 holds $F^{[\boldsymbol{n}]} \in \mathscr{C}^{\rho' \otimes \rho''}(X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}, \mathbf{E})$. We consider $X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']} \subseteq V' \oplus V''$ with $V' := V^{[\boldsymbol{n}']}$ and $V'' := V^{[\boldsymbol{n}'']}$ with $V = \mathbf{K}^d$. Let $\varepsilon > 0$. By compactness, there is $\delta > 0$ such that for all $\boldsymbol{n}', \boldsymbol{n}'' \in \mathbb{N}^d$ with $|\boldsymbol{n}'| + |\boldsymbol{n}''| = \nu$, all $x \in X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}$ and $h' \in V', h'' \in V''$ of norm at most $\delta$ holds, where defined,

$$\begin{aligned} &\|[F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x + h' + h'') - F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x + h'')] - [F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x + h') - F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x)]\| \\ \leq &\varepsilon \|h'\|^{\rho'} \|h''\|^{\rho''}. \end{aligned} \tag{$*$}$$

In particular this holds for all $\boldsymbol{n}' \in \mathbb{N}^d_{=\nu'}$ and $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$.

Fix $X'' \subseteq X$ compact cartesian with factors free of isolated points, $\varepsilon > 0$ and $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$. We have $\mu \star f(y) = \mu \cdot F(\_, y)$ for all $y \in X$. Hence for all $y + h'', y \in X''^{]\boldsymbol{n}''[}$ holds by $\mathbf{K}$-linearity of $\mu \colon \mathscr{C}^{r'}(X, \mathbf{E}) \to \mathbf{K}$ that

$$\begin{aligned} |(\mu \star f)^{]\boldsymbol{n}''[}(y + h'') - (\mu \star f)^{]\boldsymbol{n}''[}(y)| =& |\mu \cdot F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y + h'') - \mu \cdot F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y)| \\ =& |\mu \cdot (F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y + h'') - F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y))| \\ \leq& \|H\|_{\mathscr{C}^{r'}}, \end{aligned} \tag{$**$}$$

where $H := F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y + h'') - F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y) \in \mathscr{C}^r(X, \mathbf{E}) \subseteq \mathscr{C}^{r'}(X, \mathbf{E})$; and up to multiplying the distribution $\mu \colon \mathscr{C}^(X, \mathbf{K})^{r'} \to \mathbf{K}$ by a scalar $\lambda \in \mathbf{K}^*$, we assumed $\|\mu\|_{\mathscr{D}^{r'}} \leq 1$.

We just saw in Remark 2.25 that the topology of $\mathscr{C}^{r'}(X, \mathbf{E})$ is up to equivalence given by some norm $\|\cdot\|_{\mathscr{C}^{r'}} := \max_{X'} \|\cdot\|_{\mathscr{C}^{r'}, X'}$ for a finite covering of compact cartesian open subsets $X' \subseteq X$ whose factors have no isolated points. We may assume their diameters to be at most $\delta$.

We recall the above definition of the function $H = H(y + h'', y) \in \mathscr{C}^{r'}(X, \mathbf{E})$. Then by the above Inequality $(**)$, to conclude the proof, it remains in the following to find $\delta'' > 0$ such that

$$\|H(y + h'', y)\|_{\mathscr{C}^{r'}} \leq \varepsilon \|h''\|^{\rho''} \quad \text{for all } \; y + h'', y \in X''^{]\boldsymbol{n}''[} \text{ with } \|h''\| \leq \delta''.$$

Since a fortiori $F^{[\boldsymbol{n}',\boldsymbol{n}'']} \in \mathscr{C}^{\rho''}(X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}, \mathbf{E})$ for all $X' \subseteq X$ compact cartesian without isolated points and $\boldsymbol{n}' \in \mathbb{N}^d_{\leq \nu'}$, we find for $\tilde{\varepsilon} = \varepsilon \delta^{\rho'} \leq \varepsilon$ by compactness $\tilde{\delta} > 0$ such that

$$\|H^{[\boldsymbol{n}']}\|_{X'^{[\boldsymbol{n}']}} \leq \|F^{[\boldsymbol{n}',\boldsymbol{n}'']}(\_, y + h'') - F^{[\boldsymbol{n}',\boldsymbol{n}'']}(\_, y)\|_{X'^{[\boldsymbol{n}']}} \leq \tilde{\varepsilon} \cdot \|h''\|^{\rho''} \tag{$\dagger$}$$

for all $y+h'', y \in X''^{]\boldsymbol{n}''[}$ with $\|h''\| \leq \tilde{\delta}$.

Fix $y+h'', y \in X''^{]\boldsymbol{n}''[}$ with $\|h''\| \leq \delta'' := \delta \wedge \tilde{\delta}$. Let moreover $X' \subseteq X$ be a compact cartesian subset with factors free of isolated points, $\boldsymbol{n}' \in \mathbb{N}^d_{=\nu'}$ and $x+h', x \in X'^{[\boldsymbol{n}']}$. Then

$$\begin{aligned}&\mathrm{H}^{[\boldsymbol{n}']}(x+h') - \mathrm{H}^{[\boldsymbol{n}']}(x)\\ =&[\mathrm{F}^{[\boldsymbol{n}',\boldsymbol{n}'']}(x+h', y+h'') - \mathrm{F}^{[\boldsymbol{n}',\boldsymbol{n}'']}(x+h', y)] - [\mathrm{F}^{[\boldsymbol{n}',\boldsymbol{n}'']}(x, y+h'') - \mathrm{F}^{[\boldsymbol{n}',\boldsymbol{n}'']}(x,y)].\end{aligned}$$

As $\|h''\| \leq \delta$, we find by $(*)$ that

$$\|\mathrm{H}^{[\boldsymbol{n}']}(x+h') - \mathrm{H}^{[\boldsymbol{n}']}(x)\| \leq \mathrm{M} \cdot \|h'\|^{\rho'} \quad \text{for all } \ x+h', x \in X'^{[\boldsymbol{n}']} \text{ with } \|h'\| \leq \delta; \qquad (\dagger\dagger)$$

where we set $\mathrm{M} := \varepsilon \cdot \|h''\|^{\rho''}$. Putting $\tilde{\mathrm{H}} := \mathrm{H}^{[\boldsymbol{n}']}$, we see moreover

$$\begin{aligned}&\|\mathrm{H}^{[\boldsymbol{n}']}\|_{\mathscr{C}^{\rho'}, X'^{[\boldsymbol{n}']}}\\ =&\|\tilde{\mathrm{H}}\|_{X'^{[\boldsymbol{n}']}} \vee |||\tilde{\mathrm{H}}^{[\rho']}|||_{X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']}}\\ =&\|\tilde{\mathrm{H}}\|_{X'^{[\boldsymbol{n}']}} \vee |||\tilde{\mathrm{H}}^{[\rho']}|||_{\{(x+h',x) \in X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']} : \|h'\| \leq \delta\}} \vee |||\tilde{\mathrm{H}}^{[\rho']}|||_{\{(x+h',x) \in X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']} : \|h'\| > \delta\}}\\ \leq&\|\tilde{\mathrm{H}}\|_{X'^{[\boldsymbol{n}']}} \vee |||\tilde{\mathrm{H}}^{[\rho']}|||_{\{(x+h',x) \in X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']} : \|h'\| \leq \delta\}} \vee \|\tilde{\mathrm{H}}\|_{X'^{[\boldsymbol{n}']}}/\delta^{\rho'}\\ \leq&\tilde{\varepsilon} \cdot \|h''\|^{\rho''} \vee \mathrm{M} \vee \tilde{\varepsilon} \cdot \|h''\|^{\rho''}/\delta^{\rho'} = \varepsilon \|h''\|^{\rho''};\end{aligned}$$

the last inequality as because of $\|h''\| \leq \tilde{\delta}$, we can invoke Inequality $(\dagger)$ for the outer terms, and as $\|h'\| \leq 1$, Inequality $(\dagger\dagger)$ for the one in-between.

We can therefore conclude the proof by

$$\begin{aligned}\|\mathrm{H}\|_{\mathscr{C}^{r'}} &= \max_{X' \subseteq X \text{ cpt. cart.}} \|\mathrm{H}\|_{\mathscr{C}^{r'}, X'}\\ &= \max_{X' \subseteq X \text{ cpt. cart.}} \big( \max_{\boldsymbol{n}' \in \mathbb{N}^d_{<\nu'}} \|\mathrm{H}^{[\boldsymbol{n}']}\|_{X'^{[\boldsymbol{n}']}} \vee \max_{\boldsymbol{n}' \in \mathbb{N}^d_{=\nu'}} \|\mathrm{H}^{[\boldsymbol{n}']}\|_{\mathscr{C}^{\rho'}, X'^{[\boldsymbol{n}']}} \big) \leq \varepsilon \cdot \|h''\|^{\rho''}. \qquad \square\end{aligned}$$

**Theorem 2.27.** *Let $r, s \in \mathbb{R}_{\geq 0}$ and $X \subseteq \mathbf{K}^d$ a compact locally cartesian group with $\mathscr{C}^{r+s}$-multiplication whose local factors contain no isolated point. Then we can define the* ***convolution product*** $\mu \star \lambda \in \mathscr{D}^{r+s}(X, \mathbf{K})$ *of two distributions* $\mu \in \mathscr{D}^r(X, \mathbf{K})$ *and* $\lambda \in \mathscr{D}^s(X, \mathbf{K})$ *as the continuous* $\mathbf{K}$*-linear form on* $\mathscr{C}^{r+s}(X, \mathbf{E})$ *given by*

$$(\mu \star \lambda) \cdot f = \lambda \cdot (\mu \star f).$$

*Proof:* By the preceding Lemma 2.26, we find $\mu \star f \in \mathscr{C}^s(X, \mathbf{K})$ for all $f \in \mathscr{C}^{r+s}(X, \mathbf{E})$. Thence $\mu \star \lambda$ is well defined on $\mathscr{C}^{r+s}(X, \mathbf{E})$. $\square$

## The Mahler basis of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$

In this section, we will prove the $\mathscr{C}^r$-functions on $\mathbb{Z}_p^d$ to be characterized in terms of their Mahler coefficients $(a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ by obeying the convergence condition $|a_{\boldsymbol{n}}||\boldsymbol{n}|^r \to 0$ for $|\boldsymbol{n}| \to \infty$.

*Interlude: Orthogonal bases of* **K***-Banach spaces.* This brief interlude recalls the general theoretical background to describe $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ as an intersection of complete tensor products, whereby we can easily describe their Mahler coefficients through their already obtained description for one-variable functions.

Given a topological Hausdorff abelian group X, recall that a series $\sum_{i \in \mathrm{I}} x_i$ over an arbitrary index set I is defined as the unique element $x \in \mathrm{X}$ such that for every neighborhood $\mathrm{U} \ni x$ in X, there is a finite subset $\mathrm{F} \subseteq \mathrm{I}$ such that for all finite subsets $\tilde{\mathrm{F}} \supseteq \mathrm{F}$ in I we have $\sum_{i \in \tilde{\mathrm{F}}} x_i \in \mathrm{U}$.

**Definition.**

(i) For a sequence $(w_i)_{i \in \mathrm{I}}$ of weights in $\mathbb{R}_{>0}$, define the **K**-Banach space of **weighted zero sequences with respect to** $(w_i)$ by

$$c_0((w_i)_{i\in \mathrm{I}}) := \{\text{all sequences } (\lambda_i) \text{ in } \mathbf{K} \text{ such that, for any } \varepsilon > 0, \text{ only finitely often } |\lambda_i| w_i \geq \varepsilon \text{ for } i \in \mathrm{I}\}$$

with the maximum-norm

$$\|(\lambda_i)\| := \max_{i \in \mathrm{I}} |\lambda_i| w_i.$$

(ii) Given a **K**-Banach space **E**, we will call the subset $\{e_i\} \subseteq \mathbf{E}$ an **orthogonal basis** if the following map is an (isometric) isomorphism of **K**-Banach spaces:

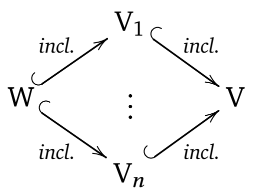
$$c_0((w_i)_{i\in \mathrm{I}}) \to \mathbf{E}$$
$$(\lambda_i) \mapsto \sum_i \lambda_i e_i,$$

were $w_i := \|e_i\|$ is the canonical weight associated to the basis vector $e_i$.

For the notion of the **completed tensor product** $\mathrm{V}\hat{\otimes}\mathrm{W}$ of two **K**-Banach spaces V and W, we refer the reader to [vR78, Chapter IV, Section "The Tensor Product"].

*Remark* 2.28. If $\{e_{i_1}\} \subseteq \mathrm{E}_1, \ldots, \{e_{i_d}\} \subseteq \mathrm{E}_d$ are orthogonal bases, then $\{e_{i_1} \otimes \cdots \otimes e_{i_d}\} \subseteq \mathrm{E}_1 \hat{\otimes} \cdots \hat{\otimes} \mathrm{E}_d$ will be an orthogonal basis. This can be seen by applying the criterion of [vR78, Comment following Cor. 4.31]. See [Nag11, Corollary 3.34] for details.

**Lemma 2.29.** *Let* W *be the initial* **K***-Banach space with respect to finitely many inclusion mappings*




*for* **K***-Banach spaces* $\mathrm{V}_1, \ldots, \mathrm{V}_n$ *and* V*. That is,* $\mathrm{W} = \mathrm{V}_1 \cap \ldots \cap \mathrm{V}_n$ *as an abstract* **K***-vector space and its norm* $\|\cdot\|_{\mathrm{W}}$ *on* W *is given by the pointwise maximum* $\|\cdot\|_{\mathrm{W}} = \|\cdot\|_{\mathrm{V}_1} \vee \ldots \vee \|\cdot\|_{\mathrm{V}_n}$.

*(i) If* $\{e_i\} \subseteq \mathrm{W}$ *is an orthogonal family of* $\mathrm{V}_1, \ldots, \mathrm{V}_n$*, then* $\{e_i\}$ *will be an orthogonal family of* W.

(ii) *If* $\{e_i\} \subseteq \mathrm{W}$ *is an orthogonal basis of* $\mathrm{V}_1, \ldots, \mathrm{V}_n$ *and* V*, then* $\{e_i\}$ *will be an orthogonal basis of* W.

*Proof:* Ad (i): Let $\{e_i\}$ be orthogonal in $\mathrm{V}_1, \ldots, \mathrm{V}_n$. We prove $\{e_i\} \subseteq \mathrm{W}$ to be orthogonal by the following computation:

$$\begin{aligned} \|\sum_i \lambda_i e_i\|_{\mathrm{W}} &= \|\sum_i \lambda_i e_i\|_{\mathrm{V}_1} \vee \ldots \vee \|\sum_i \lambda_i e_i\|_{\mathrm{V}_n} \\ &= \max_i |\lambda_i| \|e_i\|_{\mathrm{V}_1} \vee \ldots \vee \max_i |\lambda_i| \|e_i\|_{\mathrm{V}_n} \\ &= \max_i |\lambda_i| \|e_i\|_{\mathrm{W}}. \end{aligned}$$

Ad (ii): Let $x \in \mathrm{W}$. Then we can write $x = \sum_{i \geq 0} \lambda_i e_i$ in $\mathrm{V}_j$ for $j \in \{1, \ldots, n\}$. This implies $x = \sum_{i \geq 0} \lambda_i e_i$ in V. By orthogonality of $\{e_i\} \subseteq \mathrm{V}$, the coefficients $\lambda_i$ are uniquely determined, so the same equality holds in $\mathrm{V}_1, \ldots, \mathrm{V}_n$ and therefore as well in W. The orthogonality of $\{e_i\} \subseteq \mathrm{W}$ has been proved in $(i)$. □

*The initial* **K**-*Banach space* $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ *of thought topological tensor products* $\mathscr{C}^{\boldsymbol{r}}(\mathrm{X}, \mathbf{K})$. We will describe $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ as an intersection of subspaces $\mathscr{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ for $\boldsymbol{r} \in \mathbb{R}_{\geq 0}^d$ to be introduced in this subsection. Their definition is such that they resemble the completed tensor product of $\mathscr{C}^r$-function spaces in one variable.

*Assumption.* We let $\mathrm{X} \subseteq \mathbf{K}^d$ denote a compact cartesian subset whose factors contain no isolated points.

**Definition 2.30.** For a $d$-tuple $\boldsymbol{s} \in \mathbb{R}_{\geq 0}^d$, we put $|\boldsymbol{s}| = s_1 + \cdots + s_d$. For $r \in \mathbb{R}_{\geq 0}$, we define finite sets of $d$-tuples

$$\mathbb{N}_{=r}^d = \{\boldsymbol{s} \in \mathbb{R}_{\geq 0}^d : |\boldsymbol{s}| = r \text{ and } s_k \in \mathbb{N} \text{ for all but one coordinate } k \in \{1, .., d\}\}.$$

Let $\boldsymbol{r} = \boldsymbol{\nu} + \rho \cdot \boldsymbol{e}_k \in \mathbb{N}_{=r}^d$ with $\boldsymbol{\nu} \in \mathbb{N}^d$ and $k \in \{1, \ldots, d\}$. Then we define a mapping $f \colon \mathrm{X} \to \mathbf{E}$ to be a $\mathscr{C}^{\boldsymbol{r}}$**-function** if the following holds:

(i) For all $\boldsymbol{n} \leq \boldsymbol{\nu}$ with $n_k < \nu_k$, the mapping $f^{]\boldsymbol{n}[} \colon \mathrm{X}^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a continuous function $f^{[\boldsymbol{n}]} \colon \mathrm{X}^{[\boldsymbol{n}]} \to \mathbf{E}$.

(ii) For all $\boldsymbol{n} \leq \boldsymbol{\nu}$ with $n_k = \nu_k$, the mapping $f^{]\boldsymbol{n}[} \colon \mathrm{X}^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a $\mathscr{C}^{\rho \cdot {}^k\boldsymbol{e}}$-function $f^{[\boldsymbol{n}]} \colon \mathrm{X}^{[\boldsymbol{n}]} \to \mathbf{E}$; here we consider $\mathrm{X}^{[\boldsymbol{n}]} = {}^1\mathrm{X} \times \cdots \times {}^d\mathrm{X}$ for $\boldsymbol{n} \in \mathbb{N}_{\geq 0}^d$ as the cartesian product of the metric spaces ${}^k\mathrm{X} := \mathrm{X}_k^{[n_k]}$, and so put ${}^k\boldsymbol{e} = \boldsymbol{e}_k$ for $k = 1, \ldots, d$.

The **K**-vector space of all $\mathscr{C}^{\boldsymbol{r}}$-functions $f \colon \mathrm{X} \to \mathbf{E}$ will be denoted by $\mathscr{C}^{\boldsymbol{r}}(\mathrm{X}, \mathbf{E})$. We equip it with the norm $\|\cdot\|_{\mathscr{C}^{\boldsymbol{r}}}$ defined by

$$\|f\|_{\mathscr{C}^{\boldsymbol{r}}} = \max_{\boldsymbol{n} \leq \boldsymbol{\nu}] \text{ with } n_k < \nu_k} \|f^{[\boldsymbol{\nu}]}\|_{\sup} \vee \max_{\boldsymbol{n} \in \boldsymbol{\nu} \text{ with } n_k = \nu_k} \|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^{\rho \cdot {}^k\boldsymbol{e}}}.$$

**Proposition 2.31.** *The* **K**-*vector space* $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ *is a* **K**-*Banach space and initial with respect to the inclusion mappings* $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \overset{incl.}{\hookrightarrow} \mathscr{C}^{\boldsymbol{r}}(\mathrm{X}, \mathbf{E})$ *for* $\boldsymbol{r} \in \mathbb{N}_{=r}^d$.

*Remark* 2.32. In the other direction, we have by definition a norm-nonincreasing inclusion of $\mathbf{K}$-Banach spaces $\mathscr{C}^{\vec{\nu}}(\mathrm{X},\mathbf{K}) \subseteq \mathscr{C}^{\nu}(\mathrm{X},\mathbf{K})$ with $\vec{\nu} := (\nu,\dots,\nu)$ for $\nu \in \mathbb{N}$.

The following observation will be used to deduce the description of the Mahler basis of $\mathscr{C}^r(\mathbb{Z}_p,\mathbf{K})$ later on. One can infer from it the Mahler polynomials to constitute an orthogonal family and deduce their norms from the one variable case. Then to see that it is indeed a base, it rests to show the density of the polynomial functions in the space of $r$-times differentiable ones. This will be carried out in Section 2.

**Lemma 2.33.** *Let $\mathrm{X}_1,\dots,\mathrm{X}_d \subseteq \mathbf{K}$ be nonempty compact subsets without isolated points. For $\boldsymbol{r} \in \mathbb{N}^d_{=r}$, consider the mapping*

$$\mathscr{C}^{r_1}(\mathrm{X}_1,\mathbf{K}) \times \cdots \times \mathscr{C}^{r_d}(\mathrm{X}_d,\mathbf{K}) \xrightarrow{\Psi} \mathscr{C}^{\boldsymbol{r}}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_d,\mathbf{K}),$$
$$(f_1,\dots,f_d) \mapsto f := [(x_1,\dots,x_d) \mapsto f_1(x_1)\cdots f_d(x_d)].$$

*If $\{e_{i_1}\} \subseteq \mathscr{C}^{r_1}(\mathrm{X}_1,\mathbf{K}),\dots,\{e_{i_d}\} \subseteq \mathscr{C}^{r_d}(\mathrm{X}_d,\mathbf{K})$ are orthogonal families, so $\{e_{i_1} \odot \cdots \odot e_{i_d}\} \subseteq \mathscr{C}^{\boldsymbol{r}}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_d,\mathbf{K})$ will be an orthogonal family with $\|e_{i_1} \odot \cdots \odot e_{i_d}\|_{\mathscr{C}^{\boldsymbol{r}}} = \|e_{i_1}\|_{\mathscr{C}^{r_1}} \cdots \|e_{i_d}\|_{\mathscr{C}^{r_d}}$.*

*Proof:* By definition of the norm on $\mathscr{C}^{\boldsymbol{r}}(\mathrm{X},\mathbf{K})$. See [Nag11, Corollary 3.41] for details. □

*Density of (locally) polynomial functions in $\mathscr{C}^r(\mathrm{X},\mathbf{K})$.* To conclude that $\mathscr{C}^r(\mathbb{Z}_p^d,\mathbf{K})$ can indeed be described as the intersection of the completed tensor products $\mathscr{C}^{r_1}(\mathbb{Z}_p,\mathbf{K})\widehat{\otimes}\cdots\widehat{\otimes}\mathscr{C}^{r_d}(\mathbb{Z}_p,\mathbf{K})$ for $\boldsymbol{r} = (r_1,\dots,r_d) \in \mathbb{N}^d_{=r}$, we need as a last step the density of the subspace of (locally) polynomial functions inside $\mathscr{C}^r(\mathbb{Z}_p^d,\mathbf{K})$. This is proved here for a general domain $\mathrm{X} \subseteq \mathbf{K}^d$.

*Remark* 2.34. We remark that on open domains locally polynomial functions are in particular locally analytic. By Proposition 2.12, these are $\mathscr{C}^r$-functions for any $r \in \mathbb{R}_{\geq 0}$. Hence a fortiori locally polynomial functions defined on a general subset $\mathrm{X} \subseteq \mathbf{K}$ lie in $\mathscr{C}^r(\mathrm{X},\mathbf{K})$ for any $r \in \mathbb{R}_{\geq 0}$.

*Assumption.* Let $\mathrm{X} \subseteq \mathbf{K}^d$ be a nonempty compact cartesian subset whose factors contain no isolated point.

**Lemma 2.35.** *Let $\mathrm{X} \subseteq \mathbf{K}^d$ be cartesian, $\boldsymbol{n} \in \mathbb{N}^d$ and $\delta > 0$. We define*

$$\mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta} := \{x = ({}^1x; -; {}^dx) \in \mathrm{X}^{[\boldsymbol{n}]} : \operatorname{dia}{}^1x,\dots,\operatorname{dia}{}^dx \leq \delta\},$$

*where $\operatorname{dia}{}^kx := \operatorname{dia}\{{}^kx_0,\dots,{}^kx_{n_k}\}$ for $k = 1,\dots,d$. Let $p \in \mathrm{X}$ and put $\mathrm{P} := \mathrm{B}_{\leq\delta}(p) \subseteq \mathrm{X}$. Then $\mathrm{P}^{[\boldsymbol{n}]} = \mathrm{B}_{\leq\delta}(\vec{p}) \subseteq \mathrm{X}^{[\boldsymbol{n}]}$ with $\vec{p} = (\vec{p}_1;\dots;\vec{p}_d) \in \mathrm{X}^{[\boldsymbol{n}]}$ and*

$$\mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta} = \bigcup_{p\in\mathrm{X}} \mathrm{P}^{[\boldsymbol{n}]}.$$

*Likewise for $\mathrm{X}^{]\boldsymbol{n}[}_{\leq\delta} := \mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta} \cap \mathrm{X}^{]\boldsymbol{n}[}$.*

*Proof:* The first assertion holds by definition. We have

$$X^{[\boldsymbol{n}]}_{\leq\delta} = \bigcup_{\boldsymbol{p}\in X^{[\boldsymbol{n}]}} B_{\leq\delta}(\boldsymbol{p}).$$

Let $a, b \in X^{[\boldsymbol{n}]}_{\leq\delta}$ and $\|\boldsymbol{b} - \boldsymbol{a}\| \leq \delta$. Then by the ultrametric triangle inequality holds $B_{\leq\delta}(\boldsymbol{a}) = B_{\leq\delta}(\boldsymbol{b})$ in $X^{[\boldsymbol{n}]}_{\leq\delta}$. Let $\boldsymbol{p} \in X^{[\boldsymbol{n}]}_{\leq\delta}$. Then we can find $p \in X$ such that $\|\boldsymbol{p} - \vec{p}\| \leq \delta$ where we put $\vec{p} = (\vec{p}_1; \dots; \vec{p}_d) \in X^{[\boldsymbol{n}]}$. (For example, $p = ({}^1p_0, \dots, {}^dp_0) \in X$ for $\boldsymbol{p} = ({}^1\boldsymbol{p}; -; {}^d\boldsymbol{p}) \in X^{[\boldsymbol{n}]}_{\leq\delta}$.) We can therefore conclude $B_{\leq\delta}(\boldsymbol{p}) = P^{[\boldsymbol{n}]}$ with $P = B_{\leq\delta}(x)$. □

**Lemma 2.36.** *Let $f \in \mathscr{C}^n(X, \mathbf{E})$. Fix $\delta, \varepsilon > 0$. Suppose that for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ holds*

$$\|f^{[\boldsymbol{n}]}(x)\| \leq \varepsilon \quad \text{for all } x \in X^{[\boldsymbol{n}]}_{\leq\delta}.$$

*Then for all $\boldsymbol{m} \in \mathbb{N}^d_{=n-1}$, we find*

$$\|f^{[\boldsymbol{m}]}(x) - f^{[\boldsymbol{m}]}(\vec{a})\| \leq \varepsilon \cdot \delta \quad \text{for all } x, \vec{a} \in X^{[\boldsymbol{m}]} \text{ with } \|x - \vec{a}\| \leq \delta.$$

*Proof:* Fix $\boldsymbol{m} \in \mathbb{N}^d_{=n-1}$ and let $k \in \{1, \dots, d\}$. Then for all $x \in X^{[\boldsymbol{m}]}_{\leq\delta}$ and $t \in \mathbf{K}$ with ${}^kx_0 \in X_k$ and $|t| \leq \delta$ holds by assumption

$$\begin{aligned} &\|f^{[\boldsymbol{m}]}(-; {}^kx + t \cdot \boldsymbol{e}_1; -) - f^{[\boldsymbol{m}]}(-; {}^kx; -)\| \\ =&\|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(-; {}^kx_0 + t, {}^kx_0, {}^kx_1, \dots, {}^kx_{m_k}; -) \cdot t\| \leq \varepsilon \cdot |t| \leq \varepsilon\delta. \end{aligned} \tag{$*$}$$

By Lemma 2.35, we have

$$X^{[\boldsymbol{m}]}_{\leq\delta} = \bigcup_{a\in X} A^{[\boldsymbol{m}]} \tag{$**$}$$

with $A = B_{\leq\delta}(a) \subseteq X$ for $a \in X$. The set $A^{[\boldsymbol{m}]}$ is symmetric in its $A_k$-coordinates for $k = 1, \dots, d$ and, as cartesian, also telescopic. By Corollary 1.14 for $\rho = 1$, Inequality $(*)$ for $k = 1, \dots, d$ implies $\|f^{[\boldsymbol{m}]}(x) - f^{[\boldsymbol{m}]}(y)\| \leq \varepsilon\|x - y\|$ for all $x, y \in A^{[\boldsymbol{m}]}$.

We notice that $\vec{a} \in X^{[\boldsymbol{m}]}_{\leq\delta}$ for any $a \in X$. Moreover, if $\|x - \vec{a}\| \leq \delta$ for $x \in X^{[\boldsymbol{m}]}$, then by the non-Archimedean triangle inequality $x \in A^{[\boldsymbol{m}]}$ with $A = B_{\leq\delta}(a)$. By Equality $(**)$, we thus find for all $\vec{a}, x \in X^{[\boldsymbol{m}]}$ with $\|x - \vec{a}\| \leq \delta$ that

$$\|f^{[\boldsymbol{m}]}(x) - f^{[\boldsymbol{m}]}(\vec{a})\| \leq \varepsilon\|x - \vec{a}\| \leq \varepsilon \cdot \delta.$$

**Lemma 2.37.** *Let $f \in \mathscr{C}^n(X, \mathbf{K})$ and $\boldsymbol{n} \in \mathbb{N}^d_{=n}$. Fix $\delta, \varepsilon > 0$. If*

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \quad \text{for all } x, \vec{a} \in X^{[\boldsymbol{n}]} \text{ with } \|x - \vec{a}\| \leq \delta,$$

*then there will exist $\delta$-constant $g\colon X \to \mathbf{K}$ such that $\tilde{f} := f - g*^{\boldsymbol{n}}$ satisfies*

$$\|\tilde{f}^{[\boldsymbol{n}]}(x)\|_{X^{[\boldsymbol{n}]}_{\leq\delta}} \leq \varepsilon.$$

*Proof:* For all $x, \vec{a} \in X^{[\boldsymbol{n}]}$, we have by assumption

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \quad \text{if } \|x - \vec{a}\| \leq \delta.$$

In particular for all $a, b \in X$, we have $|D_{\boldsymbol{n}} f(a) - D_{\boldsymbol{n}} f(b)| \leq \varepsilon$ if $\|a - b\| \leq \delta$. By Lemma 1.4, there is $\delta$-constant $g \colon X \to \mathbf{K}$ such that $\|D_{\boldsymbol{n}} f - g\|_{\sup} \leq \varepsilon$. By Lemma 2.11(ii) we find $D_{\boldsymbol{n}}(g*^{\boldsymbol{n}}) = g$. Hence $\tilde{f} := f - g*^{\boldsymbol{n}}$ satisfies

$$\|D_{\boldsymbol{n}} \tilde{f}\|_{\sup} = \|D_{\boldsymbol{n}} f - D_{\boldsymbol{n}}(g*^{\boldsymbol{n}})\|_{\sup} = \|D_{\boldsymbol{n}} f - g\|_{\sup} \leq \varepsilon.$$

Let $x, \vec{a} \in X^{[\boldsymbol{n}]}$. Then $\|x - \vec{a}\| \leq \delta$ implies

$$\begin{aligned}
|\tilde{f}^{[\boldsymbol{n}]}(x) - \tilde{f}^{[\boldsymbol{n}]}(\vec{a})| =& |(f - g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(x) - (f - g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(\vec{a})| \\
=& |(f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})) - ((g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(x) - (g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(\vec{a}))| \\
\leq& |f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \vee |(g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(x) - (g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(\vec{a})| \\
\leq& \varepsilon \vee |g(a) - g(a)| = \varepsilon;
\end{aligned}$$

the last equality by Lemma 2.11(ii) as $x \in B_{\leq\delta}(a)^{[\boldsymbol{n}]}$. Since $\|D_{\boldsymbol{n}} \tilde{f}\|_{\sup} \leq \varepsilon$, it follows

$$|\tilde{f}^{[\boldsymbol{n}]}(x)| \leq |\tilde{f}^{[\boldsymbol{n}]}(x) - \tilde{f}^{[\boldsymbol{n}]}(\vec{a})| \vee |\tilde{f}^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon.$$

By Lemma 2.11(ii), we find $X^{[\boldsymbol{n}]}_{\leq\delta} = \cup_{a \in X} A^{[\boldsymbol{n}]}$ with $A := B_{\leq\delta}(a) \subseteq X$. Hence $|\tilde{f}^{[\boldsymbol{n}]}(x)| \leq \varepsilon$ for all $x \in X^{[\boldsymbol{n}]}_{\leq\delta}$. □

**Lemma 2.38.** *Let $f \in \mathscr{C}^{\nu}(X, \mathbf{E})$ and $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. For $k = 1, \ldots, d$, we define the function $|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| \colon X^{[\boldsymbol{n}+\boldsymbol{e}_k]} \to \mathbb{R}_{\geq 0}$ by*

$$\begin{aligned}
&|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-; {}^k\tilde{x}_0, {}^k x_0, {}^k x_1, \ldots, {}^k x_{n_k}; -) \\
:=& \|f^{[\boldsymbol{n}]}(-; {}^k\tilde{x}_0, {}^k x_1, \ldots, {}^k x_{n_k}; -) - f^{[\boldsymbol{n}]}(-; {}^k x_0, {}^k x_1, \ldots, {}^k x_{n_k}; -)\| / |{}^k\tilde{x}_0 - {}^k x_0|^{\rho}
\end{aligned}$$

*if ${}^k\tilde{x}_0 \neq {}^k x_0$ and zero otherwise. Then $f^{[\boldsymbol{n}]} \in \mathscr{C}^{\rho}(X^{[\boldsymbol{n}]}, \mathbf{E})$ implies $|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|$ to be a continuous function for $k = 1, \ldots, d$ and*

$$\|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^{\rho}} = \|f^{[\boldsymbol{n}]}\|_{\sup} \vee \||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_1]}|\|_{\sup} \vee \ldots \vee \||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|\|_{\sup}.$$

*Proof:* Recall $X^{[\boldsymbol{n}]} = X_1^{[n_1]} \times \cdots \times X_d^{[n_d]}$ and ${}^k\boldsymbol{e}_0 := (\mathbf{0}; \ldots; \boldsymbol{e}_0; \ldots; \mathbf{0}) \in \mathbb{N}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0 = (1, 0, \ldots) \in \mathbb{N}^{[n_k]}$ with $N^{[n_k]} = \mathbb{N}^{\{0,\ldots,n_k\}}$ at the $k$-th place. We view $\tilde{X} = X^{[\boldsymbol{n}]} \subseteq \mathbf{K}^{[\boldsymbol{n}]}$. Denote by

$$I_k = \{(k, 0), \ldots, (k, n_k)\} = \{\, X_k\text{-coordinate indices of } X^{[\boldsymbol{n}]} \,\}.$$

Then the only nonzero entry of ${}^k\boldsymbol{e}_0$ is at the $i_k$-th coordinate for a representative $i_k \in I_k$. By Lemma 2.1(i), the function $f^{[\boldsymbol{n}]} \colon X^{[\boldsymbol{n}]} \to \mathbf{K}$ is symmetric in its coordinates indexed by $I_1, \ldots, I_d$. By Corollary 1.15, we find

$$\|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^{\rho}} = \|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^{\rho\cdot{}^1\boldsymbol{e}_0}} \vee \ldots \vee \|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^{\rho\cdot{}^d\boldsymbol{e}_0}}.$$

By Definition 1.5, we have $\|\tilde{f}\|_{\mathscr{C}^{\rho\cdot{}^k\boldsymbol{e}_0},\tilde{X}} = \|\tilde{f}\|_{\tilde{X}} \vee |||\tilde{f}^{[\rho\cdot{}^k\boldsymbol{e}_0]}|||_{\tilde{X}^{[\rho\cdot{}^k\boldsymbol{e}_0]}}$ for $\tilde{X} := X^{[\boldsymbol{n}]}$ and $\tilde{f} := f^{[\boldsymbol{n}]}\colon \tilde{X} \to \mathbf{E}$. Now identifying $x = (-;{}^k\tilde{x}_0, {}^kx_0; {}^kx_1; -) \in \tilde{X}^{[{}^k\boldsymbol{e}_0]}$ with the element $x = (-;{}^k\tilde{x}_0, {}^kx_0, {}^kx_1, \cdots; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$,

$$|\tilde{f}^{[\rho\cdot{}^k\boldsymbol{e}_0]}|(x) = |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x).$$

Therefore $\|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^\rho} = \|f^{[\boldsymbol{n}]}\|_{\sup} \vee |||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_0]}|||_{\sup} \vee \ldots \vee |||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|||_{\sup}$. □

**Lemma 2.39.** *Given $f \in \mathscr{C}^r(X, \mathbf{E})$, let $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $k \in \{1, \ldots, d\}$. If $x \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $|{}^lx_i - {}^lx_j| > \delta$ for some coordinate $l \in \{1, \ldots, d\}$ and $i, j \in \{0, \ldots, n_l + \delta_{kl}\}$, then for all $\tilde{\boldsymbol{n}} \in \mathbb{N}^d_{=\nu}$, we have*

$$|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x) < \|f^{[\tilde{\boldsymbol{n}}]}\|_{\sup}/\delta^\rho.$$

*Proof:* We distinguish three cases in increasing generality.

Case 1: $|{}^kx_0 - {}^kx_1| > \delta$. Then by definition

$$\begin{aligned}
&|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x)\\
=&|f^{[\boldsymbol{n}]}(-;{}^kx_0, {}^kx_2, \ldots, {}^kx_{n_k+1}; -) - f^{[\boldsymbol{n}]}(-;{}^kx_1, {}^kx_2, \ldots, {}^kx_{n_k+1}; -)|/|{}^kx_0 - {}^kx_1|^\rho\\
<&\|f^{[\boldsymbol{n}]}\|_{\sup}/\delta^\rho.
\end{aligned}$$

Case 2: $|{}^lx_0 - {}^lx_1| > \delta$ for some $l \in \{1, \ldots, d\}$. If this holds for $l = k$, we will be in Case 1. If this only holds for $l \neq k$, then we can write $\boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_l$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu-1}$ and we will assume without loss of generality $l < k$. Let $(-;{}^kx_0 + s, {}^kx_0, {}^kx_1, \ldots, {}^kx_{n_k}; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $s \in \mathbf{K}$, and put $x = (-;{}^kx_0, {}^kx_1, \ldots, {}^kx_{n_k}; -) \in X^{[\boldsymbol{n}]}$. Then

$$\begin{aligned}
&|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^kx_0 + s, {}^kx_0, {}^kx_1, \ldots, {}^kx_{n_k}; -)\\
=&|f^{[\boldsymbol{n}]}(-;{}^kx_0 + s, {}^kx_1, \ldots, {}^kx_{n_k}; -) - f^{[\boldsymbol{n}]}(-;{}^kx_0, {}^kx_1, \ldots, {}^kx_{n_k}; -)|/|s|^\rho\\
=&|f^{[\boldsymbol{n}]}(x + s\cdot{}^k\boldsymbol{e}_0) - f^{[\boldsymbol{n}]}(x)|/|s|^\rho
\end{aligned}$$

with ${}^k\boldsymbol{e}_0 := (\boldsymbol{0}; \ldots; \boldsymbol{e}_0; \ldots; \boldsymbol{0}) \in \mathbf{K}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0 = (1, 0, \ldots) \in \mathbf{K}^{[n_k]} = \mathbf{K}^{\{0,\ldots,n_k\}}$ at the $k$-th place. Let $(-;{}^lx_0 + t, {}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -) \in X^{[\boldsymbol{n}]}$ with $t \in \mathbf{K}$. Put $x = (-;{}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -) \in X^{[\boldsymbol{m}]}$ and $\tilde{f} := f^{[\boldsymbol{m}]}$. Then by definition, we have

$$\begin{aligned}
&f^{[\boldsymbol{n}]}(-;{}^lx_0 + t, {}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -)\\
=&f^{[\boldsymbol{m}+\boldsymbol{e}_l]}(-;{}^lx_0 + t, {}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -)\\
=&[f^{[\boldsymbol{m}]}(-;{}^lx_0 + t, {}^lx_1, \ldots, {}^lx_{m_l}; -) - f^{[\boldsymbol{m}]}(-;{}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -)]/t\\
=&[\tilde{f}(x + t\cdot{}^l\boldsymbol{e}_0) - \tilde{f}(x)]/t.
\end{aligned}$$

Let $(-;{}^lx_0 + t, {}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -;{}^kx_0 + s, {}^kx_0, {}^kx_1, \ldots, {}^kx_{m_k}; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $t, s \in \mathbf{K}$, and put as before $x = (-;{}^lx_0, {}^lx_1, \ldots, {}^lx_{m_l}; -;{}^kx_0, {}^kx_1, \ldots, {}^kx_{m_k}; -) \in X^{[\boldsymbol{m}]}$. Combining both obtained equalities, we infer

$$\begin{aligned}&|f^{[\boldsymbol{m}+\boldsymbol{e}_l+\rho\cdot\boldsymbol{e}_k]}|(-;{}^lx_0+t,{}^lx_0,{}^lx_1,\dots,{}^lx_{m_l};-;{}^kx_0+s,{}^kx_0,{}^kx_1,\dots,{}^kx_{m_k};-)\\ =&|[\tilde f(x+t\cdot{}^l\boldsymbol{e}_0+s\cdot{}^k\boldsymbol{e}_0)-\tilde f(x+s\cdot{}^k\boldsymbol{e}_0)]-[\tilde f(x+t\cdot{}^l\boldsymbol{e}_0)-\tilde f(x)]|/|t||s|^\rho\\ \le&[|\tilde f(x+t\cdot{}^l\boldsymbol{e}_0+s\cdot{}^k\boldsymbol{e}_0)-\tilde f(x+t\cdot{}^l\boldsymbol{e}_0)|/|s|^\rho\vee|\tilde f(x+s\cdot{}^k\boldsymbol{e}_0)-\tilde f(x)|/|s|^\rho]/|t|.\qquad(*)\end{aligned}$$

We notice that for $x+s\cdot{}^k\boldsymbol{e}_0, x\in \mathrm{X}^{[\boldsymbol{m}]}$ holds

$$\begin{aligned}|\tilde f(x+s\cdot{}^k\boldsymbol{e}_0)-\tilde f(x)|/|s|^\rho=&|[\tilde f(x+s\cdot{}^k\boldsymbol{e}_0)-\tilde f(x)]/s||s|^{1-\rho}\\ =&|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(-;{}^kx_0+s,{}^kx_0,{}^kx_1,\dots,{}^kx_{m_k};-)||s|^{1-\rho}.\qquad(**)\end{aligned}$$

Let $x=(-;{}^lx_0+t,{}^lx_0,{}^lx_1,\dots,{}^lx_{m_l};-;{}^kx_0+s,{}^kx_0,{}^kx_1,\dots,{}^kx_{m_k};-)\in\mathrm{X}^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $|t|>\delta$ for $l\in\{1,\dots,d\}$. By Case 1, we may assume $|s|\le\delta$. Then under these assumptions $|s|\le\delta<|t|$, so Inequalities $(*)$ and $(**)$ yield

$$\begin{aligned}&|f^{[\boldsymbol{m}+\boldsymbol{e}_l+\rho\cdot\boldsymbol{e}_k]}|(-;{}^lx_0+t,{}^lx_0,{}^lx_1,\dots,{}^lx_{m_l};-;{}^kx_0+s,{}^kx_0,{}^kx_1,\dots,{}^kx_{m_k};-)\\ <&\|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}\|_{\sup}\cdot\delta^{1-\rho}/\delta=\|f^{[\tilde{\boldsymbol{n}}]}\|_{\sup}/\delta^\rho.\end{aligned}$$

Case 3: $|{}^lx_i-{}^lx_j|>\delta$ for some $l\in\{1,\dots,d\}$ and $i,j\in\{0,\dots,n_l\}$. We want to reduce to the second case.

Case 3.1: If $l\neq k$, then by the symmetry of $f^{[\boldsymbol{n}]}\colon \mathrm{X}^{[\boldsymbol{n}]}\to\mathbf{K}$ in its $\mathrm{X}_l$-coordinates, we may assume $i,j=0,1$ and the result follows by Case 2.

Case 3.2: If $l=k$, we may assume that $|{}^kx_0-{}^kx_1|\le\delta<|{}^kx_i-{}^kx_j|$ as otherwise the result will follow by Case 2. Let $\sigma$ be the permutation on $\mathrm{X}_k^{[n_k+1]}=\mathrm{X}_k^{\{0,\dots,n_k+1\}}$ swapping the $i$-th and $j$-th coordinate with the first and second one. We notice that by definition,

$$|f_{\boldsymbol{x}}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x)=|f^{[\boldsymbol{n}+\boldsymbol{e}_k]}(x)||{}^kx_0-{}^kx_1|^{1-\rho}.$$

Then by symmetry of $f^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ in its $\mathrm{X}_k$-coordinates, we find

$$\begin{aligned}|f_{\boldsymbol{x}}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^kx^\sigma;-)&=|f^{[\boldsymbol{n}+\boldsymbol{e}_k]}(-;{}^kx^\sigma;-)||{}^kx_i-{}^kx_j|^{1-\rho}\\ &=|f^{[\boldsymbol{n}+\boldsymbol{e}_k]}(-;{}^kx;-)||{}^kx_0-{}^kx_1|^{1-\rho}\frac{|{}^kx_i-{}^kx_j|^{1-\rho}}{|{}^kx_0-{}^kx_1|^{1-\rho}}\\ &=|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}(-;{}^kx;-)|\frac{|{}^kx_i-{}^kx_j|^{1-\rho}}{|{}^kx_0-{}^kx_1|^{1-\rho}}\\ &=|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}(-;{}^kx;-)|\left|\frac{{}^kx_0^\sigma-{}^kx_1^\sigma}{{}^kx_0-{}^kx_1}\right|^{1-\rho}.\end{aligned}$$

Since $|{}^kx_0-{}^kx_1|<|{}^kx_0^\sigma-{}^kx_1^\sigma|$, we obtain $|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^kx;-)<|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^kx^\sigma;-)$. We can therefore conclude

$$\begin{aligned}|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|({}^1x;-;{}^{k-1}x;{}^kx;{}^{k+1}x;-;{}^dx)&<|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|({}^1x;-;{}^{k-1}x;{}^kx^\sigma;{}^{k+1}x;-;{}^dx)\\ &\le\|f^{[\boldsymbol{n}]}\|_{\sup}/\delta^\rho;\end{aligned}$$

the last inequality by Case 1 as $|{}^kx_0^\sigma-{}^kx_1^\sigma|=|{}^kx_i-{}^kx_j|>\delta$. □

**Proposition 2.40.** *The locally polynomial functions of total degree at most* $\nu$ *are dense in* $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$.

*Proof:* Fix $\varepsilon > 0$ and $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{K})$. Then $f^{[\boldsymbol{n}]} \in \mathscr{C}^\rho(\mathrm{X}^{[\boldsymbol{n}]}, \mathbf{K})$. By compactness of X, there is by Proposition 2.5 some $0 < \delta \leq 1$ such that for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $x, y \in \mathrm{X}^{[\boldsymbol{n}]}$ with $\|x - y\| \leq \delta$ holds

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(y)| \leq \varepsilon \cdot \|x - y\|^\rho. \tag{$*$}$$

We will fix this $\delta > 0$ for the rest of the proof and recall $\mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta} := \{x = ({}^1x; -; {}^dx) \in \mathrm{X}^{[\boldsymbol{n}]} \text{ with } \operatorname{dia}{}^1x, \ldots, \operatorname{dia}{}^dx \leq \delta\}$ for $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$.

Step 1.: By downward induction on $n = \nu, \ldots, 0$, we will construct $\delta$-constant functions $g_{\boldsymbol{i}} \colon \mathrm{X} \to \mathbf{K}$ for $\boldsymbol{i} \in \mathbb{N}^d$ with $n \leq |\boldsymbol{i}| \leq \nu$ such that $f_n = f - \sum_{n \leq |\boldsymbol{i}| \leq \nu} g_{\boldsymbol{i}} *^{\boldsymbol{i}}$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ satisfies

$$|f_n^{[\boldsymbol{n}]}(x)| \leq \varepsilon\delta^{r-n} \quad \text{for all } \; x \in \mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta}. \tag{$**$}$$

Let $n = \nu$ and $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. By Inequality $(*)$, in particular for all $x, \vec{a} \in \mathrm{X}^{[\boldsymbol{n}]}$ with $\|x - \vec{a}\| \leq \delta$,

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \cdot \delta^\rho.$$

By Lemma 2.37, we find $\delta$-constant $g_{\boldsymbol{n}}$ such that $f_{\boldsymbol{n}} = f - g_{\boldsymbol{n}} *^{\boldsymbol{n}}$ satisfies

$$|f_{\boldsymbol{n}}^{[\boldsymbol{n}]}(x)| \leq \varepsilon\delta^\rho \quad \text{for all } \; x \in \mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta}.$$

Then we put $f_\nu := f - \sum_{\boldsymbol{n} \in \mathbb{N}^d_{=\nu}} g_{\boldsymbol{n}} *^{\boldsymbol{n}}$. We will prove Inequality $(**)$ for fixed $\boldsymbol{n}_0 \in \mathbb{N}^d_{=\nu}$. Let $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ be different from $\boldsymbol{n}_0$, so that in particular $\boldsymbol{n} \not\geq \boldsymbol{n}_0$. As $g_{\boldsymbol{n}} \colon \mathrm{X} \to \mathbf{K}$ is $\delta$-constant, we find by Lemma 2.11(ii) that $(g_{\boldsymbol{n}} *^{\boldsymbol{n}})^{[\boldsymbol{n}_0]} = 0$ and thus $f_\nu^{[\boldsymbol{n}_0]} = f_{\boldsymbol{n}_0}^{[\boldsymbol{n}_0]}$ on $\mathrm{X}^{[\boldsymbol{n}_0]}_{\leq\delta}$. Therefore by construction of $g_{\boldsymbol{n}} \colon \mathrm{X} \to \mathbf{K}$, we obtain

$$|f_\nu^{[\boldsymbol{n}_0]}(x)| \leq \varepsilon\delta^\rho \quad \text{for all } \; x \in \mathrm{X}^{[\boldsymbol{n}_0]}_{\leq\delta}.$$

Let $n < \nu$ and put $m = n + 1$. By induction hypothesis we have constructed $\delta$-constant functions $g_{\boldsymbol{i}} \colon \mathrm{X} \to \mathbf{K}$ for all $\boldsymbol{i} \in \mathbb{N}$ with $m \leq |\boldsymbol{i}| \leq \nu$ such that $f_m = f - \sum_{m \leq |\boldsymbol{i}| \leq \nu} g_{\boldsymbol{i}} *^{\boldsymbol{i}}$ for all $\boldsymbol{m} \in \mathbb{N}^d_{=m}$ satisfies

$$|f_m^{[\boldsymbol{m}]}(x)| \leq \varepsilon\delta^{r-m} \quad \text{for all } \; x \in \mathrm{X}^{[\boldsymbol{m}]}_{\leq\delta}.$$

Let $\boldsymbol{n} \in \mathbb{N}^d_{=n}$. By Lemma 2.36, for all $x, \vec{a} \in \mathrm{X}^{[\boldsymbol{n}]}$ with $\|x - \vec{a}\| \leq \delta$, we have

$$|f_m^{[\boldsymbol{n}]}(x) - f_m^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \cdot \delta^{r-m} \cdot \delta = \varepsilon \cdot \delta^{r-n}.$$

By Lemma 2.37, there is $\delta$-constant $g_{\boldsymbol{n}} \colon \mathrm{X} \to \mathbf{K}$ such that $f_{\boldsymbol{n}} = f_m - g_{\boldsymbol{n}} *^{\boldsymbol{n}}$ satisfies

$$|f_{\boldsymbol{n}}^{[\boldsymbol{n}]}(x)| \leq \varepsilon \cdot \delta^{r-n} \quad \text{for all } \; x \in \mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta}.$$

Then we put $f_n := f_m - \sum_{\boldsymbol{n} \in \mathbb{N}^d_{=n}} g_{\boldsymbol{n}} *^{\boldsymbol{n}}$. We will prove Inequality $(**)$ for fixed $\boldsymbol{n}_0 \in \mathbb{N}^d_{=n}$. Let $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ be different from $\boldsymbol{n}_0$, so that in particular $\boldsymbol{n} \not\geq \boldsymbol{n}_0$. As $g_{\boldsymbol{n}} \colon \mathrm{X} \to \mathbf{K}$ is $\delta$-constant, by Lemma 2.11(ii), we find $(g_{\boldsymbol{n}} *^{\boldsymbol{n}})^{[\boldsymbol{n}_0]} = 0$ and thus $f_n^{[\boldsymbol{n}_0]} = f_{\boldsymbol{n}_0}^{[\boldsymbol{n}_0]}$ on $\mathrm{X}_{\leq \delta}^{[\boldsymbol{n}_0]}$. Therefore by construction of $g_{\boldsymbol{n}} : \mathrm{X} \to \mathbf{K}$, we obtain

$$|f_n^{[\boldsymbol{n}_0]}(x)| \leq \varepsilon \delta^{\rho} \quad \text{for all} \;\; x \in \mathrm{X}_{\leq \delta}^{[\boldsymbol{n}_0]}.$$

This finishes the construction of the $g_{\boldsymbol{i}}$ for $\boldsymbol{i} \in \mathbb{N}_{\leq \nu}^d$.

Step 2.1.: We prove by induction on $|\boldsymbol{n}| =: n = 0, \ldots, \nu$ that $\|f_0^{[\boldsymbol{n}]}\|_{\sup} \leq \varepsilon \delta^{r-n}$ for $\boldsymbol{n} \in \mathbb{N}_{\leq \nu}^d$.

Let $n = 0$. Then $\delta\{{}^1x_0\} \vee \ldots \vee \delta\{{}^dx_0\} = 0 \leq \delta$ for all $({}^1x_0, \ldots, {}^dx_0) \in \mathrm{X}$. Hence $|f_0^{[\mathbf{0}]}({}^1x_0, \ldots, {}^dx_0)| \leq \varepsilon \delta^r$ for all $({}^1x_0, \ldots, {}^dx_0) \in \mathrm{X}$, that is, $\|f_0^{[\mathbf{0}]}\|_{\sup} \leq \varepsilon \delta^r$. Let $n \geq 1$. Then we split up

$$\|f_0^{[\boldsymbol{n}]}\|_{\sup} = \|f_0^{[\boldsymbol{n}]}\|_{\mathrm{X}_{\leq \delta}^{[\boldsymbol{n}]}} \vee \|f_0^{[\boldsymbol{n}]}\|_{\{x \in \mathrm{X}^{[\boldsymbol{n}]} \text{ s.t. } |{}^kx_i - {}^kx_j| > \delta \text{ for some coordinate } k \text{ and } i,j\}}.$$

Ad $\|f_0^{[\boldsymbol{n}]}\|_{\mathrm{X}_{\leq \delta}^{[\boldsymbol{n}]}} \leq \varepsilon \delta^{r-n}$:

Let $\boldsymbol{i} \in \mathbb{N}_{<n}^d$, so that in particular $\boldsymbol{i} \ngeq \boldsymbol{n}$. As $g_{\boldsymbol{i}} : \mathrm{X} \to \mathbf{K}$ is $\delta$-constant, by Lemma 2.11(ii), we find $(g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} = 0$ and thus $f_0^{[\boldsymbol{n}]} = f_n^{[\boldsymbol{n}]}$ on $\mathrm{X}_{\leq \delta}^{[\boldsymbol{n}]}$. Therefore restricted onto $\mathrm{X}_{\leq \delta}^{[\boldsymbol{n}]}$,

$$f_0^{[\boldsymbol{n}]} = (f - \sum_{\boldsymbol{i} \in \mathbb{N}_{\leq \nu}^d} g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} = (f - \sum_{n \leq |\boldsymbol{i}| \leq \nu} g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} = f_n^{[\boldsymbol{n}]}.$$

By construction of the $g_{\boldsymbol{i}} : \mathrm{X} \to \mathbf{K}$ for $\boldsymbol{i} \in \mathbb{N}^d$ with $n \leq |\boldsymbol{i}| \leq \nu$, we have $\|f_n^{[\boldsymbol{n}]}\|_{\mathrm{X}_{\leq \delta}^{[\boldsymbol{n}]}} \leq \varepsilon \delta^{r-n}$.

Ad $\|f_0^{[\boldsymbol{n}]}\|_{\{x \in \mathrm{X}^{[\boldsymbol{n}]} \text{ s.t. } |{}^kx_i - {}^kx_j| > \delta \text{ for some coordinate } k \text{ and } i,j\}} \leq \varepsilon \delta^{r-n}$:

Let $x \in \mathrm{X}^{[\boldsymbol{n}]}$ with $|{}^kx_i - {}^kx_j| > \delta$ for some coordinate $k \in \{1, \ldots, d\}$ and $i, j \in \{0, \ldots, n_k\}$. Assume we have shown that

$$|f_0^{[\boldsymbol{n}]}(x)| \leq \varepsilon \delta^{r-n} \quad \text{for all} \;\; x \in \mathrm{X}^{[\boldsymbol{n}]} \text{ with } |{}^kx_0 - {}^kx_1| > \delta.$$

Let $\sigma$ be the permutation on $\mathrm{X}_k^{[n_k]} = \mathrm{X}_k^{\{0, \ldots, n_k\}}$ swapping the $i$-th and $j$-th coordinate with the first and second one. Then by symmetry of $f_0^{[\boldsymbol{n}]}$ in its $\mathrm{X}_k$-coordinates, we have

$$\begin{aligned} \varepsilon \delta^{r-n} &\geq |f_0^{[\boldsymbol{n}]}(-; {}^kx^{\sigma}; -)| \\ &= |f_0^{[\boldsymbol{n}]}(-; {}^kx_i, {}^kx_j, \ldots, {}^kx_{n_k}; -)| \\ &= |f_0^{[\boldsymbol{n}]}(-; {}^kx_0, {}^kx_1, \ldots, {}^kx_{n_k}; -)|; \end{aligned}$$

here the hyphenations to the left and right of the semicolons representing the omitted arguments ${}^1x; \ldots; {}^{k-1}x$ and ${}^{k+1}x; \ldots; {}^dx$. Hence we are reduced to the case $|{}^kx_0 - {}^kx_1| > \delta$. Since $n_k \geq 1$, we can write $\boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_k$ for $\boldsymbol{m} \in \mathbb{N}_{=n-1}^d$. We compute

$$\begin{aligned} &|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(x)| \\ =&|({}^kx_0 - {}^kx_1)^{-1}[f_0^{[\boldsymbol{m}]}(-; {}^kx_0, {}^kx_2, \ldots, {}^kx_{n_k}; -) - f_0^{[\boldsymbol{m}]}(-; {}^kx_1, {}^kx_2, \ldots, {}^kx_{n_k}; -)]| \\ <&\delta^{-1} \cdot \|f_0^{[\boldsymbol{m}]}\|_{\sup} \leq \delta^{-1} \cdot \varepsilon \delta^{r-m} = \varepsilon \delta^{r-n}; \end{aligned}$$

the last inequality by the induction hypothesis for $|\boldsymbol{m}| = n-1$. This finishes the proof of $\|f_0^{[\boldsymbol{n}]}\|_{\sup} \leq \varepsilon\delta^{r-n}$ for $\boldsymbol{n} \in \mathbb{N}^d_{\leq \nu}$.

Step 2.2.: It remains to prove $\|f_0^{[\boldsymbol{n}]}\|_{\mathscr{C}^\rho} \leq \varepsilon$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. We have already proved $\|f_0^{[\boldsymbol{n}]}\|_{\sup} \leq \varepsilon\delta^\rho \leq \varepsilon$, so by Lemma 2.38, it remains to show $|||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\sup} \leq \varepsilon$ for $k = 1, \ldots, d$. We split its domain $\mathrm{X}^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ up via

$$|||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\sup} = \quad |||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\mathrm{X}^{[\boldsymbol{n}+\boldsymbol{e}_k]}_{\leq\delta}} \vee |||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\{x\in\mathrm{X}^{[\boldsymbol{n}+\boldsymbol{e}_k]} \text{ s.t. } |^k x_i - {}^k x_j|>\delta \text{ for some } k \text{ and } i,j\}}.$$

Ad $|||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\mathrm{X}^{[\boldsymbol{n}+\boldsymbol{e}_k]}_{\leq\delta}} \leq \varepsilon$:

Let $\boldsymbol{i} \in \mathbb{N}^d_{\leq\nu}$. As $g_{\boldsymbol{i}} : \mathrm{X} \to \mathbf{K}$ is $\delta$-constant, we find by Lemma 2.11(ii) that

$$(g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} = \begin{cases} g(p), & \text{if } \boldsymbol{i} = \boldsymbol{n}, \\ 0, & \text{if } \boldsymbol{i} \neq \boldsymbol{n} \end{cases}$$

on $\mathrm{P}^{[\boldsymbol{n}]}$ for a representative $p \in \mathrm{P} := \mathrm{B}_{\leq\delta}(p) \subseteq \mathrm{X}$. Therefore $|\tilde{g}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| = 0$ on $\mathrm{P}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}$ with $\tilde{g} := g_{\boldsymbol{i}} *^{\boldsymbol{i}}$ for $\boldsymbol{i} \in \mathbb{N}^d_{\leq\nu}$. By Lemma 2.35, we find $\mathrm{X}^{[\boldsymbol{n}]}_{\leq\delta} = \cup_{p\in\mathrm{X}} \mathrm{P}^{[\boldsymbol{n}]}$ with $\mathrm{P} := \mathrm{B}_{\leq\delta}(p) \subseteq \mathrm{X}$ for $p \in \mathrm{X}$ and hence $|\tilde{g}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| = 0$ on $\mathrm{X}^{[\boldsymbol{n}+\boldsymbol{e}_k]}_{\leq\delta}$. Therefore $|f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| = |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| \leq \varepsilon$ on $\mathrm{X}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}_{\leq\delta}$ by Inequality $(*)$.

Ad $\|f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}\|_{\{x \text{ s.t. } |^k x_i - {}^k x_j|>\delta \text{ for some coordinate } k \text{ and } i,j\}} \leq \varepsilon$:

Then

$$\|f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}\|_{\{x \text{ s.t. } |^k x_i - {}^k x_j|>\delta \text{ for some coordinate } k \text{ and } i,j\}} \leq \|f_0^{[\tilde{\boldsymbol{n}}]}\|_{\sup}/\delta^\rho \leq \varepsilon\delta^\rho/\delta^\rho;$$

the first inequality for $\tilde{\boldsymbol{n}} \in \mathbb{N}^d_{=\nu}$ by Lemma 2.39 and the second one by Step 2.1. This completes the proof of $\|f_0^{[\boldsymbol{n}]}\|_{\mathscr{C}^\rho} \leq \varepsilon$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$.

Step 3.: Finally put $g := \sum_{\boldsymbol{i}\in\mathbb{N}^d_{\leq\nu}} g_{\boldsymbol{i}} *^{\boldsymbol{i}}$. Then $g$ is a locally polynomial function of total degree at most $\nu$ and $f_0 = f - g$. Then $\|f - g\|_{\mathscr{C}^r} = \max_{\boldsymbol{n}\in\mathbb{N}^d_{<\nu}} \|f_0^{[\boldsymbol{n}]}\|_{\sup} \vee \max_{\boldsymbol{n}\in\mathbb{N}^d_{=\nu}} \|f_0^{[\boldsymbol{n}]}\|_{\mathscr{C}^\rho} \leq \varepsilon$, q.e.d. □

During the following proof we will use terminology introduced in the next subsection's subsubsection about topological tensor products.

**Lemma 2.41.** *The closure of the set of all polynomial functions inside the* $\mathbf{K}$*-Banach space* $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ *contains all locally constant functions.*

*Proof:* The proof is divided into two steps.

(i) Fix an indicator function $\mathbf{1}_\mathrm{B} : \mathrm{X} \to \mathbf{K}$ of a closed ball $\mathrm{B} \subseteq \mathrm{X}$ of positive radius and $\varepsilon > 0$. Then there is a polynomial function $p : \mathrm{X} \to \mathbf{K}$ such that $\|\mathbf{1}_\mathrm{B} - p\|_{\mathscr{C}^r} \leq \varepsilon$.

(ii) The closure of the polynomial functions inside $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ contains all locally constant functions.

Ad (i): By Remark 2.32, it suffices to prove that there is a polynomial function $p\colon X \to \mathbf{K}$ such that $\|\mathbf{1}_B - p\|_{\mathscr{C}^{\vec{\nu}}} \le \varepsilon$ for $\vec{\nu} := (\nu, \dots, \nu) \in \mathbb{N}^d$ with $\nu \ge r$. This is done by induction on $d \ge 1$. If $d = 1$, then this will be taken care of by [AS94, Corollary 1.3]. Let $d > 1$. Let $B = B' \times B'' \subseteq X$ with $B' = B_1 \times \cdots \times B_{d-1} \subseteq X_1 \times \cdots \times X_{d-1} =: X'$ and $B'' := B_d \subseteq X_d =: X''$. By induction, there is a polynomial function $p'\colon X' \to \mathbf{K}$ with $\|\mathbf{1}_{B'} - p'\|_{\mathscr{C}^{\vec{\nu}}} \cdot M'' \le \varepsilon$ with $M'' = \|\mathbf{1}_{B''}\|_{\mathscr{C}^{\nu}} \ge 0$. Then by the case $d = 1$, there is a polynomial function $p''\colon X'' \to \mathbf{K}$ with $\|\mathbf{1}_{B''} - p''\|_{\mathscr{C}^{\nu}} \cdot M' \le \varepsilon$ with $M' = \|p'\|_{\mathscr{C}^{\vec{\nu}}} \ge 0$. We put $p := p' \odot p''\colon X \to \mathbf{K}$ and compute

$$\begin{aligned}
\|\mathbf{1}_B - p\|_{\mathscr{C}^{\vec{\nu}}} =& \|\mathbf{1}_{B'} \odot \mathbf{1}_{B''} - p' \odot p''\|_{\mathscr{C}^{\vec{\nu}}} \\
\le& \|\mathbf{1}_{B'} \odot \mathbf{1}_{B''} - p' \odot \mathbf{1}_{B''}\|_{\mathscr{C}^{\vec{\nu}}} \vee \|p' \odot \mathbf{1}_{B''} - p' \odot p''\|_{\mathscr{C}^{\vec{\nu}}} \\
=& \|(\mathbf{1}_{B'} - p') \odot \mathbf{1}_{B''}\|_{\mathscr{C}^{\vec{\nu}}} \vee \|p' \odot (\mathbf{1}_{B''} - p'')\|_{\mathscr{C}^{\vec{\nu}}} \\
=& \|\mathbf{1}_{B'} - p'\|_{\mathscr{C}^{\vec{\nu}}} \cdot \|\mathbf{1}_{B''}\|_{\mathscr{C}^{\nu}} \vee \|p'\|_{\mathscr{C}^{\vec{\nu}}} \cdot \|\mathbf{1}_{B''} - p''\|_{\mathscr{C}^{\nu}} \le \varepsilon.
\end{aligned}$$

Ad (ii): The closed balls $B \subseteq X$ constitute a basis of the topological space $X \subseteq \mathbf{K}^d$. Hence by compactness of X, every locally constant function $g$ is the finite sum $f = \sum_i \lambda_i \mathbf{1}_{B_i}$ with $\lambda_i \in \mathbf{K}$ and indicator functions $\mathbf{1}_{B_i}$ of closed balls $B_i \subseteq X$ for $i \in I$. By (i), for every $\varepsilon > 0$, there are polynomial functions $p_i\colon X \to \mathbf{K}$ such that $\|p_i - \mathbf{1}_{B_i}\|_{\mathscr{C}^r} M_i \le \varepsilon$ with $M_i := |\lambda_i| \ge 0$. Then $p := \sum_i \lambda_i p_i\colon X \to \mathbf{K}$ satisfies $\|p - f\|_{\mathscr{C}^r} \le \max_i |\lambda_i| \|p_i - \mathbf{1}_{B_i}\|_{\mathscr{C}^r} \le \varepsilon$. □

**Corollary 2.42.** *The polynomial functions are dense in $\mathscr{C}^r(X, \mathbf{K})$.*

*Proof:* Fix $f \in \mathscr{C}^r(X, \mathbf{K})$ and $\varepsilon > 0$. By Proposition 2.40, there is a locally polynomial function $g = \sum_{\boldsymbol{i} \in \mathbb{N}^d_{\le \nu}} g_{\boldsymbol{i}} *^{\boldsymbol{i}}\colon X \to \mathbf{K}$ with locally constant $g_{\boldsymbol{i}}$ such that $\|f - g\|_{\mathscr{C}^r} \le \varepsilon$. By Lemma 2.41, there are polynomial functions $p_{\boldsymbol{i}}\colon X \to \mathbf{K}$ with $\|p_{\boldsymbol{i}} - g_{\boldsymbol{i}}\| \cdot M_{\boldsymbol{i}} \le \varepsilon$ with $M_{\boldsymbol{i}} = \|*^{\boldsymbol{i}}\|_{\mathscr{C}^r} > 0$ for all $\boldsymbol{i} \in \mathbb{N}^d_{\le \nu}$. Then the polynomial function $p := \sum_{\boldsymbol{i} \in \mathbb{N}^d_{\le \nu}} p_{\boldsymbol{i}} *^{\boldsymbol{i}}\colon X \to \mathbf{K}$ satisfies

$$\|p - g\|_{\mathscr{C}^r} \le \max_{\boldsymbol{i} \in \mathbb{N}^d_{\le \nu}} \|p_{\boldsymbol{i}} - g_{\boldsymbol{i}}\|_{\mathscr{C}^r} \cdot \|*^{\boldsymbol{i}}\|_{\mathscr{C}^r} \le \varepsilon$$

and therefore $\|p - f\|_{\mathscr{C}^r} \le \|p - g\|_{\mathscr{C}^r} \vee \|g - f\|_{\mathscr{C}^r} \le \varepsilon$. □

*The Mahler basis of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$.* We can now describe $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ as an intersection of completed tensor products of fractionally differentiable functions of one variable on $\mathbb{Z}_p$, whose Mahler coefficients have already been computed in [Nag12, Theorem 2.58].

*Assumption.* We will throughout this subsection's subsubsection on the Mahler basis of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ assume that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a normed field.

**Definition 2.43.** For $i \in \mathbb{N}$, we define $\binom{*}{i}\colon \mathbb{Z}_p \to \mathbf{K}$ by

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}.$$

Then the $\boldsymbol{i}$-th **Mahler polynomial** $\binom{*}{\boldsymbol{i}}\colon \mathbb{Z}_p^d \to \mathbf{K}$ for $\boldsymbol{i} \in \mathbb{N}^d$ will be given by $\binom{*}{\boldsymbol{i}} := \binom{*}{i_1} \odot \cdots \odot \binom{*}{i_d}$.

**Lemma 2.44.** *For $\boldsymbol{r} \in \mathbb{N}^d_{=r}$ the subset $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ is an orthogonal family with $\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^{\boldsymbol{r}}} = p^{w_{r_1}(i_1)+\cdots+w_{r_d}(i_d)}$; here $w_{r_1}(i_1), \ldots, w_{r_d}(i_d)$ as in [Nag12, Theorem 2.58].*

*Proof:* By [Nag12, Theorem 2.58], the family $\{\binom{*}{i}\} \subseteq \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})$ is in particular an orthogonal family. By Lemma 2.33, we find $\{\binom{*}{\boldsymbol{i}} = \binom{*}{i_1} \odot \cdots \odot \binom{*}{i_d}\} \subseteq \mathscr{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with

$$\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^{\boldsymbol{r}}} = \left\|\binom{*}{i_1}\right\|_{\mathscr{C}^{r_1}} \cdots \left\|\binom{*}{i_d}\right\|_{\mathscr{C}^{r_d}} = p^{w_{r_1}(i_1)} \cdots p^{w_{r_d}(i_d)};$$

the last equality by [Nag12, Theorem 2.58]. □

**Theorem 2.45.** *The family $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ is an orthogonal basis and*

$$\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^r} = p^{w_r(\boldsymbol{i})} \quad \textit{with } w_r(\boldsymbol{i}) = \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} w_{r_1}(i_1) + \cdots + w_{r_d}(i_d);$$

*here $w_{r_1}(i_1), \ldots, w_{r_d}(i_d)$ as in [Nag12, Theorem 2.58].*

*Proof:* By Lemma 2.44, we find $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with valuations $\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^{\boldsymbol{r}}} = p^{w_{r_1}(i_1)+\cdots+w_{r_d}(i_d)}$ for all $\boldsymbol{r} \in \mathbb{N}^d_{=r}$. Consequently by Proposition 2.31 and Lemma 2.29(i), we find $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with $\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r} = \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} \|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^{\boldsymbol{r}}}$. By [Sch84, Exercise 50.F], an orthogonal family whose $\mathbf{K}$-linear span is dense is an orthogonal base. It thus remains to show that the $\mathbf{K}$-linear span of $\{\binom{*}{\boldsymbol{i}}\}$ is dense in $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$.

As this span consists of all polynomial functions, it is by Corollary 2.42 indeed dense inside $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. □

**Definition.** For sequences $(u_i)$ and $(w_i)$ with values in $\mathbb{R}_{>0}$, running over the same index set I, we introduce the equivalence relation

$$(u_i) \sim (w_i) \text{ if there are } 0 < c \leq 1 \leq C \text{ with } c \cdot u_i \leq w_i \leq C \cdot u_i \text{ for all } i \in I.$$

**Lemma 2.46.** *We have $(\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r})_{\boldsymbol{i} \in \mathbb{N}^d} \sim (i_1^r \vee \ldots \vee i_d^r)_{\boldsymbol{i} \in \mathbb{N}^d}$.*

*Proof:* By [Nag12] we find for every $r \in \mathbb{R}_{\geq 0}$ positive constants $c(r) \leq 1 \leq C(r)$ with $c(r) \cdot m^r \leq p^{w_r(m)} \leq C(r) \cdot m^r$ for every $m \in \mathbb{N}$.

For $\boldsymbol{r} \in \mathbb{N}^d_{=r}$, define the positive constants

$$c(\boldsymbol{r}) := c(r_1) \cdots c(r_d) \leq 1 \leq C(\boldsymbol{r}) := C(r_1) \cdots C(r_d).$$

Then by Lemma 2.44, for all $\boldsymbol{i} \in \mathbb{N}^d$ holds

$$c(\boldsymbol{r}) \cdot i_1^{r_1} \cdots i_d^{r_d} \leq \left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^{\boldsymbol{r}}} \leq C(\boldsymbol{r}) \cdot i_1^{r_1} \cdots i_d^{r_d}.$$

Assume that $i_k = i_1 \vee \ldots \vee i_d$. Then $i_1^{r_1} \cdots i_d^{r_d} \leq i_k^{r_1} \cdots i_k^{r_d} = i_k^r$. Hence $i_1^{r_1} \cdots i_d^{r_d}$ is maximal among $\{i_1^{r_1} \cdots i_d^{r_d} : \boldsymbol{r} \in \mathbb{N}^d_{=r}\}$ if and only if $r_k = r$. Therefore $\max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} i_1^{r_1} \cdots i_d^{r_d} = i_1^r \vee$

$\ldots \vee i_d^r$. Defining the positive constants $c(r) := \min_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} c(\boldsymbol{r}) \le 1 \le \mathrm{C}(r) := \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} \mathrm{C}(\boldsymbol{r})$, it follows by the preceding Theorem 2.45 in particular that for all $\boldsymbol{i} \in \mathbb{N}^d$ holds

$$c(r) \cdot (i_1 \vee \ldots \vee i_d)^r \le \left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathscr{C}^r} \le \mathrm{C}(r) \cdot (i_1 \vee \ldots \vee i_d)^r.$$

**Corollary 2.47.** *We have* $(\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r})_{\boldsymbol{i} \in \mathbb{N}^d} \sim (|\boldsymbol{i}|^r)_{\boldsymbol{i} \in \mathbb{N}^d}$.

*Proof:* By Lemma 2.46, there are positive constants $\tilde{c} \le 1 \le \mathrm{C}$ with $\tilde{c} \cdot (i_1 \vee \ldots \vee i_d)^r \le \|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r} \le \mathrm{C} \cdot (i_1 \vee \ldots \vee i_d)^r$. Since $i_1 \vee \ldots \vee i_d \le i_1 + \cdots + i_d \le d \cdot (i_1 \vee \ldots \vee i_d)$, the asserted constriction holds for the positive constants $c := \tilde{c}/d^r \le 1 \le \mathrm{C}$. □

## Description of $\mathscr{C}^r(\mathrm{X}, \mathbf{K})$ for open $\mathrm{X} \subseteq \mathbb{Q}_p^d$ through Taylor polynomials

In this section, we use the just obtained description of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ by their Mahler coefficients to deduce a convenient characterization of $\mathscr{C}^r$-functions on open domains inside $\mathbb{Q}_p^d$ by their Taylor polynomials.

*Assumption.* We will throughout this subsection assume $\mathbf{K}$ to be a complete nontrivially non-Archimedeanly valued locally compact field.

**Definition 2.48.** Let $\mathrm{X} \subseteq \mathbf{K}^d$ be an open subset and $k \in \{1, \ldots, d\}$. We will speak of a $\mathscr{C}_\mathrm{T}^{r \cdot \boldsymbol{e}_k}$-**function** $f\colon \mathrm{X} \to \mathbf{K}$ if there are *continuous* functions $\mathrm{D}_{\mathbf{0}} f, \mathrm{D}_{1 \cdot \boldsymbol{e}_k} f, \ldots, \mathrm{D}_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X} \to \mathbf{K}$ such that if one defines $\mathrm{R}_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{K}$ on $\mathrm{X}^{[\boldsymbol{e}_k]} := \{(x; t) \in \mathrm{X} \times \mathbf{K}$ with $x + t \cdot \boldsymbol{e}_k \in \mathrm{X}\}$ by

$$\mathrm{R}_{\nu \cdot \boldsymbol{e}_k} f(x; t) := f(x + t \cdot \boldsymbol{e}_k) - \sum_{i=0, \ldots, \nu} \mathrm{D}_{i \cdot \boldsymbol{e}_k} f(x) t^i,$$

then for every point $a \in \mathrm{X}$ and any $\varepsilon > 0$, there will exist a neighborhood $\mathrm{U} \ni a$ such that

$$|\mathrm{R}_{\nu \cdot \boldsymbol{e}_k} f(x; t)| \le \varepsilon |t|^r \quad \text{for all} \quad x + t \cdot \boldsymbol{e}_k, x \in \mathrm{U}.$$

We will denote the set of all $\mathscr{C}_\mathrm{T}^{r \cdot \boldsymbol{e}_k}$-functions $f\colon \mathrm{X} \to \mathbf{K}$ by $\mathscr{C}_\mathrm{T}^{r \cdot \boldsymbol{e}_k}(\mathrm{X}, \mathbf{K})$.

*Remark.* Since $\mathrm{R}_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{K}$ vanishes on $\mathrm{X} \times \{0\}$, we see that $f = \mathrm{D}_0 f$. Moreover the continuity of $\mathrm{D}_0 f, \mathrm{D}_{1 \cdot \boldsymbol{e}_k} f, \ldots, \mathrm{D}_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X} \to \mathrm{K}$ implies the continuity of $\mathrm{R}_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{K}$. By the above convergence condition, we even have a continuous mapping $\Delta_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathbf{K}$, defined as the extension of the function $\Delta_{\nu \cdot \boldsymbol{e}_k} f(x; t) := \mathrm{R}_{\nu \cdot \boldsymbol{e}_k} f(x; y)/t^\nu$ with domain $\mathrm{X}^{]\boldsymbol{e}_k[} := \{(x; t) \in \mathrm{X} \times \mathbf{K}^*$ with $x + t \cdot \boldsymbol{e}_k \in \mathrm{X}\}$ and which will vanish if $t$ does.

**Lemma 2.49.** *The functions* $\mathrm{D}_{\mathbf{0}} f, \mathrm{D}_{1 \cdot \boldsymbol{e}_k} f, \ldots, \mathrm{D}_{\nu \cdot \boldsymbol{e}_k} f\colon \mathrm{X} \to \mathbf{K}$ *in Definition 2.48 are unique.*

*Proof:* This is proved by induction on $\nu \ge 0$, see [Nag11, Lemma 3.52]. □

**Definition.** Fix a coordinate index $k \in \{1, \ldots, d\}$.

(i) Let $f \in \mathscr{C}_T^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$. We define functions $\Delta_{\nu\cdot \boldsymbol{e}_k} f : X^{]\boldsymbol{e}_k[} \to \mathbf{K}$ and $|\Delta_{r\cdot \boldsymbol{e}_k} f| : X^{]\nu\cdot \boldsymbol{e}_k[} \to \mathbb{R}_{\geq 0}$ by putting

$$\Delta_{\nu\cdot \boldsymbol{e}_k} f(x;t) := \frac{R_{\nu\cdot \boldsymbol{e}_k} f(x;t)}{t^\nu} \quad \text{and} \quad |\Delta_{r\cdot \boldsymbol{e}_k} f|(x;t) := \frac{|R_{\nu\cdot \boldsymbol{e}_k} f(x;t)|}{|t|^r}.$$

Since $f \in \mathscr{C}_T^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$, these functions will extend continuously onto $X^{[\boldsymbol{e}_k]}$ if we let them vanish for $t = 0$. We denote these extensions likewise.

(ii) By Lemma 2.49, the functions $D_{0\cdot \boldsymbol{e}_k}, \ldots, D_{\nu\cdot \boldsymbol{e}_k} f : X \to \mathbf{K}$ of Lemma 2.49 are uniquely determined continuous functions. So it makes sense to endow $\mathscr{C}_T^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$ with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\mathscr{C}_T^{r\cdot \boldsymbol{e}_k}, C}\}$ running through all compact subsets $C \subseteq X$ defined by

$$\|f\|_{\mathscr{C}_T^{r\cdot \boldsymbol{e}_k}, C} := \|D_{\mathbf{0}} f\|_C \vee \|D_{1\cdot \boldsymbol{e}_k} f\|_C \vee \ldots \vee \|D_{\nu\cdot \boldsymbol{e}_k} f\|_C \vee \||\Delta_{r\cdot \boldsymbol{e}_k} f|\|_{C\times C}.$$

*The next five lemmata hold for a general coordinate index $k \in \{1, \ldots, d\}$, but will for notational convenience only be stated and proved for $k = 1$.*

**Lemma 2.50.** *Let $f \in \mathscr{C}^{r\cdot \boldsymbol{e}_1}(X, \mathbf{K})$. Then for $(x;t) \in X^{[\boldsymbol{e}_1]}$ holds*

$$f(x + t\cdot \boldsymbol{e}_1) = \sum_{i=0,\ldots,\nu-1} D_{i\cdot \boldsymbol{e}_1} f(x) t^i + f^{[\nu\cdot \boldsymbol{e}_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d) t^\nu$$

*with* continuous *functions $D_{i\cdot \boldsymbol{e}_1} f : X \to \mathbf{K}$ and the $\mathscr{C}^{\rho\cdot \boldsymbol{e}_1}$-function $f^{[\nu\cdot \boldsymbol{e}_1]} : X^{[\nu\cdot \boldsymbol{e}_1]} \to \mathbf{K}$.*

*Proof:* This is proved by induction on $\nu$, see [Nag11, Lemma 3.52]. □

*Remark.* We note that the preceding Lemma 2.50 yields an inclusion of locally convex $\mathbf{K}$-vector spaces $\mathscr{C}^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K}) \subseteq \mathscr{C}_T^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$ for any coordinate index $k \in \{1, \ldots, d\}$. For this, note that by uniqueness necessarily $D_{i\cdot \boldsymbol{e}_1} f = D_{i\cdot \boldsymbol{e}_1} f$ for $i = 0, \ldots, \nu$ and $\Delta_{\nu\cdot \boldsymbol{e}_k} f(x;t) = f^{[\nu\cdot \boldsymbol{e}_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d)$.

**Lemma 2.51.** *Let $f \in \mathscr{C}_T^{r\cdot \boldsymbol{e}_1}(X, \mathbf{K})$. Then $\binom{j}{j} D_{j\cdot \boldsymbol{e}_1} f, \binom{j+1}{j} D_{j+1\cdot \boldsymbol{e}_1} f, \ldots, \binom{\nu}{j} D_{\nu\cdot \boldsymbol{e}_1} f$ prove $D_{j\cdot \boldsymbol{e}_1} f$ to be in $\mathscr{C}_T^{r-j\cdot \boldsymbol{e}_1}(X, \mathbf{K})$ for $j = 0, \ldots, \nu$.*

*Proof:* Let $f \in \mathscr{C}_T^{r\cdot \boldsymbol{e}_1}(X, \mathbf{K})$. We show that the continuous functions

$$\binom{j}{j} D_{j\cdot \boldsymbol{e}_1} f, \binom{j+1}{j} D_{j+1\cdot \boldsymbol{e}_1} f, \ldots, \binom{\nu}{j} D_{\nu\cdot \boldsymbol{e}_1} f$$

prove $D_{j\cdot \boldsymbol{e}_1} f$ to be in $\mathscr{C}_T^{r-j\cdot \boldsymbol{e}_1}(X, \mathbf{K})$ for fixed $j \in \{0, \ldots, \nu\}$. Fix $\varepsilon > 0$ and $a \in X$. We find a ball $U = U_1 \times \cdots \times U_d \ni a$ such that

$$|R_\nu f(x; x_1 - y_1)| \leq \varepsilon |x_1 - y_1|^r \text{ for all } x = (x_1, x_2, \ldots, x_d), (y_1, x_2, \ldots, x_d) \in U.$$

As $U_1$ is likewise a ball, it has the $B_\nu$-property by [Nag12, Lemma 2.24]. If $t := x_1 - y_1 \neq 0$, then we will fix $x_2, \ldots, x_d$ and find by [Nag12] applied to $f_{x_2,\ldots,x_d} := f(\_, x_2, \ldots, x_d) \in \mathscr{C}^r_T(U_1, \mathbf{K})$ a uniform constant $C > 0$ (only depending on $U_1$), a finite subset $P \subseteq B_{\leq\delta}(x_1) \subseteq U_1$ with $\delta := |t| > 0$ such that

$$\begin{aligned} |R_{\nu - j\cdot \boldsymbol{e}_1} D_{j\cdot\boldsymbol{e}_1} f(x;t)| \leq & C|t|^{-j} \max_{z_0 = x_1, y_1 \text{ and } z \in P} |R_{\nu\cdot\boldsymbol{e}_1} f(z, x_2, \ldots, x_d; z_0 - z)| \\ \leq & C|t|^{-j} \max_{z_0 = x_1, y_1 \text{ and } z \in P} \varepsilon |z_0 - z|^r \leq C\varepsilon |t|^{r-j}; \end{aligned}$$

the middle inequality as $(z_0, x_2, \ldots, x_d), (z, x_2, \ldots, x_d) \in U$ and the last one since $|z_0 - z| \leq \delta$, both points $z_0 = x_1, y_1$ being the centers of $B_{\leq\delta}(x_1)$. If $t = 0$, this inequality will hold trivially. □

**Definition.** Let $\boldsymbol{g} \in \mathbb{N}^d$. Then we define a locally polynomial function $f\colon X \to \mathbf{K}$ to have **degree at most $\boldsymbol{g}$** if for every point $a \in X$, there will exist a neighborhood $U \ni a$ such that $f_{|U} = p_{|U}$ for a polynomial function $p = \sum_{\boldsymbol{i} \leq \boldsymbol{g}} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$.

We will denote the set of all locally polynomials functions $f\colon X \to \mathbf{K}$ of degree at most $\boldsymbol{g}$ by $\mathscr{C}^{\mathrm{pol}}_{\leq \boldsymbol{g}}(X, \mathbf{K})$.

**Lemma 2.52.** *We have a dense inclusion $\mathscr{C}^{\mathrm{pol}}_{\leq \nu\cdot\boldsymbol{e}_1}(X, \mathbf{K}) \subseteq \mathscr{C}^{r\cdot\boldsymbol{e}_1}_T(X, \mathbf{K})$ of the locally polynomial functions of degree $\boldsymbol{g} \leq \nu \cdot \boldsymbol{e}_1$ into the $\mathscr{C}^{r\cdot\boldsymbol{e}_1}_T$-functions.*

*Proof:* For this statement to be meaningful, recall that by Remark 2.34 and Lemma 2.50 above, we have a chain of inclusions

$$\mathscr{C}^{\mathrm{pol}}_{\leq \nu\cdot\boldsymbol{e}_1}(X, \mathbf{K}) \subseteq \mathscr{C}^r(X, \mathbf{K}) \subseteq \mathscr{C}^{r\cdot\boldsymbol{e}_1}(X, \mathbf{K}) \subseteq \mathscr{C}^{r\cdot\boldsymbol{e}_1}_T(X, \mathbf{K}).$$

Fix $f \in \mathscr{C}^{r\cdot\boldsymbol{e}_1}_T(X, \mathbf{K})$ and $\varepsilon > 0$. By definition of either function space's locally convex topology, we may assume $X \subseteq \mathbf{K}^d$ to be compact cartesian. By Lemma 2.51, we find $D_{n\cdot\boldsymbol{e}_1} f \in \mathscr{C}^{r - n\cdot\boldsymbol{e}_1}(X, \mathbf{K})$ for $n = 0, \ldots, \nu$. By compactness, we find $0 < \delta_1 \leq 1$ such that for $n = 0, \ldots, \nu$ holds

$$|R_{r - n\cdot\boldsymbol{e}_1} D_{n\cdot\boldsymbol{e}_1} f(x;t)| \leq \varepsilon |t|^{r-n} \quad \text{for all } \ x + t\cdot\boldsymbol{e}_1, x \in X \text{ with } |t| \leq \delta_1. \tag{$*$}$$

We will fix $\delta := \delta_1 > 0$ and $\boldsymbol{\delta} := (\delta_1, 0, \ldots, 0) \in [0,1]^d$ until the end of this proof. By downward induction on $n = \nu, \ldots, 0$, we will inductively construct locally $\boldsymbol{\delta}$-constant (see Definition 1.7) functions $g_{\nu\cdot\boldsymbol{e}_1}, \ldots, g_{n\cdot\boldsymbol{e}_1}\colon X \to \mathbf{K}$ such that

$$\|D_{n\cdot\boldsymbol{e}_1} f - \binom{\nu}{n} g_{\nu\cdot\boldsymbol{e}_1} *^{\nu - n\cdot\boldsymbol{e}_1} - \cdots - \binom{n+1}{n} g_{n+1\cdot\boldsymbol{e}_1} *^{\boldsymbol{e}_1} - g_{n\cdot\boldsymbol{e}_1}\|_{\sup} \leq \varepsilon \delta^{r-n}.$$

Let $n = \nu$. By $(*)$ for $n = \nu$,

$$|D_{\nu\cdot\boldsymbol{e}_1} f(x_1', x_2, \ldots, x_d) - D_{\nu\cdot\boldsymbol{e}_1} f(x_1, x_2, \ldots, x_d)| \leq \varepsilon |x_1' - x_1|^{\rho}$$

for all $(x_1', x_2, \dots, x_d), (x_1, x_2, \dots, x_d) \in \mathrm{X}$ with $|x_1' - x_1| \leq \delta_1$. By Lemma 1.9 applied to $\boldsymbol{\delta} = (\delta, 0, \dots, 0)$, we find locally $\boldsymbol{\delta}$-constant $g_{\nu\cdot\boldsymbol{e}_1} : \mathrm{X} \to \mathbf{K}$ such that

$$\|\mathrm{D}_{\nu\cdot\boldsymbol{e}_1} f - g_{\nu\cdot\boldsymbol{e}_1}\|_{\sup} \leq \varepsilon\delta^{\rho}.$$

Let $n < \nu$ and assume we constructed locally $\boldsymbol{\delta}$-constant functions $g_{\nu\cdot\boldsymbol{e}_1}, \dots, g_{n+1\cdot\boldsymbol{e}_1} : \mathrm{X} \to \mathbf{K}$ such that

$$\|\mathrm{D}_{m\cdot\boldsymbol{e}_1} f - \binom{\nu}{m} g_{\nu\cdot\boldsymbol{e}_1} *^{\nu-m\cdot\boldsymbol{e}_1} - \dots - g_{m\cdot\boldsymbol{e}_1}\|_{\sup} \leq \varepsilon\delta^{r-m} \quad \text{for } m = \nu, \dots, n+1.$$

We put $\breve{f}_{n\cdot\boldsymbol{e}_1} := \mathrm{D}_{n\cdot\boldsymbol{e}_1} f - \binom{\nu}{n} g_{\nu\cdot\boldsymbol{e}_1} *^{\nu-n\cdot\boldsymbol{e}_1} - \dots - \binom{n+1}{n} g_{n+1\cdot\boldsymbol{e}_1} *^{\boldsymbol{e}_1}$. Let us pick two points $(x_1', x_2, \dots, x_d)$ and $(x_1, x_2, \dots, x_d)$ in X. We will prove

$$|\breve{f}_{n\cdot\boldsymbol{e}_1}(x_1', x_2, \dots, x_d) - \breve{f}_{n\cdot\boldsymbol{e}_1}(x_1, x_2, \dots, x_d)| \leq \varepsilon\delta^{r-n} \quad \text{if } |x_1' - x_1| \leq \delta.$$

Then by Lemma 1.9, there is locally $\boldsymbol{\delta}$-constant $g_{n\cdot\boldsymbol{e}_1} : \mathrm{X} \to \mathbf{K}$ such that $f_{n\cdot\boldsymbol{e}_1} := \breve{f}_{n\cdot\boldsymbol{e}_1} - g_{n\cdot\boldsymbol{e}_1}$ has norm $\|f_{n\cdot\boldsymbol{e}_1}\|_{\sup} \leq \varepsilon\delta^{r-n}$. This will complete the $n$-th construction step since

$$f_{n\cdot\boldsymbol{e}_1} = \breve{f}_{n\cdot\boldsymbol{e}_1} - g_{n\cdot\boldsymbol{e}_1} = \mathrm{D}_{n\cdot\boldsymbol{e}_1} f - \binom{\nu}{n} g_{\nu\cdot\boldsymbol{e}_1} *^{\nu-n\cdot\boldsymbol{e}_1} - \dots - \binom{n+1}{n} g_{n+1\cdot\boldsymbol{e}_1} *^{\boldsymbol{e}_1} - g_{n\cdot\boldsymbol{e}_1}.$$

Put $\mathrm{X} = \mathrm{X}' \times \mathrm{X}''$ with $\mathrm{X}' = \mathrm{X}_1$ and $\mathrm{X}'' = \mathrm{X}_2 \times \dots \times \mathrm{X}_d$ and let $(x' + h', x''), (x', x'') \in \mathrm{X}' \times \mathrm{X}''$ with $h' \in \mathbf{K}$. We compute

$$\begin{aligned}
&|\breve{f}_{n\cdot\boldsymbol{e}_1}(x' + h', x'') - \breve{f}_{n\cdot\boldsymbol{e}_1}(x', x'')| \\
=&|\quad \mathrm{D}_{n\cdot\boldsymbol{e}_1} f(x' + h', x'') - \mathrm{D}_{n\cdot\boldsymbol{e}_1} f(x', x'') \\
&- \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} g_{n+i\cdot\boldsymbol{e}_1}(x', x'')((x' + h')^i - x'^i)| \\
\leq &\quad |\mathrm{D}_{n\cdot\boldsymbol{e}_1} f(x' + h', x'') - \mathrm{D}_{n\cdot\boldsymbol{e}_1} f(x', x'') - \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} \mathrm{D}_{i\cdot\boldsymbol{e}_1} f(x', x'') h'^i| \\
&\vee |\quad \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} \mathrm{D}_{n+i\cdot\boldsymbol{e}_1} f(x', x'') h'^i \\
&\quad - \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} g_{n+i\cdot\boldsymbol{e}_1}(x', x'')((x' + h')^i - x'^i)|;
\end{aligned}$$

the first equality by $g_{n+1\cdot\boldsymbol{e}_1}, \dots, g_{\nu\cdot\boldsymbol{e}_1} : \mathrm{X} \to \mathbf{K}$ being locally $\boldsymbol{\delta}$-constant. To prove the claimed inequality above, we will assume from now on $|h'| \leq \delta$. We can then estimate by Inequality $(*)$ the above maximum's first absolute value through

$$\begin{aligned}
&|\mathrm{D}_{n\cdot\boldsymbol{e}_1} f(x' + h', x'') - \mathrm{D}_{n\cdot\boldsymbol{e}_1} f(x', x'') - \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} \mathrm{D}_{n+i\cdot\boldsymbol{e}_1} f(x', x'') h'^i| \\
=&|\mathrm{R}_{r-n\cdot\boldsymbol{e}_1} \mathrm{D}_{n\cdot\boldsymbol{e}_1} f((x', x''); h)| \leq \varepsilon |h'|^{r-n} \leq \varepsilon\delta^{r-n}.
\end{aligned}$$

Regarding the second term, let us fix $n \in \mathbb{N}$. We use the binomial identity and rearrange the summation order to obtain

$$\begin{aligned}
&\sum_{i=1,\dots,\nu-n} \binom{n+i}{n} g_{n+i\cdot\boldsymbol{e}_1}(x',x'')((x'+h')^i - x'^i) \\
=& \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} g_{n+i\cdot\boldsymbol{e}_1}(x',x'') \sum_{j=1,\dots,i} \binom{i}{j} x'^{i-j} h'^j \\
=& \sum_{j=1,\dots,\nu-n} h'^j \sum_{i=0,\dots,\nu-n-j} \binom{i+j}{j}\binom{n+i+j}{n} g_{n+i+j\cdot\boldsymbol{e}_1}(x',x'')x'^i \\
=& \sum_{j=1,\dots,\nu-n} h'^j \sum_{i=0,\dots,\nu-(n+j)} \binom{n+j}{n}\binom{n+j+i}{n+j} g_{n+j+i\cdot\boldsymbol{e}_1}(x',x'')x'^i.
\end{aligned}$$

We obtain

$$\begin{aligned}
&| \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} \mathrm{D}_{n+i\cdot\boldsymbol{e}_1} f(x',x'')h'^i \\
&- \sum_{i=1,\dots,\nu-n} \binom{n+i}{n} g_{n+i\cdot\boldsymbol{e}_1}(x',x'')((x'+h')^i - x'^i)| \\
=&| \sum_{j=1,\dots,\nu-n} h'^j \binom{n+j}{n} [\mathrm{D}_{n+j\cdot\boldsymbol{e}_1} f(x',x'') \\
&- \sum_{i=0,\dots,\nu-(n+j)} \binom{n+j+i}{n+j} g_{n+j+i\cdot\boldsymbol{e}_1}(x',x'')x'^i]| \\
\leq& \max_{m=n+1,\dots,\nu} \|\mathrm{D}_{m\cdot\boldsymbol{e}_1} f - \binom{\nu}{m} g_{\nu\cdot\boldsymbol{e}_1} *^{\nu-m\cdot\boldsymbol{e}_1} - \dots - g_{m\cdot\boldsymbol{e}_1}\|_{\sup} |h'|^{m-n} \\
\leq& \max_{m=n+1,\dots\nu} \varepsilon\delta^{r-m}\delta^{m-n} = \varepsilon\delta^{r-n};
\end{aligned}$$

the last inequality by the induction hypothesis for $m = n+1, \dots, \nu$ (and since $|h'| \leq \delta$).

Having found $g_{0\cdot\boldsymbol{e}_1}, \dots, g_{\nu\cdot\boldsymbol{e}_1} : \mathrm{X} \to \mathbf{K}$, we claim that the locally ($\boldsymbol{\delta}$-)polynomial function $g := g_{\nu\cdot\boldsymbol{e}_1} *^{\nu\cdot\boldsymbol{e}_1} + \dots + g_{1\cdot\boldsymbol{e}_1} *^{\boldsymbol{e}_1} + g_{0\cdot\boldsymbol{e}_1}$ accomplishes $\|\tilde{f}\|_{\mathscr{C}_\mathrm{T}^{r\cdot\boldsymbol{e}_1}} \leq \varepsilon$ with $\tilde{f} := f - g$. For this, we prove firstly $\|\mathrm{D}_{n\cdot\boldsymbol{e}_1}\tilde{f}\|_{\sup} \leq \varepsilon\delta^{r-n}$ for $n = 0, \dots, \nu$.

Since the derivative of a locally constant function vanishes, by the product rule in Lemma 2.1(ii), we obtain

$$\mathrm{D}_{n\cdot\boldsymbol{e}_1} g = \binom{\nu}{n} g_{\nu\cdot\boldsymbol{e}_1} *^{\nu-n\cdot\boldsymbol{e}_1} - \dots - \binom{n+1}{n} g_{n+1\cdot\boldsymbol{e}_1} *^{\boldsymbol{e}_1} - g_{n\cdot\boldsymbol{e}_1}.$$

By construction of $g_{\nu\cdot\boldsymbol{e}_1}, \dots, g_{0\cdot\boldsymbol{e}_1} : \mathrm{X} \to \mathbf{K}$,

$$\|\mathrm{D}_{n\cdot\boldsymbol{e}_1}\tilde{f}\|_{\sup} = \|\mathrm{D}_{n\cdot\boldsymbol{e}_1} f - \mathrm{D}_{n\cdot\boldsymbol{e}_1} g\|_{\sup} \leq \varepsilon\delta^{r-n} \quad \text{for } n = 0, \dots, \nu. \tag{$**$}$$

It now remains to prove

$$|\Delta_{r\cdot \boldsymbol{e}_1}\tilde{f}|(x;t) \leq \varepsilon \quad \text{for all } x + t\cdot \boldsymbol{e}_1, x\in \mathrm{X}.$$

First off, assume $|t| \leq \delta$. Then by construction, the $g_{\nu\cdot \boldsymbol{e}_1}, \dots, g_{0\cdot \boldsymbol{e}_1}\colon \mathrm{X} \to \mathbf{K}$ are locally $\boldsymbol{\delta}$-constant. Hence $\mathrm{R}_{\nu\cdot \boldsymbol{e}_1} g(x;t) = 0$ for all $x + t\cdot \boldsymbol{e}_1, x \in \mathrm{X}$ with $|t| \leq \delta$. Therefore $|\Delta_{r\cdot \boldsymbol{e}_1}\tilde{f}|(x;t) = |\Delta_{r\cdot \boldsymbol{e}_1} f|(x;t) \leq \varepsilon$ by Inequality $(*)$ for $n = 0$. Otherwise $|t| > \delta$. We estimate, using $(**)$ for the ultimate inequality

$$\begin{aligned}
&|\Delta_{r\cdot \boldsymbol{e}_1}\tilde{f}|(x;t)\\
=&|\mathrm{R}_{\nu\cdot \boldsymbol{e}_1}\tilde{f}(x;t)|/|t|^r\\
=&|f(x+t\cdot \boldsymbol{e}_1) - g(x+t\cdot \boldsymbol{e}_1) - \sum_{i=0,\dots,\nu} (\mathrm{D}_{i\cdot \boldsymbol{e}_1} f(x) - \mathrm{D}_{i\cdot \boldsymbol{e}_1} g(x)) t^i|/|t|^r\\
\leq&(\|\tilde{f}\|_{\sup} \vee \max_{i=0,\dots,\nu} \|\mathrm{D}_{i\cdot \boldsymbol{e}_1} f - \mathrm{D}_{i\cdot \boldsymbol{e}_1} g\|_{\sup}\delta^i)/\delta^r\\
\leq&(\varepsilon\delta^r \vee \varepsilon \max_{i=0,\dots,\nu} \delta^{r-i}\delta^i)/\delta^r = \varepsilon.
\end{aligned}$$

□

**Lemma 2.53.** *Let $\mathrm{X}_1, \dots, \mathrm{X}_d \subseteq \mathbf{K}$ be compact open subsets. Consider the mapping*

$$\begin{aligned}
\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}_1, \mathbf{K}) \times \mathscr{C}^0(\mathrm{X}_2, \mathbf{K}) \times \cdots \times \mathscr{C}^0(\mathrm{X}_d, \mathbf{K}) &\to \mathscr{C}^{r\cdot \boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_d, \mathbf{K}),\\
(f_1, \dots, f_d) &\mapsto f := [\boldsymbol{x} \mapsto f_1(x_1)\cdots f_d(x_d)].
\end{aligned}$$

*It induces an (isometric) isomorphism of* **K***-Banach spaces*

$$\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}_1, \mathbf{K}) \widehat{\otimes} \mathscr{C}^0(\mathrm{X}_2, \mathbf{K}) \widehat{\otimes} \cdots \widehat{\otimes} \mathscr{C}^0(\mathrm{X}_d, \mathbf{K}) \to \mathscr{C}^{r\cdot \boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_d, \mathbf{K}).$$

*Proof:* Firstly, notice that putting $\mathrm{X}' := \mathrm{X}_1 \subseteq \mathbf{K}$ and $\mathrm{X}'' := \mathrm{X}_2 \times \cdots \mathrm{X}_d \subseteq \mathbf{K}^{d-1}$, the above mapping is given by

$$\begin{aligned}
\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}', \mathbf{K}) \times \mathscr{C}^0(\mathrm{X}'', \mathbf{K}) &\overset{\Psi}{\to} \mathscr{C}^{r\cdot \boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}' \times \mathrm{X}'', \mathbf{K}),\\
(f, g) &\mapsto f \odot g.
\end{aligned}$$

We prove $\operatorname{im}\Psi \subseteq \mathscr{C}^{r\cdot \boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}, \mathbf{K})$ with $\mathrm{X} := \mathrm{X}' \times \mathrm{X}''$. Let us assume $f \in \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}', \mathbf{K}), g \in \mathscr{C}^0(\mathrm{X}'', \mathbf{K})$. Let $h = f \odot g$ be the image of $(f, g)$. We suppose that the continuous functions $\mathrm{D}_0 f, \dots, \mathrm{D}_\nu f\colon \mathrm{X}' \to \mathbf{K}$ prove $f$ to be a $\mathscr{C}^r_{\mathrm{T}}$-function. We claim that the maps $\mathrm{D}_{n\cdot \boldsymbol{e}_1} h := \mathrm{D}_n f \odot g\colon \mathrm{X} \to \mathbf{K}$ for $n = 0, \dots, \nu$ prove $h\colon \mathrm{X} \to \mathbf{K}$ to be a $\mathscr{C}^{r\cdot \boldsymbol{e}_1}$-function: The maps $\mathrm{D}_{0\cdot \boldsymbol{e}_1} h, \dots, \mathrm{D}_{\nu\cdot \boldsymbol{e}_1} h$ are continuous. It suffices to prove that for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|\mathrm{R}_{\nu\cdot \boldsymbol{e}_1} f(x;t)| \leq \varepsilon|t|^r \quad \text{for all } (x;t) \in \mathrm{X}^{[\boldsymbol{e}_1]} \text{ with } |t| < \delta.$$

Since $f \in \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{K})$, there is such $\delta > 0$ such that $|\mathrm{R}_\nu f(x';t)|\|g\|_{\sup} \leq \varepsilon|t|^r$ for $x'+t, x' \in \mathrm{X}'$ with $|t| < \delta$. Then for $x + t\cdot \boldsymbol{e}_1, x \in \mathrm{X}$ with $|t| < \delta$ and $x = (x', x'') \in \mathrm{X} = \mathrm{X}' \times \mathrm{X}''$, we compute

$$\begin{aligned}
|\mathrm{R}_{\nu\cdot \boldsymbol{e}_1} f(x;t)| =&|f(x'+t) \odot g(x'') - \sum_{i=0,\dots,\nu} \mathrm{D}_i f(x') \odot g(x'')|\\
=&|\mathrm{R}_\nu f(x';t)||g(x'')| \leq |\mathrm{R}_\nu f(x';t)|\|g\|_{\sup} \leq \varepsilon|t|^r.
\end{aligned}$$

Secondly, by the criterion of [vR78, Comment following Cor. 4.31], we have to check the following:

(i) The mapping $\Psi$ is bilinear and norm-nonincreasing.

(ii) The $\mathbf{K}$-linear span of $\operatorname{im}\Psi$ is dense in $\mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}'\times\mathrm{X}'',\mathbf{K})$.

(iii) Let $0 < t \leq 1$. If $f_1, \cdots, f_n \in \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}',\mathbf{K})$ are $t$-orthogonal and moreover $g_1, \ldots, g_n \in \mathscr{C}^0(\mathrm{X}'',\mathbf{K})$, then their products $f_1 \odot g_1, \ldots, f_n \odot g_n \in \mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}'\times\mathrm{X}'',\mathbf{K})$ will be $t$-orthogonal.

Ad 1.: The map $\Psi$ is quickly checked to be bilinear. We find

$$\begin{aligned}\|f\odot g\|_{\mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}} =&\|\mathrm{D}_{0\cdot\boldsymbol{e}_1}f\odot g\|_{\sup}\vee\ldots\vee\|\mathrm{D}_{\nu\cdot\boldsymbol{e}_1}f\odot g\|_{\sup}\vee\||\Delta_{r\cdot\boldsymbol{e}_1}f\odot g|\|_{\sup}\\ =&\|\mathrm{D}_0 f\odot g\|_{\sup}\vee\ldots\vee\|\mathrm{D}_\nu f\odot g\|_{\sup}\vee\||\Delta_r f|\odot|g|\|_{\sup}\\ =&(\|\mathrm{D}_0 f\|_{\sup}\vee\ldots\vee\|\mathrm{D}_\nu f\|_{\sup}\vee\||\Delta_r f|\|_{\sup})\cdot\|g\|_{\sup}\\ =&\|f\|_{\mathscr{C}^r_{\mathrm{T}}}\cdot\|g\|_{\mathscr{C}^0}.\end{aligned}$$

Ad 2.: All locally monomial functions with a ball as support lie in $\operatorname{im}\Psi$. Hence all locally polynomial functions supported on a ball lie in the $\mathbf{K}$-linear span of $\operatorname{im}\Psi$. Since these balls form a basis of the topological space X and this space is compact, we find all locally polynomial functions to be in the $\mathbf{K}$-linear span of $\operatorname{im}\Psi$. By Lemma 2.52 those of degree $\vec{g} \leq \boldsymbol{\nu}\cdot\boldsymbol{e}_1$ are already dense in $\mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}(\mathrm{X}'\times\mathrm{X}'',\mathbf{K})$.

Ad 3.: We compute

$$\begin{aligned}&\|f_1\odot g_1+\cdots+f_n\odot g_n\|_{\mathscr{C}^{r\boldsymbol{e}_1}_{\mathrm{T}}}\\ =&\max_{i=0,\ldots,\nu}\|\mathrm{D}_{i\cdot\boldsymbol{e}_1}(f_1\odot g_1+\cdots+f_n\odot g_n)\|_{\sup}\\ &\vee\||\Delta_{r\cdot\boldsymbol{e}_1}(f_1\odot g_1+\cdots+f_n\odot g_n)|\|_{\sup}\\ \geq&\max_{i=0,\ldots,\nu}\|\mathrm{D}_{i\cdot\boldsymbol{e}_1}f_1\cdot g_1(x'')+\cdots+\mathrm{D}_{i\cdot\boldsymbol{e}_1}f_n\cdot g_n(x'')\|_{\sup}\\ &\vee\||\Delta_{r\cdot\boldsymbol{e}_1}(f_1\cdot g_1(x'')+\cdots+f_n\cdot g_n(x''))|\|_{\sup}\qquad\text{for any fixed } x''\in\mathrm{X}''.\end{aligned}$$

We see that the last term for fixed $x'' \in \mathrm{X}''$ equals $\|f_1\cdot g_1(x'')+\cdots+f_n\cdot g_n(x'')\|_{\mathscr{C}^r_{\mathrm{T}}}$. Since $f_1, \ldots, f_n$ are $t$-orthogonal with respect to $\|\cdot\|_{\mathscr{C}^r_{\mathrm{T}}}$, we find for all $x'' \in \mathrm{X}''$ that

$$\|f_1\cdot g_1(x'')+\cdots+f_n\cdot g_n(x'')\|_{\mathscr{C}^r_{\mathrm{T}}}/t \geq |g_1(x'')|\cdot\|f_1\|_{\mathscr{C}^r_{\mathrm{T}}}\vee\ldots\vee|g_n(x'')|\cdot\|f_n\|_{\mathscr{C}^r_{\mathrm{T}}}.$$

In particular it holds for $j = 1, \ldots, n$ that

$$\|f_1\cdot g_1(x'')+\cdots+f_n\cdot g_n(x'')\|_{\mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}}/t \geq \sup_{x''\in\mathrm{X}''}|g_j(x'')|\|f_j\|_{\mathscr{C}^r_{\mathrm{T}}} = \|g_j\|_{\sup}\cdot\|f_j\|_{\mathscr{C}^r_{\mathrm{T}}}.$$

We conclude

$$\|f_1g_1+\cdots+f_ng_n\|_{\mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}} \geq t\cdot\max_{j=1,\ldots,n}\|g_j\|_{\sup}\|f_j\|_{\mathscr{C}^r_{\mathrm{T}}} = t\cdot\max_{j=1,\ldots,n}\|f_j\odot g_j\|_{\mathscr{C}^{r\cdot\boldsymbol{e}_1}_{\mathrm{T}}}.$$

*Assumption.* We will assume until the end of this subsection that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a normed field.

**Lemma 2.54.** *The family* $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal basis with*

$$(\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}})_{\boldsymbol{i}\in\mathbb{N}^d} \sim (i_1^r)_{\boldsymbol{i}\in\mathbb{N}^d}.$$

*Proof:* Denote $\mathrm{X}' = \mathbb{Z}_p$ and $\mathrm{X}'' = \mathbb{Z}_p^{d-1}$. We consider the composition of morphisms

$$\mathscr{C}^r(\mathrm{X}', \mathbf{K})\widehat{\otimes}\, \mathscr{C}^0(\mathrm{X}'', \mathbf{K}) \to \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}', \mathbf{K})\widehat{\otimes}\, \mathscr{C}^0(\mathrm{X}'', \mathbf{K}) \to \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathrm{X}' \times \mathrm{X}'', \mathbf{K}).$$

The first arrow is the induced map

$$\iota\widehat{\otimes}\mathrm{id}\colon\ \mathscr{C}^r(\mathrm{X}', \mathbf{K})\widehat{\otimes}\, \mathscr{C}^0(\mathrm{X}'', \mathbf{K}) \to \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}', \mathbf{K})\widehat{\otimes}\, \mathscr{C}^0(\mathrm{X}'', \mathbf{K})$$

of the topological $\mathbf{K}$-vector space morphisms given by the inclusion $\iota : \mathscr{C}^r(\mathrm{X}', \mathbf{K}) \hookrightarrow \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}', \mathbf{K})$ and the identity on $\mathscr{C}^0(\mathrm{X}'', \mathbf{K})$. Since $\mathrm{X}' \subseteq \mathbb{Q}_p$ is a ball, it has the $\mathrm{B}_\nu$-property by [Nag12, Lemma 2.24]. By [Nag12, Theorem 2.29], noting $\mathbb{Q}_p$ being locally compact, together with [Nag12, Theorem 2.22] the canonical inclusion $\mathscr{C}^r(\mathrm{X}', \mathbf{K}) \hookrightarrow \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}', \mathbf{K})$ is a topological isomorphism of $\mathbf{K}$-vector spaces. By functoriality, the first map is therefore an isomorphism of topological $\mathbf{K}$-vector spaces. The (left|right) hand-isomorphism of $\mathbf{K}$-Banach spaces is given by the preceding Lemma 2.53. Therefore its composition is an isomorphism of topological $\mathbf{K}$-vector spaces.

We conclude $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K})$, regarding the identification

$$\mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \simeq \mathscr{C}^r(\mathrm{X}', \mathbf{K})\widehat{\otimes}\, \mathscr{C}^0(\mathrm{X}'', \mathbf{K}) \simeq \mathscr{C}^r(\mathbb{Z}_p, \mathbf{K})\widehat{\otimes}\, \mathscr{C}^0(\mathbb{Z}_p)\widehat{\otimes} \cdots \widehat{\otimes}\, \mathscr{C}^0(\mathbb{Z}_p, \mathbf{K}),$$

by [Nag12, Theorem 2.58] and Remark 2.28 to be an orthogonal basis with

$$(\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}})_{\boldsymbol{i}\in\mathbb{N}^d} \sim (\|\binom{*}{i_1}\|_{\mathscr{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d} = (p^{w_r(i_1)})_{\boldsymbol{i}\in\mathbb{N}^d}.$$

Then by [Nag12, Lemma 2.61], it holds $(p^{w_r(i_1)})_{i_1\in\mathbb{N}} \sim (i_1^r)_{i_1\in\mathbb{N}}$. □

**Lemma 2.55.** *The* $\mathbf{K}$*-Banach space* $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ *is the initial topological* $\mathbf{K}$*-vector space with respect to the inclusion mappings*

$$\begin{array}{ccc} & & \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \\ & \overset{incl.}{\nearrow} & \\ \mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K}) & & \vdots \\ & \underset{incl.}{\searrow} & \\ & & \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K}) \end{array}$$

*That is,* $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K}) = \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \cap \ldots \cap \mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K})$ *as an abstract* $\mathbf{K}$*-vector space and its norm* $\|\cdot\|_{\mathscr{C}^r}$ *on* $\mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ *is equivalent to* $\|\cdot\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}} \vee \ldots \vee \|\cdot\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}}$.

*Proof:* We consider the canonical commutative diagram

$$\begin{array}{ccc} c_0((\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d}) & \xrightarrow{\sim} & c_0((\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_1}} \vee \ldots \vee \|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_d}})_{\boldsymbol{i}\in\mathbb{N}^d}) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathscr{C}^r(\mathbb{Z}_p^d, \mathbf{K}) & \xhookrightarrow{\text{incl.}} & \mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \cap \ldots \cap \mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K}) \end{array}$$

Here the $\mathbf{K}$-Banach space at the bottom right is defined as the initial $\mathbf{K}$-Banach space of $\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_k}(\mathbb{Z}_p^d, \mathbf{K})$ for $k = 1, \ldots, d$ (inside $\mathscr{C}^0(\mathbb{Z}_p^d, \mathbf{K})$) and the lower inclusion mapping is given by Lemma 2.50. By Theorem 2.45, the left-hand map is an isomorphism of $\mathbf{K}$-Banach spaces and by the preceding Lemma 2.54 for $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ together with Lemma 2.29(ii) the right-hand map is an isomorphism of $\mathbf{K}$-Banach spaces. By Lemma 2.46 and Lemma 2.54 for $k = 1, \ldots, d$, we find

$$\left(\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathscr{C}^r}\right)_{\boldsymbol{i}\in\mathbb{N}^d} \sim (i_1^r \vee \ldots \vee i_d^r)_{\boldsymbol{i}\in\mathbb{N}^d} \sim \left(\left\|\binom{*}{i}\right\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_1}} \vee \ldots \vee \left\|\binom{*}{i}\right\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_d}}\right)_{\boldsymbol{i}\in\mathbb{N}^d}.$$

So $(\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d} \sim (\|\binom{*}{i}\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_1}} \vee \ldots \vee \|\binom{*}{i}\|_{\mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_d}})_{\boldsymbol{i}\in\mathbb{N}^d}$ and the upper map is an isomorphism of topological $\mathbf{K}$-vector spaces. The commutativity of the diagram can by $\mathbf{K}$-linearity and continuity be checked on all $\boldsymbol{e}_{\boldsymbol{i}} \in c_0((\|\binom{*}{\boldsymbol{i}}\|_{\mathscr{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d})$ whose only nonzero entry is 1 at the $\boldsymbol{i}$-th position. There it holds by definition of the above maps. All together, the bottom map is also an isomorphism of topological $\mathbf{K}$-vector spaces. □

**Definition.** Let $\mathrm{X} \subseteq \mathbb{Q}_p^d$ be an open subset. We define $\mathscr{C}_{\mathrm{T}}^r(\mathrm{X}, \mathbf{K})$ as the initial locally convex $\mathbf{K}$-vector space with respect to the inclusion mappings given by

$$\begin{array}{ccc} & \overset{\text{incl.}}{\nearrow} & \mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_1}(\mathrm{X}, \mathbf{K}) \\ \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}, \mathbf{K}) & & \vdots \\ & \underset{\text{incl.}}{\searrow} & \mathscr{C}_{\mathrm{T}}^{r\cdot\boldsymbol{e}_d}(\mathrm{X}, \mathbf{K}). \end{array}$$

**Theorem 2.56.** *Let* $\mathrm{X} \subseteq \mathbb{Q}_p^d$ *be an open subset. Then the natural inclusion* $\mathscr{C}^r(\mathrm{X}, \mathbf{K}) \hookrightarrow \mathscr{C}_{\mathrm{T}}^r(\mathrm{X}, \mathbf{K})$ *is an isomorphism of locally convex* $\mathbf{K}$*-vector spaces.*

*Proof:* Because any open ball in $\mathbb{Q}_p^d$ arises up to dilation and translation from $\mathbb{Z}_p^d$, we can infer the result from the preceding Lemma 2.55, see [Nag11, Lemma 3.60]. □

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