non-Archimedean Fractional Differential Calculus

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We define, for a real number $r \geq 0$, an r-times differentiable function of several variables over a non-Archimedean field K into a non-Archimedean Banach space. We first decompose $r = \nu + \rho$ into its integer part ν and fractional part ρ, then combine a definition of ν-fold differentiability, given by partial difference quotients, and of ρ-fold differentiability, given by a Hölderian condition.

An obstacle is the failure of the intermediate-value theorem over K (in its formulation that a continuous function maps closed balls onto closed balls). Still, we recover non-Archimedean versions of many of its corollaries by a non-Archimedean differentiability condition that is stricter than its Archimedean version.

Non-Archimedean differentiability is not only stricter, but also more labored than Archimedean differentiability and is in degree ν checked by a non-Archimedean differential of 2^ν variables. We reduce it, for a function over an open subset of K or $\mathbb{Q}_p^d$, to a check on a non-Archimedean Taylor polynomial expansion of two variables. The latter rests on a description of r-fold differentiability by the *Mahler* basis, a distinguished orthogonal basis over $\mathbb{Z}_p$.

Contents

Introduction

Calculus rests on the notions of the integral, defined via a measure, and the differential. Either notion is meaningful over a topological field K, that is, for a function that takes arguments and values in a topological vector space over K. The following desiderata put restrictions on the choice of K though:

- Integrability and differentiability of a function should only depend on its image but not on its codomain E. If E is complete, then the convergence of a sequence (a_n) only depends on the topology of the subspace $\{a_n\}$ inside E. We therefore assume E complete and, since we allow E to be a finite-dimensional K-vector space, we assume K complete (see Section 2).

- The derivative of a function f should be unique: If no point in its domain is isolated, then the derivative in a point a is a limit that is uniquely determined by the values of f around a. This holds because the topology of K is Hausdorff.

- There is a nonconstant differentiable function. We therefore assume that the topology of K is non-discrete.

Every complete valued field is either topologically isomorphic to the real or complex numbers or a non-Archimedeanly valued field. We will deal with complete non-Archimedeanly valued fields. Let us call a field *non-Archimedean* if it has a non-trivial non-Archimedean valuation and is complete. We fix once and for all such a non-Archimedean field K.

non-Archimedean integral calculus

The integral over the real numbers is defined by a *measure*, a function $\mu\colon \mathcal{B} \to [0, \infty[$ over a family $\mathcal{B}$ of sets of real numbers into the nonnegative real numbers. Every countable union of closed balls is in $\mathcal{B}$, and μ is translation invariant and *countably additive*, that is in particular, if $\{A_n\}$ is a countable family of disjoint closed balls such that $\sum \mu(A_n)$ converges, then $\mu(\cup A_n) = \sum \mu(A_n)$. If K is non-Archimedean then larger and larger disjoint unions $\mathcal{U}$ of closed balls yield smaller and smaller absolute values $|\mu(\mathcal{U})|$; hence every translation invariant countably additive function that is bounded over all the closed balls inside the closed unit ball is the zero function and we conclude that there is no distinguished integral over a non-Archimedean field.

non-Archimedean differential calculus

A priori, Archimedean differentiability can be formulated for a general topological field K as follows: Given two finite-dimensional vector spaces V and W over K and an open subset X of V, a function $f\colon X \to W$ is *differentiable* at a in X if there is a K-linear map $A\colon V \to W$ such that for every $\epsilon > 0$, there is a neighborhood U inside X around a such that, for all x in U,

$$\|f(x) - f(a) - A(x - a)\| \leq \epsilon \|x - a\|.$$

However, the mean-value theorem is at the heart of many fundamental properties of the Archimedeanly differentiable functions. It is an application of the intermediate-value theorem (to the derivative of a function) which asserts that a continuous function maps every closed ball onto a closed ball. This no longer holds for example for a locally (not everywhere) constant function that takes values in a non-Archimedean field. Therefore with our current Archimedean definition of differentiability, we run into various pathologies: for example, the differentiable functions with their natural norm form no longer a Banach space and a function with invertible derivative is no longer locally invertible (cf. [Sch84, Exercise 26.F and Example 26.6]. See also Section 1.)

To make up for the lack of the mean-value theorem, we strengthen our differentiability definition. Schikhof observed in [Sch78] that a real-valued function f over an open interval X (of real numbers) is continuously differentiable if and only if

$$f^{]1[}(x,y) = \frac{f(x) - f(y)}{x - y},$$

defined for all distinct x and y in X, extends to a continuous function over all of $X \times X$. This observation is equivalent to the mean-value theorem over the real numbers but its formulation is completely general, it makes no reference to the real numbers. We turn this characterization of differentiability into a definition: The function $f\colon X \to K$ over an open subset X of a non-Archimedean non-trivially field K is *continuously differentiable* if $f^{]1[}(x,y)$, defined for all distinct x and y in X, extends to a continuous function over all of $X \times X$.

With this differentiability definition, the above pathologies disappear: the differentiable functions form a Banach space and a function with invertible derivative is locally invertible.

Non-Archimedean Differential Calculus in many variables, that is, the function's domain is a finite-dimensional vector space over a non-Archimedean field and the degree of differentiability is integral, was first touched upon by Schikhof in [Sch84] and then more systematically treated by de Smedt; for example [DS98] established that a differentiable multivariate function with invertible derivative is locally invertible. More recently, [BGN04] gave a differentiability condition on a function on a topological vector space over a topological field which generalizes that by Schikhof and de Smedt (cf. [Glö07]). Schikhof and de Smedt's definition is the *intersection* of Bertram, Glöckner, Neeb's and our Definition 1.3. That is, Schikhof and de Smedt's definition is applicable if and only if both latter definitions are applicable; if so then all definitions agree.

Non-Archimedean Fractional Differential Calculus, that is, the degree of differentiability is a real number $r \geq 0$, was introduced in [BB10] by Berger and Breuil. The authors defined an r-times differentiable function, for short *$\mathscr{C}^r$-function*, on the unit ball $\mathbb{Z}_p$ of $\mathbb{Q}_p$ via the *Mahler basis*, a distinguished orthonormal basis of the continuous functions over $\mathbb{Z}_p$. They compute a Banach space V that is characterized by a certain uniformly continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-action and achieve a description of V by $\mathscr{C}^r$-functions over $\mathbb{P}^1(\mathbb{Q}_p)$ (covered by two copies of $\mathbb{Z}_p$).

This description is a key result in the p-adic Langlands program.

To extend it, that is, to compute the analogue of V for a general p-adic reductive group such as $\mathrm{GL}_n(\mathbb{F})$ for a finite extension $\mathbb{F}$ of $\mathbb{Q}_p$, we have to be able to speak of a $\mathscr{C}^r$-function over the analogue of $\mathbb{P}^1(\mathbb{Q}_p)$ for $\mathrm{GL}_n(\mathbb{F})$, the flag variety of $\mathrm{GL}_n(\mathbb{F})$.

We present here (developed in [Nag11]) a theory of r-times differentiable functions, for short $\mathscr{C}^r$*-functions*, on a $\mathscr{C}^r$-manifold over K (such as a p-adic reductive group) that take values in a non-Archimedean K-Banach space.

One hurdle is that the Mahler basis is no longer an orthogonal basis of the continuous functions over the unit ball of a proper extension of $\mathbb{Q}_p$. De Ieso gave in [DI13] a definition of r-fold differentiability of a function over a finite extension $\mathbb{F}$ of $\mathbb{Z}_p$ via its Taylor polynomial (which generalizes Colmez's definition on $\mathbb{Z}_p$ in [Col10]). His differentiability condition is equivalent to ours after an identification of $\mathbb{F}$ with $[F : \mathbb{Q}_p]$ copies of $\mathbb{Z}_p$ ([Nag16, Section 4]).

Outline

In Section 1 we state our definition of r-fold differentiability by decomposing $r = \nu + \rho$ into its integral part $\nu = \lfloor r \rfloor$ and its fractional part $\rho = \{r\}$. For the integral part, we build upon the classical approach by Schikhof and de Smedt (cf. [Sch84] and [DS98]) and use iterated partial difference quotients. For the fractional part, we use a strengthened Hölder-condition.

In Section 2 we show that every locally analytic function is a $\mathscr{C}^\infty$-function and the $\mathscr{C}^r$-functions for a given domain and codomain form a topological space. We show that this topological space includes all locally polynomial functions of degree at most r and all polynomial functions as dense subsets.

In Section 3 we study the injective limit $\mathscr{D}(X)$ over the duals $\mathscr{D}^r(X)$ of the r-times differentiable functions on a compact open group X; it consists of all continuous linear forms $\mu\colon \mathscr{C}^\infty(X) \to K$ over the arbitrarily often differentiable functions that extend to a continuous linear form $\mu\colon \mathscr{C}^r(X) \to K$ over the r-times differentiable functions for some $r \geq 0$. We show that $\mathscr{D}(X)$ is a *filtered algebra*, that is, it has multiplication given by the convolution product, and a natural filtration over $\mathbb{R}_{\geq 0}$, such that it is an algebra whose operations, in particular the multiplication, respect this filtration.

Finally in Section 4 we give simpler characterizations of a $\mathscr{C}^r$-function,

- when its domain is an open subset of K or of $\mathbb{Q}_p^d$, via its Taylor polynomial, and
- when its domain is $\mathbb{Z}_p^d$ via the Mahler basis. This recovers Berger and Breuil's seminal notion of r-fold differentiability ([BB10]) for $d = 1$.

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Notations

Let us recall that a *non-Archimedean field* is a field that has a non-trivial non-Archimedean valuation and is complete. A *non-Archimedean* norm $\|\cdot\|$ on a vector space V is norm that satisfies the strong triangle inequality, that is, $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all x and y in V. A *non-Archimedean Banach space* is a vector space that has a non-Archimedean norm $\|\cdot\|$ and is complete.

Henceforth once for all

- let K denote a non-Archimedean field, and
- let E denote a non-Archimedean K-Banach space.

1 Definition of r-fold differentiability for $r \in \mathbb{R}_{\geq 0}$

sec:FracDiffDefn

In this section, we define r-fold differentiability. First we decompose $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ into its integer part $\nu \in \mathbb{N}$ and its fractional part $\rho \in [0, 1[$. Then we define ν-fold differentiability by iteratively taking partial difference quotients, and ρ-fold differentiability by a strengthened Hölder-continuity condition. Finally, an r-times differentiable function is a ν-times differentiable function such that each of its partial difference quotients is ρ-times differentiable.

$\mathscr{C}^\nu$-functions for a natural number ν

sec:Cvfunctions sec:Pathologies

Pathologies under the Archimedean approach. To see that the Archimedean derivative does not suffice to describe differentiability of a function over a non-Archimedeanly valued domain, in particular in higher degrees, we exhibit a function f that is

- infinitely often Archimedeanly differentiable, but its Taylor polynomial expansion of degree greater than 1 does not converge, and
- is injective, but its derivative is zero everywhere.

(At the other extreme, there is also a function whose derivative is everywhere invertible, but nowhere injective. See [Sch84, Example 26.6].) Let

{eq:pathology}

$$\begin{aligned} f\colon \mathbb{Z}_p &\to \mathbb{Z}_p \\ \sum_{n\in\mathbb{N}} a_n p^n &\mapsto \sum_{n\in\mathbb{N}} a_n p^{2n}. \end{aligned} \tag{1.1}$$

Because $|f(x+h) - f(x)| = |h|^2$, it is a differentiable function whose derivative f' vanishes everywhere and is thus infinitely often Archimedeanly differentiable. However, its Taylor polynomial expansion up to degree 2 does not converge, for example at $a = 0$. That is, let

$$\mathrm{T}(h) = f(0) + f'(0)h + f''(0)h^2 = 0$$

be the Taylor polynomial of f at 0, up to degree 2, and

$$\mathrm{R}(h) = f(h) - \mathrm{T}(h) = f(h)$$

its rest term. Then $|\mathrm{R}(h)|/|h|^2 = 1$ for every h in the domain. In particular, if $h \to 0$, then $|\mathrm{R}(h)|/|h|^2 \not\to 0$.

sec:LackMeanValueThm

Lack of the Mean-Value Theorem. These pathologies are excluded if we assume that a function $f \colon \mathrm{X} \to \mathrm{K}$ over an open subset X of K is *non-Archimedeanly* (or *strictly*) continuously differentiable. That is, its *differential*

$$f^{]1[}(x, y) = \frac{f(x) - f(y)}{x - y} \qquad \text{\{eq:differential\}} \quad (1.2)$$

defined for all distinct x and y in X, extends to a continuous function over all of $\mathrm{X} \times \mathrm{X}$.

For a real-valued function over the real numbers, the mean-value theorem shows that the non-Archimedean and Archimedean differentiability condition are equivalent. In a way, the non-Archimedean differentiability condition is more natural than the Archimedean one: If we use the non-Archimedean differentiability definition then general facts in Archimedean Calculus like

(i) the local invertibility around a point in which the derivative is invertible

(ii) the existence of the Taylor polynomial, and

(iii) the completeness of the normed space of differentiable functions

follow from the definition, whereas if we use the Archimedean differentiability definition then they are proved by a detour either via the mean-value theorem (like in (i)) or via the fundamental theorem of calculus (like in (ii) and (iii)).

sec:DiffConceptualApproach

$\mathscr{C}^1$-functions — Coordinate-free approach. Let us recall *non-Archimedean continuous* (or *strict*) differentiability. Let V and E be two K-Banach spaces, X an open subset of V. The function $f : \mathrm{X} \to \mathrm{E}$ is *continuously differentiable* or $\mathscr{C}^1$ at a point a in X if there is a continuous K-linear map $\mathrm{A}: \mathrm{V} \to \mathrm{E}$ such that for every $\epsilon > 0$, there is a neighborhood U of a such that for, all $x + h, x \in \mathrm{U}$,

$$\|f(x + h) - f(x) - \mathrm{A}(h)\| \leq \epsilon\|h\|.$$

This condition is stricter than the Archimedean differentiability condition, because here the offset h and the expansion point x varies, there h varies but x is fixed.

sec:DiffCoordinatewiseApproach

$\mathscr{C}^1$-functions — Coordinate-wise approach. We want to define two-fold (and eventually ν-fold) differentiability by applying strict differentiability to the differential (and eventually iterate).

Pathology (1.1) showed that the Archimedean derivative does not yield a coherent theory of non-Archimedean Calculus. Instead, similar to Definition (1.2) in one variable, we define a differential $f^{]1[}$ that computes all the difference quotients around x.

Let V be a finite-dimensional K-vector space and f a function over an open subset of V. We will define a differential of f that takes a point x together with a set of points in X around x whose differences span V and returns a linear map that approximates f. To formulate it conveniently, we introduce coordinates on V by choosing an ordered basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$ of V.

Definition. Let V be a finite-dimensional K-vector space and X an open subset of V and $f : \mathrm{X} \to \mathrm{E}$. The differential $f^{]1[}(x + h, x)$ of f at $x + h, x$ in X with $h \in \mathrm{K}^{*d}$ is the K-linear map A determined by

$$\mathrm{A}\boldsymbol{e}_k := \frac{f(x + h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1} + h_k\boldsymbol{e}_k) - f(x + h_1\boldsymbol{e}_1 + \cdots + h_{k-1}\boldsymbol{e}_{k-1})}{h_k}$$

for $k = 1, \ldots, d$. The function f is a *$\mathscr{C}^1$-function* if $f^{]1[}$ extends to a continuous function $f^{[1]} : \mathrm{X} \times \mathrm{X} \to \mathrm{Hom}^1(\mathrm{V}, \mathrm{E})$.

sec:CnFunctions

$\mathscr{C}^\nu$-functions for a natural number ν. Let $f \in \mathscr{C}^1(\mathrm{X}, \mathrm{E})$. Let us compare the domain and codomain of $f^{[1]}$ with that of f. The domain $\mathrm{X}^{[1]} := \mathrm{X} \times \mathrm{X}$ of $f^{[1]}$ is included in the finite-dimensional K-vector space $\mathrm{V}^{[1]} = \mathrm{V} \times \mathrm{V}$ with a canonical ordered basis, like the domain X of f, and the codomain $\mathrm{E}^{[1]} := \mathrm{Hom}_{\mathrm{K}}(\mathrm{V}, \mathrm{E})$

of $f^{[1]}$ is a K-Banach space, like the codomain E of f. We can therefore iterate the non-Archimedean differentiability definition by applying it to $f^{[1]}$, that is, f is in $\mathscr{C}^2(\mathrm{X}, \mathrm{E})$ if, first $f^{[1]}$ exists, and second its differential

$$f^{]2[} = (f^{[1]})^{]1[} \colon (\mathrm{X}^{[1]})^{]1[} \to (\mathrm{E}^{[1]})^{[1]}$$

extends to a continuous function $f^{[2]}$ over $\mathrm{X}^{[2]} := (\mathrm{X}^{[1]})^{[1]}$ (with values in $\mathrm{E}^{[2]} := (\mathrm{E}^{[1]})^{[1]}$).

Definition. Let $f \colon \mathrm{X} \to \mathrm{E}$ be a function over an open subset X of the finite-dimensional K-vector space V with values in the K-Banach space E. Let ν be a natural number. The function f is a $\mathscr{C}^{\nu+1}$-*function*

- if f is a $\mathscr{C}^{\nu}$-function, and
- if $\mathfrak{X} = \mathrm{X}^{[\nu]}, \mathfrak{E} = \mathrm{E}^{[\nu]}$ and $\mathfrak{f} = f^{[\nu]}$, then $\mathfrak{f}^{]1[}$ extends to a continuous function $\mathfrak{f}^{[1]} \colon \mathfrak{X} \times \mathfrak{X} \to \mathrm{Hom}_{\mathrm{K}}(\mathfrak{E}, \mathrm{E})$.

$\mathscr{C}^{\rho}$-functions for $\rho \in [0, 1[$

sec:FctRhoDefProp

Let $\rho \in [0, 1[$. Roughly, ρ-fold differentiability is stricter Hölder-continuity.

defn:Crho

Definition 1.1. Let X and Y be metric spaces with metrics d and **d**. Let A be a subset of X and $f \colon \mathrm{A} \to \mathrm{Y}$. Let a be a point in X. The function f is $\mathscr{C}^{\rho}$ at a if for every $\varepsilon > 0$ there is a neighborhood $\mathrm{U} \ni a$ in X such that

$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathbf{d}(x, y)^{\rho} \quad \text{for all } x, y \in \mathrm{U} \cap \mathrm{A}.$$

The function f is a $\mathscr{C}^{\rho}$-*function* if f is $\mathscr{C}^{\rho}$ at all points $a \in \mathrm{A}$. Let $\mathscr{C}^{\rho}(\mathrm{A}, \mathrm{Y})$ denote the set of all $\mathscr{C}^{\rho}$-functions $f \colon \mathrm{A} \to \mathrm{Y}$.

For later use, we record that if a is in the boundary of A inside X and Y is complete, then f extends uniquely to a.

prop:FctRhoExtToClosure

Proposition 1.2 ([Nag11, Proposition 1.6])**.** *Let* X *be a metric space and let* A *be a subset of* X*, let* Y *a complete metric space and* $f \colon \mathrm{A} \to \mathrm{Y}$. *If* B *denotes the set of* $\mathscr{C}^{\rho}$*-points of* f *included in the closure of* A *inside* X*, then* f *extends to a unique* $\mathscr{C}^{\rho}$*-function over* B.

$\mathscr{C}^r$-functions for a real number $r \geq 0$

Henceforth we fix a real number $r \geq 0$ and its decomposition

$$r = \nu + \rho$$

into • an integral part $\nu := \lfloor r \rfloor \in \mathbb{N}$, and • a fractional part $\rho := \{r\} \in [0, 1[$. ∎

Definition. Let $f \colon \mathrm{X} \to \mathrm{E}$ be a function over an open subset X of a finite-dimensional K-vector space with values in E. The function f is a *$\mathscr{C}^r$-function* if f is a $\mathscr{C}^\nu$-function, and $f^{[\nu]}$ is a $\mathscr{C}^\rho$ function.

$\mathscr{C}^r$-functions — by partial differentials. The symmetry of the differential allows us to reduce for increasing degree of differentiability ν the exponential growth in the number of variables of the total differential $f^{[\nu]}$ to a linear growth in the number of variables of certain partial differentials $f^{[\boldsymbol{n}]}$.

To define these partial differentials, we fix some notation. For a subset X of K and $n \in \mathbb{N}$, let

$$\mathrm{X}^{[n]} := \mathrm{X}^{\{0,\dots,n\}} \quad \text{and} \quad \mathrm{X}^{]n[} := \{(x^{[0]}, \dots, x^{[n]}) \in \mathrm{X}^{[n]} : x^{[i]} = x^{[j]} \text{ only if } i = j\}.$$

For subsets $\mathrm{X}_1, \dots, \mathrm{X}_d$ of K and $\boldsymbol{n} \in \mathbb{N}^d$, let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ and

$$\mathrm{X}^{[\boldsymbol{n}]} := \mathrm{X}_1^{[n_1]} \times \cdots \times \mathrm{X}_d^{[n_d]} \quad \text{and} \quad \mathrm{X}^{]\boldsymbol{n}[} := \mathrm{X}_1^{]n_1[} \times \cdots \times \mathrm{X}_d^{]n_d[}.$$

Given $\boldsymbol{x} \in \mathrm{X}^{[\boldsymbol{n}]}$, let $\boldsymbol{x} = (\boldsymbol{x}_1; \dots; \boldsymbol{x}_d)$ represent its entries in $\mathrm{X}_1^{[n_1]}, \dots, \mathrm{X}_d^{[n_d]}$.

Definition. Let $\mathrm{X}_1, \dots, \mathrm{X}_d$ be subsets of K, let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ be their product and $f \colon \mathrm{X} \to \mathrm{E}$. Let $\boldsymbol{n} \in \mathbb{N}^d$. We define $f^{]\boldsymbol{n}[} \colon \mathrm{X}^{]\boldsymbol{n}[} \to \mathrm{E}$ by

$$f^{]0[} = f,$$

and, by induction on $n = n_1 + \cdots + n_d$, if $\boldsymbol{n}' = \boldsymbol{n} + (0, \dots, 1, \dots, 0)$ where the latter tuple has a single nonzero entry 1 in the k-th coordinate, then

$$\begin{aligned} & f^{]\boldsymbol{n}'[}(\dots; x_k^{[0]}, x_k^{[1]}, x_k^{[2]}, \dots, x_k^{[n_k+1]}; \dots) \\ = & \frac{f^{]\boldsymbol{n}[}(\dots; x_k^{[0]}, x_k^{[2]}, \dots, x_k^{[n_k+1]}; \dots) - f^{]\boldsymbol{n}[}(\dots; x_k^{[1]}, x_k^{[2]}, \dots, x_k^{[n_k+1]}; \dots)}{x_k^{[0]} - x_k^{[1]}}. \end{aligned}$$

A subset X of K^d is *Cartesian* if it is of the form $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ with $\mathrm{X}_1, \dots, \mathrm{X}_d$ subsets of K.

defn:CrFct

Definition 1.3. Let X be a subset of K^d and let E be a K-Banach space.

- Let X be Cartesian. Let a in X. The function $f\colon \mathrm{X} \to \mathrm{E}$ is $\mathscr{C}^r$ at $a = (a_1, \dots, a_d)$ if $f^{]\boldsymbol{n}[}\colon \mathrm{X}^{]\boldsymbol{n}[} \to \mathrm{E}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a_1, \dots, a_1; \dots ; a_d, \dots, a_d)$ in $\mathrm{X}^{[\boldsymbol{n}]}$ for all $\boldsymbol{n}$ in $\mathbb{N}^d$ such that $n_1 + \cdots + n_d = \nu$.
- Let X be open. Let a in X. The function f is $\mathscr{C}^r$ at a in X if, after choosing a Cartesian neighborhood U of a included in X, the function $f_{|\mathrm{U}} : \mathrm{U} \to \mathrm{E}$ is $\mathscr{C}^r$ at a.

The function f is a $\mathscr{C}^r$*-function* if f is $\mathscr{C}^r$ at all a in X. Let $\mathscr{C}^r(\mathrm{X}, \mathrm{E})$ denote the set of all $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathrm{E}$.

Let X be an open subset of K^d and $f\colon \mathrm{X} \to \mathrm{E}$. Let a in X and $\boldsymbol{n}$ in $\mathbb{N}^d$ such that $n_1 + \cdots + n_d = \nu$. We denote by $\mathrm{D}^{\boldsymbol{n}} f(a)$, the $\boldsymbol{n}$*-th partial derivative* of f at a, the unique value to which $f^{]\boldsymbol{n}[}$ extends at $\vec{a}$ as $\mathscr{C}^\rho$-function by Proposition 1.2. If $f^{(\boldsymbol{n})}$ is the Archimedean partial derivative of f taken n_1-times along $\boldsymbol{e}_1, \dots,$ n_d-times along $\boldsymbol{e}_d$, then $n_1! \cdots n_d!\, \mathrm{D}^{\boldsymbol{n}} f = f^{(\boldsymbol{n})}$.

$\mathscr{C}^r$-manifolds

sec:CrManifolds

Definition. Let d be a natural number. A *topological manifold* of dimension d is a topological Hausdorff space M such that for every point x in M there is a *chart* inside M around x, that is,

- an open neighborhood U of x included in M, and
- a map $\phi\colon \mathrm{U} \to \mathrm{K}^d$ such that its image $\phi(\mathrm{U})$ is open inside K^d and $\phi\colon \mathrm{U} \to \phi(\mathrm{U})$ is a homeomorphism.

Definition. A $\mathscr{C}^r$*-atlas* of M is a set $\mathscr{A}$ of charts over M that

- covers M, and such that
- every pair of charts (U, ϕ) and (V, ψ) in $\mathscr{A}$ is *compatible*, that is,

$$\mathrm{K}^d \supseteq \phi(\mathrm{U} \cap \mathrm{V}) \underset{\phi\circ\psi^{-1}}{\overset{\psi\circ\phi^{-1}}{\rightleftarrows}} \psi(\mathrm{V} \cap \mathrm{U}) \subseteq \mathrm{K}^d$$

  are $\mathscr{C}^r$-functions.

An atlas $\mathscr{A}$ is *maximal* if $\mathscr{A}$ contains every chart over M that is compatible with all charts in $\mathscr{A}$.

Definition. A $\mathscr{C}^r$*-manifold* is a topological manifold with a maximal $\mathscr{C}^r$-atlas.

Definition. Let M be a $\mathscr{C}^r$-manifold. A function $f\colon \mathrm{M} \to \mathrm{E}$ is a $\mathscr{C}^r$*-function* if for every chart (U, ϕ) over M, the function $f \circ \phi^{-1}\colon \phi(\mathrm{U}) \to \mathrm{E}$ is a $\mathscr{C}^r$-function.

sec:WellDefined

Well-definedness. Let us record the closure under composition of $\mathscr{C}^r$-functions. ∎ This, together with the local nature of r-fold differentiability over K^d yields by a formal verification that the notion of a $\mathscr{C}^r$-manifold is well defined.

prop:FctRCmp

Proposition 1.4 ([Nag11, Proposition 3.23]). *Let* $f\colon \mathrm{X} \to \mathrm{K}^e$ *and* $g\colon \mathrm{Y} \to \mathrm{E}$ *be* $\mathscr{C}^r$*-functions. If* $\operatorname{im} f \subseteq \mathrm{Y}$ *and either*

- $r \geq 1$*, or*
- $r < 1$ *and* f *or* g *is locally Lipschitz,*

then $g \circ f\colon \mathrm{X} \to \mathrm{E}$ *is a* $\mathscr{C}^r$*-function.*

In consequence, if $r \geq 1$, then r-fold differentiability over a topological manifold does not depend on the choice of $\mathscr{C}^r$-atlas, that is,

- if a topological manifold has *one* $\mathscr{C}^r$-atlas then it has a (unique) *maximal* atlas, and
- if a function $f\colon \mathrm{M} \to \mathrm{E}$ is a $\mathscr{C}^r$-function for *one* atlas then it is a $\mathscr{C}^r$-function for the *maximal* (or, equivalently, to every) atlas.

If $r < 1$, then the latter point stays true provided f is locally Lipschitz.

The locally convex topology of $\mathscr{C}^r$-functions

Henceforth, given a real valued function f over a compact topological space, let $\|f\|_{\sup}$ always denote the supremum norm of f.

Definition. Let X and Y be metric spaces with metric d and $\mathbf{d}$ and $f\colon \mathrm{X} \to \mathrm{Y}$. The ρ-th differential $|f^{]\rho[}|\colon \triangledown \mathrm{X} \times \mathrm{X} \to \mathbb{R}_{\geq 0}$ of f is defined by

$$|f^{]\rho[}|(x, y) = \frac{\mathbf{d}(f(x), f(y))}{\mathrm{d}(x, y)^{\rho}}.$$

The function $f\colon \mathrm{X} \to \mathrm{Y}$ is a $\mathscr{C}^\rho$-function if and only if $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|$ on all of $\mathrm{X}\times\mathrm{X}$ that vanishes on the diagonal of $\mathrm{X}\times\mathrm{X}$ (and which is *unique* provided X is free of isolated points).

Definition. Let X be compact and free of isolated points. The norm $\|\cdot\|_{\mathscr{C}^\rho}$ over $\mathscr{C}^\rho(\mathrm{X}, \mathrm{E})$ is defined by

$$\|f\|_{\mathscr{C}^\rho} = \max\{\|f\|_{\sup}, \|\,|f^{[\rho]}|\,\|_{\sup}\}.$$

If a function f is r-times differentiable then its partial differentials $f^{]\boldsymbol{n}[}$ extend to $\mathscr{C}^\rho$-functions. We will use the $\mathscr{C}^\rho$-norm of the partial derivatives of f to define the $\mathscr{C}^r$-norm of f.

Proposition ([Nag11, Proposition 3.8])**.** *Let* X *be a Cartesian subset of* K^d *and* $f\colon \mathrm{X} \to \mathrm{E}$*. The function* f *is* r*-times differentiable if and only if for all* $\boldsymbol{n}$ *in* $\mathbb{N}^d$ *such that* $n_1 + \cdots + n_d \leq \nu$ *the function* $f^{]\boldsymbol{n}[}\colon \mathrm{X}^{]\boldsymbol{n}[} \to \mathrm{E}$ *extends to a* $\mathscr{C}^\rho$*-function* $f^{[\boldsymbol{n}]}\colon \mathrm{X}^{[\boldsymbol{n}]} \to \mathrm{E}$ *(and which is* unique *provided the factors of* X *are free of isolated points)*

Definition. Let C be a compact Cartesian subset of K^d whose factors are free of isolated points. The *norm* $\|\cdot\|_{\mathscr{C}^r}$ over $\mathscr{C}^r(\mathrm{C}, \mathrm{E})$ is defined by

$$\begin{aligned}\|f\|_{\mathscr{C}^r} = \max\{&\|f^{[\boldsymbol{n}]}\|_{\sup} : \boldsymbol{n} \text{ in } \mathbb{N}^d \text{ such that } n_1 + \cdots + n_d < \nu\}\\ &\cup \{\|f^{[\boldsymbol{n}]}\|_{\mathscr{C}^\rho} : \boldsymbol{n} \text{ in } \mathbb{N}^d \text{ such that } n_1 + \cdots + n_d\}\end{aligned}$$

Let X be an open subset of K^d. The locally convex topology of $\mathscr{C}^r(\mathrm{X}, \mathrm{E})$ is defined as the initial topology given by the restriction mappings

$$\begin{aligned}\mathscr{C}^r(\mathrm{X}, \mathrm{E}) &\to \mathscr{C}^r(\mathrm{C}, \mathrm{E}),\\ f &\mapsto f_{|\mathrm{C}}\end{aligned}$$

for all compact Cartesian subsets C of X whose factors are free of isolated points.

- The locally convex K-vector space $\mathscr{C}^r(\mathrm{X}, \mathrm{E})$ is complete, and is a complete locally convex K-algebra if E is a K-Banach algebra.
- If $r \geq s$, then there is a continuous inclusion $\mathscr{C}^r(\mathrm{X}, \mathrm{E}) \subseteq \mathscr{C}^s(\mathrm{X}, \mathrm{E})$.
- The locally convex vector space $\mathscr{C}^r(\mathrm{X}, \mathrm{E})$ is Fréchet if and only if X is σ-compact. (A locally convex vector space is *Fréchet* if its topology is given by a metric. A topological space X is *σ-compact* if it is the countable union of compact subsets.)

- Let $f\colon X \to Y$ be either a $\mathscr{C}^r$-function, if $r \geq 1$, or a locally Lipschitz function otherwise. The precomposition operator

$$\begin{aligned} \mathscr{C}^r(Y, E) &\to \mathscr{C}^r(X, E) \\ f &\mapsto f \circ g \end{aligned}$$

is continuous (and well defined by Proposition 1.4).

2 Examples of $\mathscr{C}^r$-functions

sec:Examples

Let X be an open subset of K^d.

Locally analytic functions

ssec:LocAn

Definition. A function $f\colon X \to E$ is *locally analytic* if for every point $\boldsymbol{a}$ in X there is

- a ball B included in X around $\boldsymbol{a}$, and
- a power series $F(X) = \sum_{\boldsymbol{i} \in \mathbb{N}^d} a_{\boldsymbol{i}} X^{\boldsymbol{i}}$

such that $f(\boldsymbol{a} + \boldsymbol{x}) = F(\boldsymbol{x})$ for all $\boldsymbol{x}$ in B.

prop:LocAnInFctR

Proposition 2.1 ([Nag11, Proposition 3.18])**.** *A locally analytic function $f\colon X \to E$ is arbitrarily often differentiable.*

The proof rests on the completeness of E (which implies that of $\mathscr{C}^r(X, E)$). By completeness, a series in $\mathscr{C}^r(X, E)$ converges if and only if its entries converge to zero. Therefore, if the function f is analytic over the open ball B of radius ϵ given by the power series $\sum a_{\boldsymbol{i}} x^{\boldsymbol{i}}$, then f is r-times differentiable if and only if $a_{\boldsymbol{i}} x^{\boldsymbol{i}}$ converges to zero in $\mathscr{C}^r(B, E)$. That is, if and only if $\|x^{\boldsymbol{i}}\|_{\mathscr{C}^r} \leq \epsilon^i$.

This existence criterion gives another reason for the assumed completeness of the codomain of our differentiable functions.

Density of (locally) polynomial functions

Definition. A function $f\colon X \to E$ is *locally polynomial* of degree ν if for every point $\boldsymbol{a}$ in X there is

- a ball B included in X around $\boldsymbol{a}$, and

- a polynomial $F(X) = \sum_{\boldsymbol{i} \in \mathbb{N}^d \text{ with } i=0,\dots,\nu} a_{\boldsymbol{i}} X^{\boldsymbol{i}}$

such that $f(\boldsymbol{a} + \boldsymbol{x}) = F(\boldsymbol{x})$ for all $\boldsymbol{x}$ in B.

Every locally polynomial function is locally analytic and, by Proposition 2.1, every locally analytic function is arbitrarily often differentiable. That is,

$$\mathscr{C}^{\mathrm{lp}}(X, K) \subseteq \mathscr{C}^{\mathrm{la}}(X, K) \subseteq \mathscr{C}^{\infty}(X, K).$$

The locally polynomial functions are prototypic in the following sense.

prop:FctRMVFctPolDense

Proposition 2.2 ([Nag11, Proposition 3.30]). *The locally polynomial functions of maximal degree* ν *are dense inside* $\mathscr{C}^r(X, K)$.

Because every locally constant function is a limit of polynomial functions in $\mathscr{C}^r(X, K)$, the following variant obtains:

cor:FctRPolDense

Corollary 2.3 ([Nag11, Corollary 3.32]). *The polynomial functions are dense inside* $\mathscr{C}^r(X, K)$.

3 The dual of $\mathscr{C}^r$-functions

sec:Dual

Let X be an open subset of K^d.

Definition. Let $\mathscr{D}^r(X, E)$ denote all $\mathscr{C}^r$*-distributions* over X, that is,

$$\mathscr{D}^r(X, E) = \{\text{all continuous K-linear mappings } \mu \colon \mathscr{C}^r(X, E) \to K\}.$$

and let $\|\cdot\|_{\mathscr{D}^r}$ denote the *operator norm* over $\mathscr{D}^r(X, E)$ defined by

$$\|\mu\|_{\mathscr{D}^r} = \inf\{C \in \mathbb{R}_{\geq 0} : |\mu(f)| \leq C\|f\|_{\mathscr{C}^r} \quad \text{for all } f \in \mathscr{C}^r(X, E)\}.$$

Let $\mathscr{D}(X, E)$ be the direct limit over $r \geq 0$ of all $\mathscr{D}^r(X, E)$.

Let $\mathscr{C}^{\infty}(X, E)$ be the K-vector space of all arbitrarily often differentiable functions over X. inside its dual, which includes $\mathscr{D}(X, E)$ and all $\mathscr{D}^r(X, E)$ for $r \geq 0$ by Proposition 2.2, we have

$$\mathscr{D}(X, E) = \bigcup_{r \geq 0} \mathscr{D}^r(X, E).$$

That is, $\mathscr{D}(X, E)$ consists of all the K-linear maps $\mu \colon \mathscr{C}^{\infty}(X, E) \to K$ that, for some real number $r \geq 0$, extend to a continuous map over $\mathscr{C}^r(X, E)$.

The space of distributions on $\mathscr{C}^r(\mathrm{X},\mathrm{E})$ for a compact group X

ssec:Dist

If X is a (compact) open group inside K^d then $\mathscr{D}(\mathrm{X},\mathrm{E})$ carries a product, the convolution product that turns it into a K-algebra. We show that $\mathscr{D}(\mathrm{X},\mathrm{E})$ is a *filtered* K-*algebra* in the sense that its algebra operations, in particular its product, respect the natural filtration of $\mathscr{D}^r(\mathrm{X},\mathrm{E})$.

prop:ConvProd

Proposition 3.1 ([Nag11, Lemma 3.64]). *Let $r', r'' \geq 0$ and $r = r' + r''$. If the multiplication over* X *is either r-times differentiable if $r \geq 1$ (or Lipschitz continuous if $r < 1$) then*

- *given* μ *in* $\mathscr{D}^{r'}(\mathrm{X},\mathrm{E})$ *and* f *in* $\mathscr{C}^r(\mathrm{X},\mathrm{E})$*, their* convolution $\mu \star f : \mathrm{X} \to \mathrm{K}$ *given by*
$$\mu * f(x) := \mu\,(f(\_ \cdot x))$$
is in $\mathscr{C}^{r''}(\mathrm{X},\mathrm{E})$*, and*

- *there is a well-defined* convolution product
$$\mathscr{D}^r(\mathrm{X},\mathrm{E}) \times \mathscr{D}^s(\mathrm{X},\mathrm{E}) \to \mathscr{D}^{r+s}(\mathrm{X},\mathrm{E})$$
that sends (μ, ν) *to the continuous* K*-linear form on* $\mathscr{C}^{r+s}(\mathrm{X},\mathrm{E})$ *given by*
$$(\mu \star \lambda) \cdot f := \lambda \cdot (\mu \star f).$$

4 Explicit descriptions

sec:ExpDesc

… by the Mahler basis

subsec:Mahler

In this Section 4, we assume that the non-Archimedean field K includes $\mathbb{Q}_p$.

The Mahler basis of $\mathscr{C}^0(\mathbb{Z}_p,\mathrm{K})$. Let ${}_{\mathrm{K}} := \{x \in \mathrm{K} : |x| \leq 1\}$ be the closed unit ball of K. Let $\mathscr{C}^0(\mathbb{Z}_p, {}_{\mathrm{K}})$ be the continuous functions $f : \mathbb{Z}_p \to {}_{\mathrm{K}}$. The set $\mathscr{C}^{\mathrm{lc}}(\mathbb{Z}_p, {}_{\mathrm{K}})$ of locally constant functions over $\mathbb{Z}_p$ that take values in is dense inside $\mathscr{C}^0(\mathbb{Z}_p, {}_{\mathrm{K}})$. We have $\mathscr{C}^{\mathrm{lc}}(\mathbb{Z}_p, {}_{\mathrm{K}}) = \bigcup {}_{\mathrm{K}}[\mathbb{Z}/p^n\mathbb{Z}]$ and dually
$$\mathscr{D}^0(\mathbb{Z}_p, {}_{\mathrm{K}}) = \varprojlim\, [\mathbb{Z}/p^n\mathbb{Z}] =: {}_{\mathrm{K}}[[\mathbb{Z}_p]].$$

If we endow the group algebra ${}_{\mathrm{K}}[[\mathbb{Z}_p]]$ with the projective limit topology, then the morphism of topological ${}_{\mathrm{K}}$-algebras
$$\begin{aligned} {}_{\mathrm{K}}[[\mathbb{Z}_p]] &\to {}_{\mathrm{K}}[[\mathrm{X}]] \\ 1 &\mapsto 1 + \mathrm{X} \end{aligned}$$

that maps the generator 1 of the topological group $\mathbb{Z}_p$ to $1+\mathrm{X}$ is an isomorphism (the *Iwasawa isomorphism*).

Since the power series $_{\mathrm{K}}[[\mathrm{X}]]$ with coefficients in $_{\mathrm{K}}$ are the continuous dual of the zero sequences $c^0(\mathbb{N}, _{\mathrm{K}})$ with entries in $_{\mathrm{K}}$, the Iwasawa isomorphism corresponds by (*Schikhof*) duality to the (well-defined) topological isomorphism

$$c^0(\mathbb{N}, _{\mathrm{K}}) \to \mathscr{C}^0(\mathbb{Z}_p, _{\mathrm{K}})$$
$$e_n \mapsto \binom{\cdot}{n}$$

where $e_n := (\dots, 0, 1, 0 \dots)$ is the sequence whose sole nonzero entry is 1 at the n-th position and $\binom{\cdot}{n} \colon \mathbb{Z}_p \to \mathrm{K}$ is the *n-th Mahler polynomial* defined by $\binom{x}{n} = x(x-1)\cdots(x-n+1)/n!$.

In particular, every continuous function f over $\mathbb{Z}_p$ is a uniformly convergent sum of scaled Mahler polynomials, that is, there is a zero sequence (a_n) with entries in K, the *Mahler coefficients* of f, such that $f = \sum a_n \binom{\cdot}{n}$.

The Mahler basis of $\mathscr{C}^r(\mathbb{Z}_p^d, \mathrm{K})$. The convergence condition on the Mahler coefficients that characterizes the r-times differentiable functions of many variables over $\mathbb{Z}_p$ is described by the following *weighted zero sequences*.

Definition. Let $\boldsymbol{w} = (w_i : i \in \mathrm{I})$ be a sequence of positive real numbers. The K-Banach space of *weighted zero sequences* $c^{\boldsymbol{w}}(\mathrm{I}, \mathrm{K})$ is defined by

$$c^{\boldsymbol{w}}(\mathrm{I}, \mathrm{K}) = \{ \text{ all sequences } \lambda = (\lambda_i : i \in \mathrm{I}) \text{ with entries in K such that}$$
$$\text{for every } \varepsilon > 0 \text{ only finitely many } \lambda_i \text{ fulfill } |\lambda_i| w_i \geq \varepsilon \}$$

with the maximum-norm $\|(\lambda_i)\| := \max\{|\lambda_i| w_i : i \in \mathrm{I}\}$.

The following description of a many-variable $\mathscr{C}^r$-function over $\mathbb{Z}_p$ extends the initial definition of a one-variable $\mathscr{C}^r$-function over $\mathbb{Z}_p$ by a convergence condition on its Mahler coefficients from [BB10].

thm:MahlerBase

Theorem 4.1 ([Nag11, Corollary 3.49]). *Let $r \geq 0$ be a real number and $\boldsymbol{r} = ((n_1 + \cdots + n_d)^r : \boldsymbol{n} \in \mathbb{N}^d)$. Let $e_{\boldsymbol{n}} := (\dots, 0, 1, 0 \dots)$ be the sequence whose sole nonzero entry is 1 at place $\boldsymbol{n}$ and $\binom{\cdot}{\boldsymbol{n}} \colon \mathbb{Z}_p \to \mathrm{K}$ the Mahler polynomial given by $\binom{x}{\boldsymbol{n}} = x(x-1)\cdots(x-n+1)/n!$. Then*

$$c^{\boldsymbol{r}}(\mathbb{N}^d, \mathrm{K}) \to \mathscr{C}^r(\mathbb{Z}_p^d, \mathrm{K})$$
$$e_{\boldsymbol{n}} \mapsto \binom{\cdot}{\boldsymbol{n}}$$

is a well-defined isomorphism of topological K*-vector spaces.*

cor:MahlerBase

Corollary 4.2. *The function* $f\colon \mathbb{Z}_p^d \to \mathrm{K}$ *is r-times differentiable if and only if its Mahler coefficients* $(a_{\boldsymbol{n}})$ *fulfill* $|a_{\boldsymbol{n}}|(n_1 + \cdots + n_d)^r \to 0$ *as* $n_1 + \cdots + n_d \to \infty$.

… by the Taylor polynomial of one variable

subsec:Taylor lem:TaylorExp

Lemma 4.3 (Taylor-polynomial)**.** *Let* X *be an open subset of* K*. If* $f\colon \mathrm{X} \to \mathrm{K}$ *is r-times differentiable, then there are uniquely determined* $\mathscr{C}^{r-i}$*-functions* $\mathrm{D}^i f\colon \mathrm{X} \to \mathrm{K}$ *for* $i = 0, \ldots, \nu - 1$*, the derivatives of* f *as defined after Definition 1.3, and a* $\mathscr{C}^{\rho}$*-function* $f^{[\nu]}\colon \mathrm{X}^{[\nu]} \to \mathrm{K}$ *such that*

$$f(x) = \sum_{i=0,\ldots,\nu-1} \mathrm{D}^i f(y)(x-y)^i + f^{[\nu]}(x, y, \ldots, y)(x-y)^\nu \quad \textit{for all } x, y \textit{ in } \mathrm{X}$$

The *remainder* $\mathrm{R}f\colon \mathrm{X} \times \mathrm{X} \to \mathrm{K}$ of the Taylor polynomial of f is defined as

$$\mathrm{R}^\nu f(x, y) = [f^{[\nu]}(x, y, \ldots, y) - \mathrm{D}^\nu f(y)](x-y)^\nu$$

It is a $\mathscr{C}^{\rho}$-function that vanishes on the diagonal. In particular, for every $\varepsilon > 0$ and every point a in X, there is a neighborhood U inside X × X around (a, a) such that

$$|\mathrm{R}^\nu f(x, y) - \mathrm{R}^\nu f(y, y)| \leq \varepsilon |(x, y) - (y, y)|^\rho \quad \text{for all } (x, y), (y, y) \text{ in } \mathrm{U} \times \mathrm{U}.$$

defn:TaylorP

Definition 4.4. Let X be an open subset of K. A function $f\colon \mathrm{X} \to \mathrm{K}$ is a $\mathscr{C}^r_{\mathrm{T}}$-*function* if there are functions $\mathrm{D}^0 f, \ldots, \mathrm{D}^\nu f\colon \mathrm{X} \to \mathrm{K}$ and $\mathrm{R}^\nu f\colon \mathrm{X} \times \mathrm{X} \to \mathrm{K}$ such that

$$f(x+y) = \sum_{i=0,\ldots,\nu} \mathrm{D}^i f(x) y^i + \mathrm{R}^\nu f(x+y, x) \quad \text{for all } x+y, x \text{ in X},$$

and for every $a \in \mathrm{X}$ and $\varepsilon > 0$ there is a neighborhood U inside X around a such that

$$|\mathrm{R}^\nu f(x+y, x)| \leq \varepsilon |y|^r \quad \text{for all } x+y, x \text{ in U}.$$

Lemma (4.3')**.** *If* X *is an open subset of* K *then every* $\mathscr{C}^r$*-function* $f\colon \mathrm{X} \to \mathrm{K}$ *is a* $\mathscr{C}^r_{\mathrm{T}}$*-function.*

Sufficiency. The converse of Lemma 4.3', that is, if X is an open subset of K then every $\mathscr{C}^r_{\mathrm{T}}$-function $f\colon \mathrm{X} \to \mathrm{K}$ is a $\mathscr{C}^r$-function, is also true; in fact the equality between the sets of $\mathscr{C}^r_{\mathrm{T}}$-functions and $\mathscr{C}^r$-functions is an equality between locally convex K-vector spaces if we endow the $\mathscr{C}^r_{\mathrm{T}}$-functions with a locally convex topology as follows.

Let f be a $\mathscr{C}^r_{\mathrm{T}}$-function and let us keep the notations of Definition 4.4. The functions $\mathrm{D}^0 f$, $\mathrm{D}^1 f$, ..., $\mathrm{D}^\nu f\colon \mathrm{X} \to \mathrm{K}$ are uniquely determined by f (cf. [Nag11, Corollary 2.23]) and thence the function $|\Delta^r f|\colon \triangledown \mathrm{X} \times \mathrm{X} \to \mathbb{R}_{\geq 0}$ defined by

$$|\Delta^r f|(x, y) = \frac{|\mathrm{R}^\nu f(x, y)|}{|x - y|^r}$$

is uniquely determined. It extends to a function over $\mathrm{X} \times \mathrm{X}$ that vanishes continuously on the diagonal of $\mathrm{X} \times \mathrm{X}$. By [loc.cit.] the functions $\mathrm{D}^0 f$, $\mathrm{D}^1 f$, ..., $\mathrm{D}^\nu f$ are continuous. Thence $|\Delta^r f|$ is continuous.

Definition. Let C be a compact open subset of K. The *norm* $\|\cdot\|_{\mathscr{C}^r_{\mathrm{T}}}$ over $\mathscr{C}^r_{\mathrm{T}}(\mathrm{C}, \mathrm{E})$ is defined by

$$\|f\|_{\mathscr{C}^r_{\mathrm{T}}} = \max\{\|\mathrm{D}^0 f\|_{\sup}, \dots, \|\mathrm{D}^\nu f\|_{\sup}, \|\ |\Delta^r f|\ \|_{\sup}\}$$

Let X be an open subset of K. The locally convex topology of $\mathscr{C}^r(\mathrm{X}, \mathrm{E})$ is defined as the initial topology that is given by the restriction mappings

$$\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathrm{E}) \to \mathscr{C}^r_{\mathrm{T}}(\mathrm{C}, \mathrm{E}),$$
$$f \mapsto f_{|\mathrm{C}}$$

over all compact open subsets C of X.

thm:FctTEqFctTppLocally

Theorem 4.5 ([Nag11, Corollary 2.32]). *If* X *is an open subset of* K *then the natural inclusion of locally convex* K*-algebras* $\mathscr{C}^r(\mathrm{X}, \mathrm{K}) \hookrightarrow \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathrm{K})$ *is an isomorphism.*

If X is a subset of K without isolated points but not necessarily open, then Theorem 4.5 is true for $r \leq 2$ by [Sch84, Proposition 28.4] and false for $r > 2$ by [loc.cit., Example 83.2]. It remains true however with the following stronger variant of Definition 4.4.

Definition. A function $f\colon \mathrm{X} \to \mathrm{K}$ is a $\mathscr{C}^r_{\mathrm{T}}$*-function* if there are functions $\mathrm{D}^0 f$, ..., $\mathrm{D}^\nu f\colon \mathrm{X} \to \mathrm{K}$ such that for every $n = 0, \dots, \nu$ the function $\mathrm{F} = \mathrm{D}^n f$ is a $\mathscr{C}^{r-n}_{\mathrm{T}}$-function with $\mathrm{D}^0\mathrm{F} = \binom{n}{n}\mathrm{D}^n f$, $\mathrm{D}^1\mathrm{F} = \binom{n+1}{n}\mathrm{D}^{n+1} f$, ..., $\mathrm{D}^{\nu-n}\mathrm{F} = \binom{\nu}{n}\mathrm{D}^\nu f$.

Over certain *locally* B_ν*-subsets* of K the notion of a $\mathscr{C}^r_T$-function and $\mathscr{C}^r_T$-function coincide. Since by [Nag11, Lemma 2.27] every ball of K is B_ν for all ν in $\mathbb{N}$ and so every open subset of K is locally B_ν for all ν in $\mathbb{N}$, the notion of a $\mathscr{C}^r_T$-function and $\mathscr{C}^r_T$-function coincide over an open subset of K.

defn:BnuProp

Definition 4.6. Let ν be a natural number. A subset X of K is B_ν if either $\nu < 3$ or there is a positive constant $c \leq 1$ such that for every x_0 and x_1 in X there are $x_2, \ldots, x_\nu$ in the closed ball of radius $\delta = |x_1 - x_0|$ around x_0 inside X such that

$$\min\{|x_i - x_j| : i, j = 0, \ldots, \nu \text{ distinct}\} \geq c\delta.$$

A subset X of K is *locally* B_ν if for every x in X there is a neighborhood inside X around x which is B_ν.

Theorem 4.7 ([Nag11, Corollary 2.25]). *If* X *is a* B_ν*-subset of* K *without isolated points, then the natural inclusion of locally convex* K*-vector spaces* $\mathscr{C}^r_T(X, K) \hookrightarrow \mathscr{C}^r_T(X, K)$ *is an isomorphism.*

… by the Taylor polynomial of many variables over $\mathbb{Q}_p$

subsec:TaylorPol

In this section, we combine the descriptions of a $\mathscr{C}^r$-function

- over $\mathbb{Z}_p^d$ by its Mahler coefficients in Section 4, and
- over an open subset X of K by its Taylor polynomial in Section 4

to infer a description of a $\mathscr{C}^r$-function over an open subset X of $\mathbb{Q}_p^d$ by its Taylor polynomial.

sssec:totally

The total Taylor polynomial. Let $\mathrm{Mult}^n(K^d, E)$ be the symmetric multilinear forms of n arguments over K^d that take values in E. They identify by diagonal evaluation with homogeneous polynomials of d variables of total degree n.

defn:CrTaylorTotal

Definition 4.8. Let X be an open subset of K^d. The function $f : X \to E$ is a $\mathscr{C}^r_T$*-function* if there are continuous functions $D^n f : X \to \mathrm{Mult}^n(K^d, E)$ for $n = 0, \ldots, \nu$ and $R^\nu f : X \times X \to E$ such that

$$f(x + h) = \sum_{n=0,\ldots,\nu} D^n f(x)(h, \ldots, h) + R^\nu f(x, h)$$

and for every a in X and every $\varepsilon > 0$, there is a neighborhood U inside X around a such that

$$|R^\nu f(x, h)| \leq \varepsilon |h|^r \quad \text{for all } x + h, x \text{ in U}.$$

By [Nag16, Corollary 3.6] every $\mathscr{C}^r$-function is a $\mathscr{C}^r_{\mathrm{T}}$-function. The converse holds over $\mathrm{K} = \mathbb{Q}_p$ (and rests open otherwise). We show that even more holds.

thm:TaylorTotal

Theorem 4.9. *Let* X *be an open subset of* $\mathbb{Q}_p^d$ *and* $f\colon \mathrm{X} \to \mathrm{E}$*. Then* f *is a* $\mathscr{C}^r_{\mathrm{T}}$*-function if and only if* f *is a* $\mathscr{C}^r$*-function.*

sssec:Partially

The partial Taylor polynomial. If $f\colon \mathrm{X} \to \mathrm{E}$ is n-times differentiable, then $\mathrm{D}^n f\colon \mathrm{X} \to \mathrm{Mult}^n(\mathrm{K} \times \cdots \times \mathrm{K}, \mathrm{E})$ is the total differential of order n of f given by

$$\mathrm{D}^n f(x)(h,\dots,h) = \sum_{\boldsymbol{n}\in\mathbb{N}^d \text{ with } n_1+\cdots+n_d=n} \mathrm{D}^{\boldsymbol{n}} f(x) h_1^{n_1}\cdots h_d^{n_d}.$$

Let $k \in \{1,\dots,d\}$. The following Definition 4.8' is the special case of Definition 4.8 where, in the notation of loc.cit., the vector h in K^d has a single nonzero entry in the k-th coordinate.

Definition (4.8'). Let X be an open subset of K^d. The function $f\colon \mathrm{X} \to \mathrm{E}$ is a $\mathscr{C}^{r\cdot\boldsymbol{e}_k}_{\mathrm{T}}$*-function* if there are continuous functions $\mathrm{D}^0 f, \mathrm{D}^{1\cdot\boldsymbol{e}_k} f, \dots, \mathrm{D}^{\nu\cdot\boldsymbol{e}_k} f\colon \mathrm{X} \to \mathrm{E}$ and $\mathrm{R}^{\nu\cdot\boldsymbol{e}_k} f\colon \mathrm{X}^{[\boldsymbol{e}_k]} \to \mathrm{E}$ over $\mathrm{X}^{[\boldsymbol{e}_k]} := \{(x;t) \in \mathrm{X}\times\mathrm{K} : x + t\cdot\boldsymbol{e}_k \in \mathrm{X}\}$ such that

$$f(x + t\cdot\boldsymbol{e}_k) = \sum_{i=0,\dots,\nu} \mathrm{D}^{i\cdot\boldsymbol{e}_k} f(x) t^i + \mathrm{R}^{\nu\cdot\boldsymbol{e}_k} f(x;t)$$

and for every a in X and every $\varepsilon > 0$, there is a neighborhood U inside $\mathrm{X}^{[\boldsymbol{e}_k]}$ around a such that

$$|\mathrm{R}^{\nu\cdot\boldsymbol{e}_k} f(x;t)| \le \varepsilon|t|^r \quad \text{for all } x + t\cdot\boldsymbol{e}_k, x \text{ in U}.$$

Fix a coordinate index $k \in \{1,\dots,d\}$. Let $f\colon \mathrm{X} \to \mathrm{E}$ be a $\mathscr{C}^{r\cdot\boldsymbol{e}_k}_{\mathrm{T}}$-function and let us keep the notation of Definition 4.8'. The function $|\Delta^{r\cdot\boldsymbol{e}_k} f|\colon \mathrm{X}^{]\nu\cdot\boldsymbol{e}_k[} \to \mathbb{R}_{\ge 0}$ is defined by

$$|\Delta^{r\cdot\boldsymbol{e}_k} f|(x;t) = \frac{|\mathrm{R}^{\nu\cdot\boldsymbol{e}_k} f(x;t)|}{|t|^r}.$$

The continuity of $\mathscr{D}_0 f, \mathrm{D}^{1\cdot\boldsymbol{e}_k} f, \dots, \mathrm{D}^{\nu\cdot\boldsymbol{e}_k} f\colon \mathrm{X} \to \mathrm{K}$ implies the continuity of $|\Delta|_{\nu\cdot\boldsymbol{e}_k} f$ over $\mathrm{X}^{]\boldsymbol{e}_k[}$. Since f is a $\mathscr{C}^{r\cdot\boldsymbol{e}_k}_{\mathrm{T}}$-function, $|\Delta^{r\cdot\boldsymbol{e}_k} f|$ extends to a continuous function over $\mathrm{X}^{[\boldsymbol{e}_k]}$ that vanishes if t vanishes.

The functions $\mathrm{D}^{0\cdot\boldsymbol{e}_k}, \dots, \mathrm{D}^{\nu\cdot\boldsymbol{e}_k} f\colon \mathrm{X} \to \mathrm{E}$ and thus $|\Delta^{r\cdot\boldsymbol{e}_k} f|$ are uniquely determined by f by [Nag11, Lemma 3.52].

Definition. Let C be a compact open subset of $\mathbb{Q}_p^d$. The *norm* $\|\cdot\|_{\mathscr{C}_\mathrm{T}^{r\cdot e_k}}$ over $\mathscr{C}^{r\cdot e_k}(\mathrm{C},\mathrm{E})$ is defined by

$$\|f\|_{\mathscr{C}_\mathrm{T}^{r\cdot e_k},\mathrm{C}} = \max\{\|\mathrm{D}^0 f\|_{\sup}, \|\mathrm{D}^{1\cdot e_k} f\|_{\sup}, \ldots, \|\mathrm{D}^{\nu\cdot e_k} f\|_{\sup}, \| \, |\Delta^{r\cdot e_k} f| \, \|_{\sup}\}.$$

Let X be an open subset of $\mathbb{Q}_p^d$. The locally convex topology of $\mathscr{C}^{r\cdot e_k}(\mathrm{X},\mathrm{E})$ is defined as the initial topology that is given by the restriction mappings

$$\begin{aligned}\mathscr{C}^r(\mathrm{X},\mathrm{E}) &\to \mathscr{C}^{r\cdot e_k}(\mathrm{C},\mathrm{E}),\\ f &\mapsto f_{|\mathrm{C}}\end{aligned}$$

for all compact open subsets C of X.

The following definition of a $\mathscr{C}_\mathrm{T}^r$-function is the case of a $\mathscr{C}_\mathrm{T}^r$-function where, in the notation of Definition 4.8, the vector h in K^d has a single nonzero entry.

defn:CrTaylorCoord

Definition. Let X be an open subset of $\mathbb{Q}_p^d$. Then $\mathscr{C}_\mathrm{T}^r(\mathrm{X},\mathrm{E})$ is the initial normed K-vector space of $\mathscr{C}_\mathrm{T}^{r\cdot e_1}(\mathrm{X},\mathrm{E}), \ldots, \mathscr{C}_\mathrm{T}^{r\cdot e_d}(\mathrm{X},\mathrm{E})$. That is, $\mathscr{C}_\mathrm{T}^r(\mathrm{X},\mathrm{E}) = \mathscr{C}^{r\cdot e_1}(\mathrm{X},\mathrm{E}) \cap \ldots \cap \mathscr{C}_\mathrm{T}^{r\cdot e_d}(\mathrm{X},\mathrm{E})$ with norm $\|\cdot\|_{\mathscr{C}_\mathrm{T}^r} = \max\{\|\cdot\|_{\mathscr{C}_\mathrm{T}^{r\cdot e_1}}, \ldots, \|\cdot\|_{\mathscr{C}_\mathrm{T}^{r\cdot e_d}}\}$.

We conclude that it suffices to check the convergence condition in Theorem 4.9 on all h with a single nonzero entry (see [Glö13, Corollary 9.5] for a variant).

Theorem ([Nag11, Corollary 3.60]). *Let* X *be an open subset of* $\mathbb{Q}_p^d$. *The natural inclusion of locally convex* K*-vector spaces* $\mathscr{C}^r(\mathrm{X},\mathrm{E}) \hookrightarrow \mathscr{C}_\mathrm{T}^r(\mathrm{X},\mathrm{E})$ *is an isomorphism.*

References

[BB10] L. Berger and C. Breuil, *Sur quelques représentations potentiellement cristallines de* $\mathrm{GL}_2(\mathrm{Q}_p)$, Astérisque **330** (2010), 155–211.

[BGN04] W. Bertram, H. Glöckner, and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. **22** (2004), no. 3, 213–282. MR 2069671. DOI 10.1016/S0723-0869(04)80006-9.

[Col10] P. Colmez, *Fonctions d'une variable p-adique*, Astérisque (2010), no. 330, 13–59. MR 2642404.

[DI13] M. De Ieso, *Espaces de fonctions de classe* C^r *sur* $\mathcal{O}_F$, Indag. Math. (N.S.) **24** (2013), no. 3, 530–556. MR 3064559. DOI 10.1016/j.indag.2013.02.006.

[DS98] S. De Smedt, *Local invertibility of non-archimedian vector-valued functions*, Ann. Math. Blaise Pascal **5** (1998), 13–23.

[Glö07] H. Glöckner, *Comparison of some notions of* C^k*-maps in multi-variable non-Archimedian analysis*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 5, 877–904. MR 2378995. Confer http://projecteuclid.org/euclid.bbms/1197908901.

[Glö13] ———, *Exponential laws for ultrametric partially differentiable functions and applications*, p-Adic Numbers Ultrametric Anal. Appl. **5** (2013), no. 2, 122–159. MR 3056778. DOI 10.1134/S2070046613020039.

[Nag11] E. Nagel, *Fractional non-Archimedean differentiability*, Univ. Münster, Mathematisch-Naturwissenschaftliche Fakultät (Diss.), 2011. zbMATH 1223.26011. Confer http://nbn-resolving.de/urn:nbn:de:hbz:6-75409405856.

[Nag16] E. Nagel, *p-adic Taylor Polynomials*, Indagationes Mathematicae **1** (2016), –. DOI 10.1016/j.indag.2015.12.003. Confer http://www.sciencedirect.com/science/article/pii/S0019357715001275.

[Sch78] W. H. Schikhof, *Non-Archimedean calculus*, Report, vol. 7812, Katholieke Universiteit, Mathematisch Instituut, Nijmegen, 1978, Lecture notes. MR 522166.

[Sch84] ———, *Ultrametric calculus*, Cambridge Studies in Advanced Mathematics, vol. 4, Cambridge University Press, Cambridge, 1984, An introduction to p-adic analysis. MR 791759.