p-adic Taylor Polynomials

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For a real number $r \geq 0$, we define r-fold differentiability of a function on a p-adic vector space by the convergence of its Taylor polynomial expansion, and compare this differentiability definition with that by iterated divided differences, the textbook approach (from the 80's) to define p-adic differentiability.

This comparison applies to a recent definition of r-fold differentiability over a p-adic number field $\mathbf{K}$ that arises from the p-adic Langlands program over $\mathrm{GL}_2(\mathbf{K})$; yielding that this differentiability condition is equivalent to that via divided differences on $\mathbf{K}$ as vector space over $\mathbb{Q}_p$.

Contents

Introduction

p-adic Calculus done right must use a differentiability definition stronger than that of ordinary Calculus to account for the total disconnectedness of the range.

The prevalent definition (see [Sch84, Section 26ff.]), though conceptually sound, uses a differential whose number of variables doubles with each degree of differentiability and quickly becomes unwieldy. Since (fractional) p-adic differentiability has resurged in representation theory of p-adic Lie groups, interest arose in a different approach that keeps the number of variables at bay.

We study a definition by the Taylor polynomial expansion on normed vector spaces, a function of two arguments (expansion point and offset), and compare this definition to those recently proposed in the representation theory of p-adic Lie groups.

non-Archimedean differential Calculus

Let us first discuss

1. how the differentiability condition over the real numbers falls short over the p-adic numbers and instead

2. is replaced over the p-adic numbers by a stronger differentiability condition via iterated divided differences, which however

3. in each iteration doubles the number of variables of the divided difference, but how a handier differentiability condition via the Taylor polynomial often comes to rescue.

Shortcomings of ordinary differentiability. Let $\mathbf{K}$ be a *non-Archimedean field*, that is, a field with an absolute value that is non-Archimedean, non-trivial and complete.

The definition of differentiability over $\mathbb{R}$ transfers verbatim to $\mathbf{K}$: A function $f\colon \mathrm{X} \to \mathbf{K}$ on an open subset X of $\mathbf{K}$ is *(ordinarily) differentiable* at a in X if there is $f'(a)$ in $\mathbf{K}$ such that, for every sequence (x_n) in X, if $x_n \to a$ then

$$\frac{f(x_n) - f(a)}{x_n - a} \to f'(a)$$

Completeness and non-discreteness of $\mathbf{K}$ ensure that the *derivative* $f'(a)$ exists in $\mathbf{K}$ and is unique. Still, this definition defies our intuition through the function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$\sum_{n\in\mathbb{N}} a_n p^n \mapsto \sum_{n\in\mathbb{N}} a_n p^{2n}. \qquad (*)$$

It fulfills $|f(\cdot)| = |\cdot|^2$ and so is differentiable with $f' = 0$. We diagnose though:

- it is injective, but its derivative is 0, and
- it is infinitely often differentiable but its Taylor polynomial expansion of degree greater than 1 does not converge.

These pathologies occur because the intermediate-value theorem fails on the totally disconnected range $\mathbf{K}$ (in the wording that every continuous function f maps the smallest ball containing two points a and b onto that containing $f(a)$ and $f(b)$). To compensate this failure, we define more strictly a function $f\colon \mathrm{X} \to \mathbf{K}$ on an open subset X of $\mathbf{K}$ as *(non-Archimedeanly) differentiable* if its *divided difference*

$$f^{]1[}(x,y) := \frac{f(x) - f(y)}{x - y},$$

defined for all distinct x and y in X, extends to a continuous function $f^{[1]}\colon \mathrm{X} \times \mathrm{X} \to \mathbf{K}$. For a function $f\colon \mathrm{X} \to \mathbb{R}$ the mean-value theorem proves the non-Archimedean and ordinary differentiability condition equivalent.

Let us outline the first part of this article, where we expand this definition to functions of many variables by two different approaches and compare these:

Divided differences. In Section 1 we introduce the divided difference $f^{[1]}$ of a function $f : \mathrm{X} \to \mathbf{E}$ on an open subset X of a finite-dimensional vector space V with values in a Banach space $\mathbf{E}$. It is a function $f^{[1]} : \mathrm{X} \times \mathrm{X} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$. In particular it maps from a finite-dimensional vector space into a Banach space, like f does, and we define a differentiable function f as *twice differentiable* if the divided difference of $f^{[1]}$ is again differentiable.

For applications in representation theory of p-adic Lie groups, it is important not to halt at iterated differentiability, but to further define r-fold differentiability for a real number $r \geq 0$: We first define it for $r < 1$ by a strengthened Hölder condition (Definition 1.1). In general, we write $r = \nu + \rho$ for ν in $\mathbb{N}$ and ρ in $[0, 1[$, and ask the ν-th iterated divided difference $f^{[\nu]}$ to be ρ-times differentiable.

This non-Archimedean definition of differentiability preserves common facts in ordinary differential calculus ([Sch84, Section 27]) such as

- invertibility of a function around a point where its derivative is invertible,
- convergence of the Taylor polynomial expansion up to degree ν of a ν-times differentiable function ([Sch84, Chapter Ten]), and
- completeness of the normed space of all differentiable functions,

whereas the ordinary definition breaks each of these over $\mathbf{K}$. (Indeed these facts follow with the non-Archimedean definition of differentiability straight from the definition whereas the ordinary definition demands a detour via the mean-value theorem or the fundamental theorem of calculus.)

Taylor polynomial. Our differentiability definition by iterated divided differences leads to a sound differential calculus, yet with increasing degree of differentiability ν also to an exponential growth in the number of variables of the divided difference $f^{[\nu]}$.

In Section 2, we define instead r-fold differentiability of a many-variable function f over $\mathbf{K}$ by convergence of the Taylor polynomial expansion of f, which is a function of two arguments (the expansion point and its offset), irrespective of the degree of differentiability.

We then compare in Section 3 this differentiability definition to that by iterated divided differences: If f is differentiable then its Taylor polynomial expansion converges and vice versa, for f a one-variable function or a many-variable function over $\mathbb{Q}_p$, if its Taylor polynomial expansion converges then f is differentiable.

In the second half of this article (Section 4) we apply our results from the first half (Section 1, 2 and 3), to link the introduced notions of *fractional differentiability* to that used in the *p-adic Langlands program*.

Fractional differentiability, that is, r-fold differentiability for a real number $r \geq 0$, emerged from the p-adic Langlands program which links p-adic Galois and Lie group representations: Let $\mathbf{K}$ (the base field) be a finite extension of $\mathbb{Q}_p$ and $\mathbf{E}$ (the coefficient field) a complete extension of $\mathbb{Q}_p$. The two-dimensional *p-adic Langlands correspondence* (see [Col14] or [Ber11]) puts a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{K}}/\mathbf{K})$ of $\mathbf{K}$ on a 2-dimensional $\mathbf{E}$-vector space in correspondence with a unitary continuous action by $\mathrm{GL}_2(\mathbf{K})$ on a, generally infinite-dimensional, $\mathbf{E}$-Banach space $\widehat{\mathrm{V}}$.

The category of *crystalline* Galois actions is a full subcategory of that of all continuous actions of $\mathrm{Gal}(\overline{\mathbf{K}}/\mathbf{K})$ which is equivalent to a category of explicit linear algebraic objects, so called admissible filtered φ-modules, (see [ST02, Section 5]) and as such serves as a first test case for the p-adic Langlands correspondence.

To a crystalline Galois action corresponds a unitary continuous $\mathrm{GL}_2(\mathbf{K})$-action on a Banach space $\widehat{\mathrm{V}}$ that is the topological completion of a vector space V of certain locally algebraic functions $f : \mathbf{K} \to \mathbf{E}$. (It is the tensor product $\mathrm{V} = \mathrm{W} \otimes \mathrm{U}$ of a vector space W of locally constant functions, whose choice comes from the local Langlands program, and a vector space U of $\mathbb{Q}_p$-algebraic functions, whose choice comes from the resemblance between certain parameters classifying all crystalline Galois actions and those classifying all irreducible algebraic actions (see [BS07, Introduction]).)

The group action on $\widehat{\mathrm{V}}$ is unitary if $\widehat{\mathrm{V}}$ is the topological completion of V for a norm that is invariant under the group action. This norm is, for some real number $r \geq 0$, a quotient norm of the norm $\|\cdot\|_{\mathscr{C}^r}$ on the space of all r-times differentiable functions on $\mathbf{K}$ and $\widehat{\mathrm{V}}$ is a subquotient of the topological vector space $\mathscr{C}^r(\mathbf{K})$ of all r-times differentiable functions on $\mathbf{K}$. The real number r is determined by the action of $\mathrm{GL}_2(\mathbf{K})$ on V (read off from the action of a certain diagonal matrix on a generator of the module V over the group ring). Given V, it determines the topology of $\widehat{\mathrm{V}}$ (but the action of $\mathrm{GL}_2(\mathbf{K})$ on V only remotely).

If $r \geq 0$ is a nonintegral number, then r-fold differentiability was defined first in [BB10] for a function on $\mathbb{Z}_p$ by a growth condition on its *Mahler coefficients*, its coefficients in terms of a distinguished orthogonal basis of the continuous functions on $\mathbb{Z}_p$.

In Section 4 we first recall Colmez's equivalent approach in [Col10] via the Taylor polynomial expansion of a function on $\mathbb{Z}_p$ that de Ieso in [De 13a] extended to functions on a finite extension $\mathbf{o_K}$ of $\mathbb{Z}_p$.

We then show how Colmez's and de Ieso's definition of a $\mathscr{C}^r$-function furnishes a Banach space, that of all $\mathscr{C}^r$-functions $f: \mathbf{K} \to \mathbf{E}$, whose natural norm is invariant under the action of all triangular matrices in $\mathrm{GL}_2(\mathbf{K})$ on V.

In Theorem 4.7 we answer the question raised at [De 13a, End of Section 1.1]: How does the definition of r-fold differentiability (in Section 1) via divided differences from non-Archimedean calculus compare to de Ieso's definition (in Section 4) via Taylor polynomials from representation theory?

Acknowledgments. Thanks to Christophe Breuil, Marco de Ieso and João Pedro dos Santos (who also proofread and offered clarifications) for enlightening discussions on p-adic differentiability and its role in representation theory.

Notations

We adopt conventional terminology in non-Archimedean Functional Analysis (see [PGS10, Section 2.1]): A *non-Archimedean field* is a field that has a non-trivial non-Archimedean absolute value and is complete. A *non-Archimedean* norm $\|\cdot\|$ on a vector space V is a norm that satisfies the strong triangle inequality, that is, $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all x and y in V. A *non-Archimedean Banach space* is a vector space that has a non-Archimedean norm $\|\cdot\|$ and is complete for it.

Henceforth

- **K** is always a non-Archimedean field, and
- **E** is always a non-Archimedean **K**-Banach space.

For a continuous function f on a compact set X that takes values in a normed space let

$$\|f\|_{\sup} := \sup\{\|f(x)\| \text{ for all } x \in \mathrm{X}\}$$

denote its supremum norm; if X is not necessarily compact then this notation implies that this supremum exists.

1. Fractional differentiability via iterated divided differences

In this section, we define r-fold differentiability classically via iterated divided differences. First we decompose $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ into its integer part $\nu \in \mathbb{N}$ and its fractional part $\rho \in [0,1[$. Then we define ν-fold differentiability by iteratively building partial divided differences, and ρ-fold differentiability by a strengthened Hölder-continuity condition. Finally, an r-times differentiable function is a ν-times differentiable function such that each of its partial divided differences is ρ-times differentiable.

$\mathscr{C}^\nu$-functions for a natural number ν

Differentiable functions. Let V be a finite-dimensional $\mathbf{K}$-vector space, X a subset of V and $f\colon X \to \mathbf{E}$. Recall that a function f is *differentiable* at a in X if there is a linear map $A\colon V \to \mathbf{E}$ such that for every $\epsilon > 0$ there is a neighborhood U around a inside X such that

$$f(x+h) = f(x) + Ah + R(x,h)$$

with $\|R(x,h)\| \leq \epsilon\|h\|$ for all $x+h,x$ in U. The following, equivalent, differentiability criterion requires a choice of coordinates on V but can be iterated.

Let us fix a basis $e_1,\ldots,e_d$ of V by which V identifies with the d-fold direct sum $\mathbf{K} \oplus \cdots \oplus \mathbf{K}$.

Definition. Let X be a subset of V. The differential $f^{]1[}(x+h,x)$ of f at $x+h,x$ in X with $h \in \mathbf{K}^{*d}$ is the $\mathbf{K}$-linear map $A\colon V \to \mathbf{E}$ determined by

$$A \cdot h_k e_k = f(x + h_1e_1 + \cdots + h_{k-1}e_{k-1} + h_ke_k) - f(x + h_1e_1 + \cdots + h_{k-1}e_{k-1})$$

for all $k = 1,\ldots,d$. The function f is a $\mathscr{C}^1$*-function* if $f^{]1[}$ extends to a continuous function $f^{[1]}\colon X \times X \to \mathrm{Hom}_{\mathbf{K}}(V,\mathbf{E})$.

A subset X of V is *accumulated* if it is locally a product $U = U_1 \times \cdots \times U_d$ of subsets $U_1,\ldots,U_d$ of $\mathbf{K}$ without isolated points. For example, every open subset is accumulated. If X is accumulated then f uniquely determines the extended function $f^{[1]}$ because the domain of $f^{]1[}$ is dense inside $X^{[1]}$.

Remark. To put a topology on the set of all r-times differentiable functions, we would like to take the supremum of a differentiable function on a compact set. From this viewpoint the notion of an accumulated set suits us because a general, not necessarily locally compact, field can be covered by compact *accumulated* but not necessarily by compact *open* subsets. (For example $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, has no nonempty open subset that is compact.)

$\mathscr{C}^\nu$-functions for a natural number ν. Let X be an accumulated subset of V. Let $f\colon X \to \mathbf{E}$ be a $\mathscr{C}^1$-function. Let us compare the domain and codomain of $f^{[1]}$ with that of f. The domain $X^{[1]} := X \times X$ of $f^{[1]}$ is included in the finite-dimensional $\mathbf{K}$-vector space $V^{[1]} = V \times V$ with a canonical ordered basis, like the domain X of f, and the codomain $\mathbf{E}^{[1]} := \mathrm{Hom}_{\mathbf{K}}(V, \mathbf{E})$ of $f^{[1]}$ is a $\mathbf{K}$-Banach space, like the codomain $\mathbf{K}$ of f. We can thus apply the differentiability condition to $f^{[1]}$, and define iterated differentiability this way.

Definition. Let ν in $\mathbb{N}$. The function $f\colon X \to \mathbf{E}$ is a *$\mathscr{C}^{\nu+1}$-function*

- if f is a $\mathscr{C}^\nu$-function, and
- if $\mathfrak{X} = X^{[\nu]}$, $\mathfrak{V} = V^{[\nu]}$, $\mathfrak{E} = \mathbf{E}^{[\nu]}$ and $\mathfrak{f} = f^{[\nu]}$ then $\mathfrak{f}^{]1[}$ extends to a continuous function $\mathfrak{f}^{[1]}\colon \mathfrak{X} \times \mathfrak{X} \to \mathrm{Hom}_{\mathbf{K}}(\mathfrak{V} \times \mathfrak{V}, \mathfrak{E})$.

Definition. Let X be compact without isolated points. The norm $\|\cdot\|_{\mathscr{C}^\nu}$ on $\mathscr{C}^\nu(X, \mathbf{E})$ is defined by $\|f\|_{\mathscr{C}^\nu} := \max\{\|f\|_{\sup}, \|f^{[1]}\|_{\sup}, \ldots, \|f^{[\nu]}\|_{\sup}\}$.

$\mathscr{C}^\rho$-functions for ρ in $[0,1[$

Let ρ in $[0,1[$. Roughly, ρ-fold differentiability is stricter Hölder-continuity.

Let $\mathfrak{X}$ be a subset of a finite-dimensional $\mathbf{K}$-vector space and let $\mathfrak{E}$ be a non-Archimedean $\mathbf{K}$-Banach space.

Definition 1.1. The function $f\colon \mathfrak{X} \to \mathfrak{E}$ is a *$\mathscr{C}^\rho$-function* if for every a in $\mathfrak{X}$ and every $\varepsilon > 0$, there is a neighborhood U around a inside $\mathfrak{X}$ such that

$$\|f(x) - f(y)\| \leq \varepsilon \cdot \|x - y\|^\rho \quad \text{for all } x, y \text{ in } U.$$

The *fractional divided difference* $|f^{]\rho[}|\colon \triangledown\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_{\geq 0}$ of f is defined by

$$|f^{]\rho[}|(x,y) = \|f(x) - f(y)\| / \|x - y\|^\rho.$$

The function $f\colon \mathfrak{X} \to \mathfrak{E}$ is a $\mathscr{C}^\rho$-function if and only if $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|$ on all of $\mathfrak{X} \times \mathfrak{X}$ that vanishes on the diagonal of $\mathfrak{X} \times \mathfrak{X}$. This extension is *unique* if $\mathfrak{X}$ contains no isolated points. We define:

Definition. Let $\mathfrak{X}$ be compact without isolated points. The norm $\|\cdot\|_{\mathscr{C}^\rho}$ on all $\mathscr{C}^\rho$-functions $f\colon \mathfrak{X} \to \mathfrak{E}$ is defined by $\|f\|_{\mathscr{C}^\rho} := \max\{\|f\|_{\sup}, \| \, |f^{[\rho]}| \, \|_{\sup}\}$.

It $\mathfrak{X}$ is not necessarily compact then this notation implies that each supremum on the right-hand side, and thence their maximum, exists.

$\mathscr{C}^r$-functions for a real number $r \geq 0$

We fix henceforth a real number $r \geq 0$ and its decomposition

$$r = \nu + \rho$$

into an • integer part $\nu = \lfloor r \rfloor \in \mathbb{N}$, and a • fractional part $\rho = \{r\} \in [0,1[$.

Definition 1.2. Let X be an accumulated subset of V. The function $f\colon X \to \mathbf{E}$ is a $\mathscr{C}^r$*-function* if f is a $\mathscr{C}^\nu$-function and $f^{[\nu]}$ is a $\mathscr{C}^\rho$ function.

Definition. Let X be an accumulated compact subset of V. The norm $\|\cdot\|_{\mathscr{C}^\nu}$ on $\mathscr{C}^\nu(X, \mathbf{E})$ is defined by $\|f\|_{\mathscr{C}^\nu} := \max\{\|f^{[0]}\|_{\sup}, \ldots, \|f^{[\nu-1]}\|_{\sup}, \|f^{[\nu]}\|_{\mathscr{C}^\rho}\}$.

2. Taylor Polynomials

Let $r \geq 0$ be a real number with integer part ν and fractional part ρ. We define r-fold differentiability of a function on a non-Archimedean vector space by the convergence of its Taylor polynomial expansion up to degree ν.

Definition

Let V be a $\mathbf{K}$-vector space. Let $\mathrm{Sym}^n_{\mathbf{K}}(V, \mathbf{E})$ be all continuous symmetric $\mathbf{K}$-multilinear maps $M\colon V \times \cdots \times V \to \mathbf{E}$ of n variables. These form a non-Archimedean $\mathbf{K}$-Banach space by the operator norm

$$\|M\| = \sup\{\|M(x)\| \text{ for all } x \text{ with } \|x\| \leq 1\}.$$

That is, the supremum of M on the unit ball of $V \times \cdots \times V$ with respect to the product norm $\|v_1, \ldots, v_n\| = \max\{\|v_1\|, \ldots, \|v_n\|\}$.

The following definition generalizes that of onefold differentiability at the beginning of Section 1 to a higher differentiability degree $r \geq 0$.

Definition 2.1. Let X be an accumulated subset of V. The function $f\colon X \to \mathbf{E}$ is a $\mathscr{C}^r_{\mathrm{T}}$*-function* if there are functions $D^n f\colon X \to \mathrm{Sym}^n(V, \mathbf{E})$ for $n = 0, 1, \ldots, \nu$ and $R^\nu f\colon X \times X \to \mathbf{E}$ such that

$$f(x+h) = \sum_{n=0,\ldots,\nu} D^n f(x)(h, \ldots, h) + R^\nu f(x+h, x)$$

and for every a in X and $\varepsilon > 0$, there is a neighborhood U around a inside X such that

$$\|R^\nu f(x+h, x)\| \leq \varepsilon \|h\|^r \quad \text{for all } x+h, x \text{ in U}.$$

Symmetric multilinear forms and homogeneous polynomials. A sum of symmetric multilinear maps is determined by its restriction onto the diagonal (if the characteristic of the field of definition is equal to 0 or greater than the maximal number of arguments of these maps) and this restriction becomes after a choice of basis a polynomial map.

Given finitely many multilinear maps, the following lemma shows how to recover each one of them by the restriction of their sum onto the diagonal.

Lemma 2.2 (Polarization of an algebraic form). *Let* $\mathrm{M}^0, \dots, \mathrm{M}^n$ *be multilinear maps of* $0, \dots, n$ *arguments in* V *respectively and the functions* $m^0, \dots, m^n$ *on* V *their restrictions onto the diagonal. Let* $m = m^0 + \cdots + m^n$*. Then*

$$n!\mathrm{M}^n(x_1, \dots, x_n) = [\Delta^{x_1} \circ \cdots \circ \Delta^{x_n}(m)](0),$$

where, given $x \in \mathrm{V}$*, the* difference operator Δ^x *on all functions on* V *is defined by* $m \mapsto m(\cdot + x) - m$*.*

Proof: Given nonzero $x \in \mathrm{V}$, the difference operator Δ^x diminishes the *degree* of m by 1. That is, if m corresponds to a sum M of multilinear map in up to n arguments then $\Delta^x m$ corresponds to a sum of multilinear maps in up to $n-1$ arguments. Because, if N is a map of n arguments that is multilinear and symmetric then

$$\Delta^x \mathrm{N} = \mathrm{N}(\cdot + x, \dots, \cdot + x) - \mathrm{N} = \sum_{i=1,\dots,n} \binom{n}{i} \mathrm{N}(\underbrace{x, \dots, x}_{i\text{-times}}, \cdot, \dots, \cdot)$$

with the highest term $n\mathrm{N}(x, \cdot, \dots, \cdot)$. We conclude iteratively that $\Delta^{x_1} \circ \cdots \circ \Delta^{x_n} m$ is constant and, by the above equality, equal to $n!\mathrm{M}(x_1, \dots, x_n)$.

We recover M^{n-1} by applying the above reasoning to the diagonal map m_{n-1} given by $v \mapsto m(v) - \mathrm{M}^n(v, \dots, v) = \mathrm{M}^0 + \mathrm{M}^1(v) + \cdots + \mathrm{M}^{n-1}(v, \dots, v)$ and iteratively recover $\mathrm{M}^{n-2}, \dots, \mathrm{M}^0$ as well. □

Given $\delta > 0$, let $\|\cdot\|_\delta$ be the norm on all polynomial maps $f \colon \mathrm{V} \to \mathbf{E}$ defined by $\|f\|_\delta := \sup\{\|f(x)\| \text{ for all } x \text{ with } \|x\| \leq \delta\}$.

Corollary 2.3. *Let* $\mathrm{M}^0, \dots, \mathrm{M}^n$ *be multilinear maps on* V *of* $0, \dots, n$ *variables respectively with values in* $\mathbf{E}$*. Let* $m^0, \dots, m^n$ *be their restrictions onto the diagonal and put* $m = m^0 + \cdots + m^n$*. Then for every* $i = 0, \dots, n$ *and* δ *in* $|\mathbf{K}^*|$*,*

$$\|n! \cdots i!\mathrm{M}^i\| \leq \delta^{-i} \|m\|_\delta.$$

Proof: By Lemma 2.2, whose notation we adopt, $\|n!\mathrm{M}^n\| \leq \|m\|$. So $\|(n-1)!(n!\mathrm{M}^{n-1})\| \leq \max\{\|n!m\|, \|n!\mathrm{M}^n\|\} \leq \max\{\|n!m\|, \|m\|\} = \|m\|$. Iteratively $\|n!\cdots i!\mathrm{M}^i\| \leq \|m\|$ for $i = 0,\ldots,n$. By multilinearity $\|\mathrm{M}^i\|_\delta = \delta^i\|\mathrm{M}^i\|$. □

Differentiability of the Taylor polynomial coefficients. Let $f\colon \mathrm{X} \to \mathbf{E}$ be a $\mathscr{C}^r_\mathrm{T}$-function. The estimate in Corollary 2.3 on the polynomial coefficients applies in particular, for fixed x in X, to each polynomial coefficient $\mathrm{D}^0 f(x), \ldots, \mathrm{D}^n f(x)$.

We prove via this estimate that the convergence of the Taylor polynomial remainder of f implies that of the Taylor polynomial of $\mathrm{D}^1 f, \ldots, \mathrm{D}^n f$.

Proposition 2.4. *Let either* char K $> \nu$ *or* char $\mathbf{K} = 0$. *Let* X *be an accumulated subset of* V. *If* $f \in \mathscr{C}^r_\mathrm{T}(\mathrm{X}, \mathbf{E})$ *then* $\mathrm{D}^n f \in \mathscr{C}^{r-n}_\mathrm{T}(\mathrm{X}, \mathrm{Sym}^n(\mathrm{V}, \mathbf{E}))$ *for* $n = 0, \ldots, \nu$.

Proof: Let $x, x+y, x+y+z \in \mathrm{X}$. Write out

$$\begin{aligned}
&\mathrm{R}^\nu f(x+y+z, x) - \mathrm{R}^\nu f(x+y+z, x+y)\\
=&f(x+y+z) - \sum_{n=0,\ldots,\nu} \mathrm{D}^n f(x)(y+z, \ldots, y+z)\\
&-(f(x+y+z) - \sum_{n=0,\ldots,\nu} \mathrm{D}^n f(x+y)(z, \ldots, z)\\
=&\sum_{n=0,\ldots,\nu} \mathrm{D}^n f(x)(y+z, \ldots, y+z) - \mathrm{D}^n f(x+y)(z, \ldots, z)).
\end{aligned}$$

Fix $x = x_0$ and abbreviate $\Gamma^n = \mathrm{D}^n f(x_0)$. Recall that Γ^n is a multilinear map in n variables. Fix $y = y_0$. Then the map $\Gamma^n(y_0 + \cdot, \ldots, y_0 + \cdot)$ of n arguments is by multilinearity of Γ^n a sum of multilinear maps of m arguments for $m = 0, \ldots, n$. So

$$\Gamma^n(y_0 + \cdot, \ldots, y_0 + \cdot) = \sum_{m=0,\ldots,n} \binom{n}{m} \Gamma^n(\underbrace{y_0, \ldots, y_0}_{m\text{-times}}, \cdot, \ldots, \cdot).$$

We compute

$$\begin{aligned}
&\sum_{n=0,\ldots,\nu} \Gamma^n(y_0 + \cdot, \ldots, y_0 + \cdot)\\
=&\sum_{i,j \text{ with } i+j=0,\ldots,\nu} \binom{i+j}{i} \Gamma^{i+j}(\underbrace{y_0, \ldots, y_0}_{i\text{-times}}, \cdot, \ldots, \cdot)\\
=&\sum_{j=0,\ldots,\nu} \sum_{i=0,\ldots,\nu-j} \binom{i+j}{i} \Gamma^{i+j}(\underbrace{y_0, \ldots, y_0}_{i\text{-times}}, \cdot, \ldots, \cdot) = \sum_{j=0,\ldots,\nu} \Gamma^j_{y_0}
\end{aligned}$$

with $\Gamma^j_{y_0} = \sum_{i=0,\ldots,\nu-j} \binom{i+j}{i} \Gamma^{i+j}(\underbrace{y_0,\ldots,y_0}_{i\text{-times}},\cdot,\ldots,\cdot)$ a multilinear map in j arguments.

Together,

$$\begin{aligned}
&\mathrm{R}^\nu f(x_0+y_0+\cdot,x_0) - \mathrm{R}^\nu f(x_0+y_0+\cdot,x_0+y_0)\\
=&\sum_{n=0,\ldots,\nu}\sum_{i=0,\ldots,\nu-n}\binom{i+n}{n}\mathrm{D}^{i+n}f(x_0)(\underbrace{y_0,\ldots,y_0}_{i\text{-times}},\cdot,\ldots,\cdot) - \mathrm{D}^n f(x_0+y_0)(\cdot,\ldots,\cdot)\\
=&\sum_{n=0,\ldots,\nu}\Theta^n(x_0,y_0)
\end{aligned}$$

with $\Theta^n(x_0,y_0)$ for $n=0,\ldots,\nu$ the multilinear map of n arguments given by

$$\Theta^n(x_0,y_0) = \sum_{i=0,\ldots,\nu-n}\binom{i+n}{n}\mathrm{D}^{i+n}f(x_0)(\underbrace{y_0,\ldots,y_0}_{i\text{-times}},\cdot,\ldots,\cdot) - \mathrm{D}^n f(x_0+y_0)(\cdot,\ldots,\cdot).$$

Denote its restriction onto the diagonal by $\theta^n(x_0,y_0)$ and put

$$\theta(x_0,y_0) = \theta^\nu(x_0,y_0) + \cdots + \theta^0(x_0,y_0).$$

By Corollary 2.3, if $\mathrm{C}(n) = 1/|\nu!(\nu-1)!\cdots n!| > 0$ (well defined by our hypothesis on char $\mathbf{K}$) then for every δ in $|\mathbf{K}^*|$,

$$\begin{aligned}
&\|\mathrm{D}^n f(x_0+y_0)(\cdot,\ldots,\cdot) - \sum_{i=0,\ldots,\nu-n}\binom{i+n}{n}\mathrm{D}^{i+n}f(x_0)(\underbrace{y_0,\ldots,y_0}_{i\text{-times}},\cdot,\ldots,\cdot)\|\\
=&\|\Theta^n(x_0,y_0)\|\\
\leq&\mathrm{C}(n)\delta^{-n}\|\theta(x_0,y_0)\|_\delta\\
=&\mathrm{C}(n)\delta^{-n}\|\mathrm{R}^\nu f(x_0+y_0+\cdot,x_0) - \mathrm{R}^\nu f(x_0+y_0+\cdot,x_0+y_0)\|_\delta\\
\leq&\mathrm{C}(n)\delta^{-n}\max\{\|\mathrm{R}^\nu f(x_0+y_0+\cdot,x_0)\|_\delta, \|\mathrm{R}^\nu f(x_0+y_0+\cdot,x_0+y_0)\|_\delta\}.
\end{aligned}$$

We ultimately want to prove that $\mathrm{D}^n f \in \mathscr{C}^{r-n}_{\mathrm{T}}(\mathrm{X},\mathrm{Sym}^n(\mathrm{V},\mathbf{E}))$. Let $a \in \mathrm{X}$ and $\varepsilon > 0$. Let $x+y, x$ in X and put $\delta := \|y\|$. Then

$$\begin{aligned}
&\|\mathrm{D}^n f(x+y)(\cdot,\ldots,\cdot) - \sum_{i=0,\ldots,\nu-n}\binom{i+n}{n}\mathrm{D}^{i+n}f(x)(\underbrace{y,\ldots,y}_{i\text{-times}},\cdot,\ldots,\cdot)\|\\
\leq&\mathrm{C}(n)\delta^{-n}\max\{\|\mathrm{R}^\nu f(x+y+\cdot,x)\|_\delta, \|\mathrm{R}^\nu f(x+y+\cdot,x+y)\|_\delta\}.
\end{aligned}$$

Since f is a $\mathscr{C}^r_{\mathrm{T}}$-function, there is an open ball U around a inside X such that $\|\mathrm{R}^\nu f(x+y,x)\| \leq \varepsilon\|y\|^r$ for all $x+y,x \in \mathrm{U}$. The ball of radius equal to δ around $x+y$ is by the strong triangle inequality included in U. Thence

$$\|\mathrm{R}^\nu f(x+y+\cdot,x)\|_\delta, \|\mathrm{R}^\nu f(x+y+\cdot,x+y)\|_\delta \leq \varepsilon\|y\|^r.$$

By the canonical identification of $\mathrm{Sym}^{i+n}(\mathrm{V},\mathbf{E})$ with $\mathrm{Sym}^i(\mathrm{V},\mathrm{Sym}^n(\mathrm{V},\mathbf{E}))$ via $\mathrm{M} \mapsto [\boldsymbol{v} \mapsto \mathrm{M}_{\boldsymbol{v}} = \mathrm{M}(\boldsymbol{v},\cdot)]$, we find $\mathrm{D}^{i+n}f(x)$ to be a multilinear map in i arguments with values in $\mathrm{Sym}^n(\mathrm{V},\mathbf{E})$. We conclude

$$\begin{aligned}&\|\mathrm{D}^n f(x+y) - \sum_{i=0,\dots,\nu-n} \binom{i+n}{n} \mathrm{D}^{i+n} f(x)(\underbrace{y,\dots,y}_{i\text{-times}})\| \\ \leq{}& \mathrm{C}(n)\delta^{-n}\varepsilon\|y\|^r = \mathrm{C}(n)\varepsilon\|y\|^{r-n}.\end{aligned}$$

That is, $\mathrm{D}^n f\colon \mathrm{X} \to \mathrm{Sym}^n(\mathrm{V},\mathbf{E})$ is a $\mathscr{C}^{r-n}_{\mathrm{T}}$-function with respect to the functions $\mathrm{D}^i(\mathrm{D}^n f)\colon \mathrm{X} \to \mathrm{Sym}^i(\mathrm{Sym}^n(\mathrm{V},\mathbf{E}),\mathbf{E})$ given by $\mathrm{D}^i(\mathrm{D}^n f) = \binom{i+n}{n}\mathrm{D}^{i+n}f(x)$ for $i = 0,\dots,\nu-n$. □

Corollary 2.5. *Under the premises of Proposition 2.4 the functions* $\mathrm{D}^0 f, \dots, \mathrm{D}^\nu f$ *are uniquely determined and continuous.*

Proof: By Proposition 2.4 the functions $\mathrm{D}^0 f,\dots,\mathrm{D}^\nu f$ are in particular continuous. Therefore f is differentiable (by the convergence of its Taylor polynomial expansion) and its derivative $\mathrm{D}^1 f$ is uniquely determined. Because $\mathrm{D}^1 f$ is a $\mathscr{C}^{r-1}_{\mathrm{T}}$-function, by the same reasoning its derivative $\mathrm{D}^2 f$ is uniquely determined, and iteratively every function $\mathrm{D}^2 f, \dots, \mathrm{D}^\nu f$ is uniquely determined. □

The norm

Let $\mathscr{C}^r_{\mathrm{T}}(\mathrm{X},\mathbf{E})$ be the $\mathbf{K}$-vector space of all $\mathscr{C}^r_{\mathrm{T}}$ functions $f\colon \mathrm{X} \to \mathbf{E}$. By Corollary 2.5 the functions $\mathrm{D}^0 f, \mathrm{D}^1 f, \dots, \mathrm{D}^\nu f$ are uniquely determined and differentiable of degree r, $r-1$, …, ρ. Hence

1. in particular, the functions $\mathrm{D}^0 f, \mathrm{D}^1 f, \dots, \mathrm{D}^\nu f$ are continuous, and
2. the remainder $\mathrm{R}^\nu f$ of its Taylor polynomial expansion converges as in Definition 2.1 if and only if the function

   $$(x,y) \mapsto \|\mathrm{R}^\nu f(x,y)\|/\|x-y\|^r \quad \text{for distinct } x,y \in \mathrm{X},$$

   extends to a continuous function $|\Delta^r f|\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}_{\geq 0}$ that vanishes on the diagonal.

We conclude that the following norm on $\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ is well-defined.

Definition 2.6. Let X be a compact accumulated subset of V. The norm $\|\cdot\|_{\mathscr{C}^r_{\mathrm{T}}}$ on $\mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ is defined by $\|f\|_{\mathscr{C}^r_{\mathrm{T}}} := \max\{\|\mathrm{D}^0 f\|_{\sup}, \ldots, \|\mathrm{D}^\nu f\|\} \cup \{\| \, |\Delta^r f| \, \|_{\sup}\}$.

3. Comparison

In this Section 3 we compare differentiability via divided differences in Section 1 with that via the Taylor polynomial in Section 2 on an open subset. We show differentiability via divided differences implies that via the Taylor polynomial, and the inverse holds, that is, both differentiability conditions are equivalent, for functions of one variable and of many variables in $\mathbb{Q}_p$.

Necessity

Let in this subsection either $\operatorname{char} \mathbf{K} > \nu$ or equal to 0 and let X be an open subset of V. For a tuple $\boldsymbol{p} = (p_1, \ldots, p_d)$ of entries in $\mathbb{N}$ and $\boldsymbol{h} = (h_1, \ldots, h_d)$ of entries in $\mathbf{K}$ denote $\boldsymbol{h}^{\boldsymbol{p}} = h_1^{p_1} \cdots h_d^{p_d}$.

Lemma 3.1. *Let $f \colon \mathrm{X} \to \mathbf{E}$ be a $\mathscr{C}^r$-function. Let $x_0 \in \mathrm{X}$ such that the ball of radius ϵ around x_0 is included in X. Let h in V such that $\|v\| \leq \epsilon$ and for t in $\mathbf{K}$ with $|t| \leq 1$ put $\mathrm{F}(t) = f(x_0 + th)$. Then for every $i = 0, \ldots, \nu$,*

$$\mathrm{D}^i \mathrm{F}(t) = \sum_{\boldsymbol{p} \text{ in } \mathbb{N}^d \text{ with } p_1 + \cdots + p_d = i} \mathrm{D}^{\boldsymbol{p}} f(x_0 + th) \cdot \boldsymbol{h}^{\boldsymbol{p}}. \tag{$*$}$$

Proof: For a natural number i, let $\mathrm{F}^{(i)}$ be the i-th Archimedean derivative of F, and for a tuple $\boldsymbol{p} = (p_1, \ldots, p_d)$ of natural numbers, let $f^{(\boldsymbol{p})}$ be the Archimedean derivative of f that is taken p_1-times along the first, …, p_d-times along the last coordinate.

By [Sch84, Theorem 29.5] and by its multivariate version $i!\,\mathrm{D}^i\mathrm{F} = \mathrm{F}^{(i)}$ and $\boldsymbol{p}!\,\mathrm{D}^{\boldsymbol{p}} = f^{(\boldsymbol{p})}$. By our assumption on $\operatorname{char} \mathbf{K}$ the factors $i!$ and $p_1! \cdots p_d!$ invert and we can replace the non-Archimedean by the Archimedean derivatives. Let $\mathrm{I} = \{x \in \mathbf{K} : |x| \leq 1\}$ and define $\mathrm{T} \colon \mathrm{I} \to \mathrm{X}$ by $\mathrm{T}(t) = x_0 + t \cdot h$. We obtain Equation $(*)$ by applying the Archimedean chain rule to $\mathrm{F} = f \circ \mathrm{T}$. □

Let $\mathrm{V}_1, \ldots, \mathrm{V}_n$ be vector spaces. The normed $\mathbf{K}$-vector space

$$\mathrm{Hom}(\mathrm{V}_1, \mathrm{Hom}(\mathrm{V}_2, \ldots, \mathrm{Hom}(\mathrm{V}_n, \mathbf{E})) \cdots)$$

is canonically isomorphic to

$$\mathrm{Mult}_{\mathbf{K}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_n, \mathbf{E}) := \{ \text{ all } \mathbf{K}\text{-multilinear maps } m \colon \mathrm{V}_1 \times \cdots \times \mathrm{V}_n \to \mathbf{E}\}.$$

We identify $\mathbf{E}^{[\nu]}$ with $\mathrm{Mult}(\mathrm{V}^{[\nu-1]} \times \cdots \times \mathrm{V}, \mathbf{E})$ by this isomorphism.

Lemma 3.2. *Let* x_0 *in* X. *Let* h *in* V, *let* $\mathrm{I} = \{t \in \mathbf{K} : |t| \leq 1\}$ *and, for* t *in* I, *put* $\mathrm{T}(t) = x_0 + th$. *Let* $f \colon \mathrm{X} \to \mathbf{E}$ *and assume that* T *takes values in* X. *If* f *is* r*-times differentiable then* $f \circ \mathrm{T}$ *is* r*-times differentiable, so for every* t *in* $\mathrm{I}^{[\nu]}$ *and* λ *in* $\mathbf{K}^{[\nu-1]} \times \cdots \times \mathbf{K}^{[1]} \times \mathbf{K}$,

$$\left[(f \circ \mathrm{T})^{[\nu]}(t)\right](\lambda) = \left[f^{[\nu]}(\mathbf{T}(t))\right](\lambda \cdot \boldsymbol{h}) \tag{3.1}$$

where

- *the function* $\mathbf{T} \colon \mathrm{I}^{[\nu]} \to \mathrm{X}^{[\nu]}$ *is understood as* T *if* $\nu = 0$ *and otherwise inductively for* (t, t) *in* $\mathrm{I}^{[n+1]} = \mathrm{I}^{[n]} \times \mathrm{I}^{[n]}$ *as* $(\mathrm{T}(t), \mathrm{T}(t))$, *and*
- *the scalar product* $\lambda \cdot \boldsymbol{h}$ *for* $\lambda \in \mathbf{K}^{[\nu-1]} \times \cdots \times \mathbf{K}^{[1]} \times \mathbf{K}$ *is understood as* $\lambda \cdot h$ *if* $\nu = 1$ *and otherwise inductively for* (λ, μ) *in* $\mathbf{K}^{[n+1]} = \mathbf{K}^{[n]} \times \mathbf{K}^{[n]}$ *as* $(\lambda \cdot \boldsymbol{h}, \mu \cdot \boldsymbol{h})$.

Proof: If f is r-times differentiable then, because T is linear, in particular r-times differentiable, their composition $f \circ \mathrm{T}$ is by [Nag11, Proposition 3.23] likewise r-times differentiable.

If $\nu = 0$ then Equation (3.1) holds. For $\nu > 0$, by definition of the iterated divided difference and the chain rule for onefold divided differences,

$$\begin{aligned}(f \circ \mathrm{T})^{[\nu+1]} &= \left((f \circ \mathrm{T})^{[1]}\right)^{[\nu]} \\ &= \left((f^{[1]} \circ (\mathrm{T}, \mathrm{T})) \cdot \mathrm{T}^{[1]}\right)^{[\nu]} = \left((f^{[1]} \circ (\mathrm{T}, \mathrm{T})) \cdot (\cdot h)\right)^{[\nu]}.\end{aligned}$$

By induction

$$\begin{aligned}\left[\left((f^{[1]} \circ (\mathrm{T}, \mathrm{T})) \cdot (\cdot h)\right)^{[\nu]}(t)\right](\lambda) &= \left[\left(f^{[1]} \circ (\mathrm{T}, \mathrm{T})\right)^{[\nu]}(t)\right](\lambda, \cdot h) \\ &= \left(f^{[1]\,[\nu]}(\mathbf{T}(t))\right)(\lambda \cdot \boldsymbol{h}, \cdot h) \\ &= \left(f^{[\nu+1]}(\mathbf{T}(t))\right)(\lambda, \cdot) \cdot (\boldsymbol{h}, h)\end{aligned}$$

where the constant functions $\cdot h$ respectively $\cdot \boldsymbol{h}$ send

- every t in $\mathrm{I}^{[1]}$ to the map $\mathbf{K} \to \mathrm{V}$ given by scalar multiplication $\lambda \mapsto \lambda \cdot h$, respectively
- every t in $\mathrm{I}^{[\nu]}$ to the map

$$\mathbf{K}^{[\nu-1]} \times \cdots \times \mathbf{K}^{[1]} \to \mathrm{V}^{[\nu-1]} \times \cdots \times \mathrm{V}^{[1]}$$

given by component-wise scalar multiplication

$$(\lambda_{\nu-1}, \ldots, \lambda_1) \mapsto (\lambda_{\nu-1} \cdot h, \ldots, \lambda_1 \cdot h).$$

The function $(\boldsymbol{h}, h)$ is their tensor product on $(\mathbf{K}^{[\nu-1]} \times \cdots \times \mathbf{K}^{[1]}) \times \mathbf{K}$. □

Corollary 3.3. *Let us keep the assumptions and notation of Lemma 3.2 above. Then, if* X *is moreover compact,*

$$\|(f \circ \mathrm{T})^{[\nu]}\|_{\mathscr{C}^\rho} = \|f^{[\nu]}_{|\operatorname{im} \mathrm{T}}\|_{\mathscr{C}^\rho} / \|h\|^r.$$

Proof: Because $\|\cdot \boldsymbol{h}\| = \|h\|^\nu$, Equation $(*)$ follows from Equation (3.1). □

Lemma 3.4. *Let* $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$. *Let* $\mathrm{I} = \{x \in \mathbf{K} : |x| \leq 1\}$. *For* $x + h, x \in \mathrm{X}$ *define* $\mathrm{T}_{x,h} \colon \mathrm{I} \to \mathrm{X}$ *by* $\mathrm{T}_{x,h}(t) = x + th$. *For every* $a \in \mathrm{X}$ *and* $\varepsilon > 0$, *there is a neighborhood* U *around* a *inside* X *such that for all* $x + h, x$ *in* U,

$$\|(f \circ \mathrm{T}_{x,h})^{<\nu>}\|_{\mathscr{C}^\rho} \leq \varepsilon \|h\|^r.$$

Proof: By Proposition A.2,

$$\|(f \circ \mathrm{T}_{x,h})^{<\nu>}\|_{\mathscr{C}^\rho} = \|(f \circ \mathrm{T}_{x,h})^{[\nu]}\|_{\mathscr{C}^\rho}$$

By assumption $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$, that is, $f^{[\nu]} \in \mathscr{C}^\rho(\mathrm{X}^{[\nu]}, \mathbf{E}^{[\nu]})$. That is, for every $a \in \mathrm{X}$ and $\varepsilon > 0$ there is a neighborhood U around a inside X such that for all $x + h, x \in \mathrm{U}^{[\nu]}$,

$$\|f^{[\nu]}(x + h) - f^{[\nu]}(x)\| \leq \varepsilon \|h\|^\rho.$$

If $x + h, x$ in U then $\operatorname{im} \mathrm{T}$ is included in $\mathrm{U}^{[\nu]}$. Thus, by Corollary 3.3,

$$\|(f \circ \mathrm{T}_{x,h})^{[\nu]}\|_{\mathscr{C}^\rho} \leq \varepsilon \|h\|^r.$$

Proposition 3.5. *Let* $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$. *There are functions* $\mathrm{D}^{\boldsymbol{p}} f \colon \mathrm{X} \to \mathbf{E}$ *for* $\boldsymbol{p}$ *in* $\mathbb{N}^d$ *with* $p_1 + \cdots + p_d \leq r$ *and* $\mathrm{R}^v f \colon \mathrm{X} \times \mathrm{X} \to \mathbf{E}$ *such that*

$$f(x + h) = \sum_{\boldsymbol{p} \text{ in } \mathbb{N}^d \text{ with } p_1 + \cdots + p_d = i} \mathrm{D}^{\boldsymbol{p}} f(x_0 + th) \cdot \boldsymbol{h}^{\boldsymbol{p}} + \mathrm{R}^\nu f(x + h, x)$$

and for every $a \in \mathrm{X}$ *and* $\varepsilon > 0$ *there is a neighborhood* $\mathrm{U} \ni a$ *in* X *such that*

$$\|\mathrm{R}^\nu f(x + h, x)\| \leq \varepsilon \|h\|^r \quad \textit{for all } x + h, x \in \mathrm{U}$$

Proof: Define $T = T_{x,h} \colon I \to V$ by $T(t) = x + th$ for all t in $I = \{t \in \mathbf{K} : |t| \leq 1\}$. Define $F = F_{x,h} \colon I \to \mathbf{E}$ by $F = f \circ T$. Then there is, by [Nag11, Corollary 2.14], for all $t \in I$ a Taylor-polynomial expansion

$$t^\nu (F^{<\nu>}(t, 0, \ldots, 0) - F^{<\nu>}(0, \ldots, 0)) = F(t) - \sum_{i=0,\ldots,\nu} D^i F(0) t^i$$

for $\mathscr{C}^{r-i}$-functions $D^i F \colon X \to \mathbf{E}$ for $i = 0, \ldots, \nu$ and a $\mathscr{C}^\rho$-function $F^{<\nu>} \colon X^{<\nu>} \to \mathbf{E}$. By Lemma 3.1, for $t = 1$,

$$\begin{aligned} & F^{<\nu>}(1, 0, \ldots, 0) - F^{<\nu>}(0, 0, \ldots, 0) \\ = & F(1) - \sum_{i=0,\ldots,\nu} D^i F(0) \\ = & f(x+h) - \sum_{\boldsymbol{p} \text{ in } \mathbb{N}^d \text{ with } p_1+\cdots+p_d=i} D^{\boldsymbol{p}} f(x_0 + th) \cdot \boldsymbol{h}^{\boldsymbol{p}}. \end{aligned}$$

For every $\varepsilon > 0$ and $a \in X$, there is by Lemma 3.4 a neighborhood $U \ni a$ such that for all $x + h, x \in U$ in particular

$$\|F_{x,h}^{<\nu>}(1, 0, \ldots, 0) - F_{x,h}^{<\nu>}(0, \ldots, 0)\| \leq \varepsilon \|h\|^r.$$

We conclude

$$\|f(x+h) - \sum_{\boldsymbol{p} \text{ in } \mathbb{N}^d \text{ with } p_1+\cdots+p_d=i} D^{\boldsymbol{p}} f(x_0 + th) \cdot \boldsymbol{h}^{\boldsymbol{p}}\| \leq \varepsilon \|h\|^r.$$

Corollary 3.6. *We have* $\mathscr{C}^r(X, \mathbf{E}) \subseteq \mathscr{C}^r_T(X, \mathbf{E})$. *If* X *is a compact open subset of* V *then the inclusion* $\mathscr{C}^r(X, \mathbf{E}) \hookrightarrow \mathscr{C}^r_T(X, \mathbf{E})$ *is a monomorphism of normed vector spaces.*

Proof: By Proposition 3.5, the inclusion holds with, in the notation of Definition 2.1,

$$D^n f(x)(h, \ldots, h) := \sum_{\boldsymbol{p} \text{ in } \mathbb{N}^d \text{ with } p_1+\cdots+p_d=i} D^{\boldsymbol{p}} f(x_0 + th) \cdot \boldsymbol{h}^{\boldsymbol{p}}.$$

Let X be a compact accumulated subset of V. Given f in $\mathscr{C}^r(X, \mathbf{E})$, we show $\|f\|_{\mathscr{C}^r_T} \leq \|f\|_{\mathscr{C}^r}$, that is,

(i) $\max\{\|D^0 f\|_{\sup}, \ldots, \|D^\nu f\|\} \leq \|f\|_{\mathscr{C}^r}$ and (ii) $\| |\Delta^r f| \|_{\sup} \leq \|f\|_{\mathscr{C}^r}$.

(i) : Let n in $\{0,\dots,\nu\}$. By definition $\|D^n f\|_{\sup} = \max\{\|D^p f\|_{\sup} : p_1 + \cdots + p_d = n\}$ and $\|D^p f\|_{\sup} \leq \|f^{<p>}\|_{\sup} \leq \|f^{<n>}\|_{\sup}$. By Proposition A.2, we find $\|f^{<n>}\|_{\sup} = \|f^{[n]}\|_{\sup}$ and $\|f^{<\nu>}\|_{\mathscr{C}^\rho} = \|f^{[\nu]}\|_{\mathscr{C}^\rho}$. We conclude $\|D^n f\|_{\sup} \leq \|f\|_{\mathscr{C}^r}$.

(ii) : Let $x+h, x$ in X. For t in $\{x \in \mathbf{K} : |t| \leq 1\}$ let $T(t) = x + th$. Put $F = f \circ T$. By Proposition 3.5,

$$R^\nu f(x+h,x) = F^{<\nu>}(1,0,\dots,0) - F^{<\nu>}(0,0,\dots,0).$$

By Proposition A.2, we find $\|F^{<\nu>}\|_{\mathscr{C}^\rho} = \|F^{[\nu]}\|_{\mathscr{C}^\rho}$. By Corollary 3.3, we conclude

$$\begin{aligned}|\Delta^r f|(x+h,x) &= \|F^{<\nu>}(1,0,\dots,0) - F^{<\nu>}(0,0,\dots,0)\|/\|h\|^r \\ &\leq \|F^{<\nu>}\|_{\mathscr{C}^\rho}/\|h\|^r = \|F^{[\nu]}\|_{\mathscr{C}^\rho}/\|h\|^r = \|f^{[\nu]}_{|\operatorname{im} T}\|_{\mathscr{C}^\rho}. \qquad \square\end{aligned}$$

Sufficiency for functions of one variable

In one variable, every $\mathscr{C}^r_T$-function is also a $\mathscr{C}^r$-function. The passage from many variables to one simplifies the definition of a $\mathscr{C}^r_T$-function.

Definition 3.7. Let X be an open subset of $\mathbf{K}$. A function $f\colon X \to \mathbf{K}$ is a $\mathscr{C}^r_T$*-function* if there are functions $D^0 f,\dots,D^\nu f\colon X \to \mathbf{K}$ and $R^\nu f\colon X \times X \to \mathbf{K}$ such that

$$f(x+h) = \sum_{i=0,\dots,\nu} D^i f(x) h^i + R^\nu f(x+h,x) \quad \text{for all } x+h, x \text{ in X},$$

and for every $a \in X$ and $\varepsilon > 0$ there is a neighborhood U around a inside X such that

$$|R^\nu f(x+h,x)| \leq \varepsilon |h|^r \quad \text{for all } x+h, x \text{ in U}.$$

Let f be a $\mathscr{C}^r_T$-function and let us keep the notations of Definition 3.7. By Corollary 2.5 the functions $D^0 f$, $D^1 f$, …, $D^\nu f\colon X \to \mathbf{K}$ and the function $|\Delta^r f|\colon \nabla X \times X \to \mathbb{R}_{\geq 0}$ defined by

$$|\Delta^r f|(x,y) := |R^\nu f(x,y)|/|x-y|^r$$

are uniquely determined by f. The functions $D^0 f, D^1 f, \dots, D^\nu f$ are continuous and $|\Delta^r f|$ extends to a continuous function on $X \times X$ that vanishes on the diagonal of $X \times X$. Hence the following definition is meaningful:

Definition. Let C be a compact open subset of $\mathbf{K}$. The norm $\|\cdot\|_{\mathscr{C}^r_{\mathrm{T}}}$ on $\mathscr{C}^r_{\mathrm{T}}(\mathrm{C},\mathbf{E})$ is defined by $\|f\|_{\mathscr{C}^r_{\mathrm{T}}} := \max\{\|\mathrm{D}^0 f\|_{\sup},\ldots,\|\mathrm{D}^\nu f\|_{\sup},\| \, |\Delta^r f| \, \|_{\sup}\}$.

Theorem 3.8 ([Nag11, Corollary 2.32]). *If* X *is an open subset of* $\mathbf{K}$ *then* $\mathscr{C}^r(\mathrm{X},\mathbf{K}) = \mathscr{C}^r_{\mathrm{T}}(\mathrm{X},\mathbf{K})$ *as sets and if* X *is also compact then the identity map between sets* $\mathrm{id}\colon \mathscr{C}^r(\mathrm{X},\mathbf{K}) \to \mathscr{C}^r_{\mathrm{T}}(\mathrm{X},\mathbf{K})$ *is an isomorphism of locally convex* $\mathbf{K}$*-algebras.*

Sufficiency for functions of many variables in $\mathbb{Q}_p$

Partial Taylor polynomial. Let $k \in \{1,\ldots,d\}$. The following definition of a $\mathscr{C}^{r\cdot e_k}_{\mathrm{T}}$-function is that given in Definition 2.1 where now the vector h in V has a single nonzero entry in the k-th coordinate.

Definition (**2.1'**). Let X be an open subset of $\mathbf{K}^d$. The function $f\colon \mathrm{X} \to \mathbf{E}$ is a $\mathscr{C}^{r\cdot e_k}_{\mathrm{T}}$*-function* if there are continuous functions $\mathrm{D}^0 f$, $\mathrm{D}^{1\cdot e_k} f$, ..., $\mathrm{D}^{\nu\cdot e_k} f\colon \mathrm{X} \to \mathbf{E}$ and $\mathrm{R}^{\nu\cdot e_k} f\colon \mathrm{X}^{[e_k]} \to \mathbf{E}$ on $\mathrm{X}^{[e_k]} := \{(x;t) \in \mathrm{X}\times\mathbf{K} : x + t\cdot e_k \in \mathrm{X}\}$ such that

$$f(x+t\cdot e_k) = \sum_{i=0,\ldots,\nu} \mathrm{D}^{i\cdot e_k} f(x) t^i + \mathrm{R}^{\nu\cdot e_k} f(x;t)$$

and for every a in X and every $\varepsilon > 0$, there is a neighborhood U inside $\mathrm{X}^{[e_k]}$ around a such that

$$|\mathrm{R}^{\nu\cdot e_k} f(x;t)| \leq \varepsilon |t|^r \quad \text{for all } x+t\cdot e_k, x \text{ in U}.$$

Fix a coordinate index $k \in \{1,\ldots,d\}$. Let $f\colon \mathrm{X} \to \mathbf{E}$ be a $\mathscr{C}^{r\cdot e_k}_{\mathrm{T}}$-function and let us keep the notation of Definition 2.1'. The function $|\Delta^{r\cdot e_k} f|$ is defined by

$$(x;t) \mapsto |\mathrm{R}^{\nu\cdot e_k} f(x;t)|/|t|^r \quad \text{for all } (x;t) \text{ in } \mathrm{X}^{[e_k]} \text{ with } t \text{ in } \mathbf{K}^*$$

Continuity of $\mathrm{D}^0 f, \mathrm{D}^{1\cdot e_k} f, \ldots, \mathrm{D}^{\nu\cdot e_k} f\colon \mathrm{X} \to \mathbf{K}$ implies that of $|\Delta|^{r\cdot e_k} f$ on $\mathrm{X}^{]e_k[}$; because f is a $\mathscr{C}^{r\cdot e_k}_{\mathrm{T}}$-function, $|\Delta^{r\cdot e_k} f|$ extends to a continuous function on $\mathrm{X}^{[e_k]}$, denoted likewise, that vanishes if t vanishes.

The functions $\mathrm{D}^{0\cdot e_k}, \ldots, \mathrm{D}^{\nu\cdot e_k}$ (and thus $|\Delta^{r\cdot e_k} f|$) are uniquely determined by f by [Nag11, Lemma 3.52].

Definition. Let C be a compact open subset of $\mathbb{Q}_p^d$. The norm $\|\cdot\|_{\mathscr{C}^{r\cdot e_k}_{\mathrm{T}}}$ on $\mathscr{C}^{r\cdot e_k}(\mathrm{C},\mathbf{E})$ is defined by

$$\|f\|_{\mathscr{C}^{r\cdot e_k}_{\mathrm{T}},\mathrm{C}} := \max\{\|\mathrm{D}^0 f\|_{\sup}, \|\mathrm{D}^{1\cdot e_k} f\|_{\sup}, \ldots, \|\mathrm{D}^{\nu\cdot e_k} f\|_{\sup}\} \cup \{\|\Delta^{r\cdot e_k} f\|_{\sup}\}.$$

The following definition of a $\mathscr{C}^r_{\mathbf{T}}$-function is that of a $\mathscr{C}^r_{\mathrm{T}}$-function where, in the notation of Definition 2.1, the vector h in V has a single nonzero entry.

Definition. Let X be an open subset of $\mathbb{Q}_p^d$. The normed $\mathbf{K}$-vector space $\mathscr{C}^r_{\mathbf{T}}(\mathrm{X}, \mathbf{E})$ is the initial normed $\mathbf{K}$-vector space of $\mathscr{C}^{r,0,\ldots,0}_{\mathrm{T}}(\mathrm{X}, \mathbf{E}), \ldots, \mathscr{C}^{0,\ldots,0,r}_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$. That is,

- $\mathscr{C}^r_{\mathbf{T}}(\mathrm{X}, \mathbf{E}) := \mathscr{C}^{r,0,\ldots,0}_{\mathrm{T}}(\mathrm{X}, \mathbf{E}) \cap \ldots \cap \mathscr{C}^{0,\ldots,0,r}_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ as $\mathbf{K}$-vector space,
- and its norm is given by $\|\cdot\|_{\mathscr{C}^r_{\mathbf{T}}} := \max\{\|\cdot\|_{\mathscr{C}^{r,0,\ldots,0}_{\mathrm{T}}}, \ldots, \|\cdot\|_{\mathscr{C}^{0,\ldots,0,r}_{\mathrm{T}}}\}$.

Theorem 3.9 ([Nag11, Corollary 3.60])**.** *Let* X *be an open subset of* $\mathbb{Q}_p^d$*. The inclusion map* $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathbf{T}}(\mathrm{X}, \mathbf{E})$ *is an isomorphism of locally convex* $\mathbf{K}$*-vector spaces.*

This isomorphism results from the characterization of r-fold differentiability of a function by its *Mahler coefficients*, the coefficients of its expansion with respect to the *Mahler basis*, a canonical orthonormal basis of all continuous functions on $\mathbb{Z}_p$ (that is dual to the Iwasawa isomorphism). See also [Glö13, Corollary 9.5] for a variant of this result.

Total Taylor polynomial. Corollary 3.10 directly implies that r-fold differentiability via divided differences, as defined in Section 1, is equivalent to r-fold differentiability via the total Taylor polynomial, as defined in Section 2.

Corollary 3.10. *Let* X *be an open subset of* $\mathbb{Q}_p^d$*. The inclusion map* $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ *is an isomorphism of locally convex* $\mathbf{K}$*-vector spaces.*

Proof: Because a $\mathscr{C}^{r \cdot e_1}$-function is the special case of a $\mathscr{C}^r_{\mathrm{T}}$-function where, in the notation of Definition 2.1, the vector h in $\mathbf{K}^d$ has a single nonzero entry, we have the continuous inclusions of locally convex $\mathbf{K}$-vector spaces

$$\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathbf{T}}(\mathrm{X}, \mathbf{E}).$$

Their composition is by Theorem 3.9 an isomorphism and thus the first inclusion $\mathscr{C}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathrm{T}}(\mathrm{X}, \mathbf{E})$ too. □

4. Fractional Differentiability from Representation Theory

We introduce de Ieso's differentiability definition on a function f on $\mathbf{o}_{\mathbf{K}}$, then motivate its origins from representation theory. Finally we show that de Ieso's differentiability definition puts a convergence condition on the $\mathbb{Q}_p$-algebraic Taylor polynomial expansion of f and that this condition is equivalent to that of iterated differentiability via divided differences on $\mathbf{o_K}$ as open subset of the $\mathbb{Q}_p$-vector space $\mathbf{K}$.

Definition

Let $r \geq 0$ be a real number. There is another notion of r-fold differentiability for functions on finite extensions $\mathbf{o}$ of $\mathbb{Z}_p$ via Taylor polynomials by [De 13a]. It generalizes that initially introduced by Colmez in [Col10] on the domain $\mathbb{Z}_p$ (and which makes in turn reference to the findings in [Sch78]).

The original definition by Colmez and de Ieso. We adopt the notations of [De 13a]. Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathbf{o}$, and $\mathbf{E}$ be a finite field containing the normal closure of $\mathbf{K}$.

We denote by S the finite set of ring homomorphisms from $\mathbf{K}$ into $\mathbf{E}$. We introduce the following standard multi-index notations over S. Let $\boldsymbol{n} \in \mathbb{N}^{\mathrm{S}}$.

- We put $n = \sum_{s \in \mathrm{S}} n_s$, and
- given $z \in \mathbf{o}$, we put $z^{\boldsymbol{n}} = \prod_{s \in \mathrm{S}} s(z)^{n_s}$.

Definition 4.1 ([De 13a, Déf. 2.1]). The function $f : \mathbf{o} \to \mathbf{E}$ is a *$\mathscr{C}^r_{CdI}$-function*, that is, an r-times differentiable function à la Colmez de Ieso, if there are bounded functions $\mathrm{D}^{\boldsymbol{i}} f : \mathbf{o} \to \mathbf{E}$ for $\boldsymbol{i} \in \mathbb{N}^{\mathrm{S}}$ with $i \leq r$ such that if we define $\mathrm{R}^{\nu} f : \mathbf{o} \times \mathbf{o} \to \mathbf{E}$ by

$$\mathrm{R}^{\nu} f(x,h) = f(x+h) - \sum_{\boldsymbol{i} \in \mathbb{N}^{\mathrm{S}} \text{ with } i \leq r} \mathrm{D}^{\boldsymbol{i}} f(x) h^{\boldsymbol{i}},$$

then

$$\Delta^r f(\delta) := \sup_{x_0 \in \mathbf{o}} \sup_{|h| \leq \delta} |\mathrm{R}^{\nu} f(x_0, h)| / \delta^r$$

is a well-defined function $\Delta^r f :]0,1] \to \mathbb{R}_{\geq 0}$ which converges to 0 as δ does.

De Ieso shows in [De 12, Section 2.2] that the functions $\mathrm{D}^{\boldsymbol{i}} f$ for $\boldsymbol{i}$ in $\mathbb{N}^{\mathrm{S}}$ are uniquely determined and in particular continuous. Thus $\mathrm{R}^{\nu} f : \mathbf{o} \times \mathbf{o} \to \mathbf{E}$ is uniquely determined and it is continuous: on the diagonal by the condition on $\mathrm{R}^{\nu} f$ in Definition 4.1 and elsewhere by the continuity of all $\mathrm{D}^{\boldsymbol{i}} f$ for $\boldsymbol{i}$ in $\mathbb{N}^{\mathrm{S}}$. Hence the following definition is meaningful.

Definition. The *norm* $\|\cdot\|_{\mathscr{C}^r_{\mathbf{CdI}}}$ on $\tilde{\mathscr{C}}^r_{\mathbf{CdI}}(\mathbf{o}, \mathbf{E})$ is defined by

$$\|f\|_{\mathscr{C}^r_{\mathbf{CdI}}} = \max\{\|\mathrm{D}^{\boldsymbol{i}} f\|_{\sup} \text{ for all } \boldsymbol{i} \text{ in } \mathbb{N}^{\mathrm{S}} \text{ with } i \leq r\} \cup \{\| \, |\Delta^r f| \, \|_{\sup}\}.$$

Compactness and uniform continuity. We give a variant of Definition 4.1 by a less uniform convergence condition on the Taylor polynomial expansion and show this convergence condition to be equivalent to that of Definition 4.1 by compactness of $\mathbf{o}$.

Definition 4.2. A function $f\colon \mathbf{o} \to \mathbf{E}$ is a $\mathscr{C}^r_{\mathrm{CdI}}$*-function* if there are functions $\mathrm{D}^{\boldsymbol{i}} f\colon \mathbf{o} \to \mathbf{E}$ for $\boldsymbol{i} \in \mathbb{N}^{\mathrm{S}}$ with $i \leq r$ such that if we define $\mathrm{R}_{\lfloor r \rfloor} f\colon \mathbf{o} \times \mathbf{o} \to \mathbf{E}$ by

$$\mathrm{R}^{\nu} f(x,y) = f(x) - \sum_{\boldsymbol{i} \in \mathbb{N}^{\mathrm{S}} \text{ with } i \leq r} \mathrm{D}^{\boldsymbol{i}} f(y)(x-y)^{\boldsymbol{i}},$$

then for every a included in $\mathbf{o}$ and $\varepsilon > 0$ there is a neighborhood U around a in $\mathbf{o}$ such that

$$|\mathrm{R}^{\nu} f(x,y)| \leq \varepsilon |x-y|^r \quad \text{for all } x,y \in \mathrm{U}.$$

Proposition 4.3 (Adaption of [Nag11, Proposition 2.33]). *Let $f\colon \mathbf{o} \to \mathbf{E}$ be a function. Then f is a $\mathscr{C}^r_{\boldsymbol{CdI}}$-function if and only if f is a $\mathscr{C}^r_{\mathrm{CdI}}$-function.*

Proof: Given a function $f\colon \mathbf{o} \to \mathbf{E}$, it will suffice to show that the conditions on the functions $\mathscr{D}^{\boldsymbol{i}} f$ in Definition 4.1 respectively Definition 4.2 are equivalent.

Recall by Definition 4.1 that $f \in \mathscr{C}^r_{\mathbf{CdI}}(\mathbf{o},\mathbf{E})$ if $\boldsymbol{\Delta}^r f(\delta) \to 0$ for $\delta \to 0$. That is, for any $\varepsilon > 0$ there is a $\delta_0 > 0$ such that

$$\tilde{\boldsymbol{\Delta}}^r f(\delta_0) := \sup_{0<\delta\leq\delta_0} \boldsymbol{\Delta}^r f(\delta) < \varepsilon.$$

On the other hand $f \in \mathscr{C}^r_{\mathrm{CdI}}(\mathbf{o},\mathbf{E})$ if

$$|\Delta^r f|(x+h,x) := |\mathrm{R}^{\nu} f(x+h,x)|/|h|^r,$$

a priori defined for all different $x+h, x$ in $\mathbf{o}$, extends to a continuous function, denoted likewise, on all of $\mathbf{o}\times\mathbf{o}$ that vanishes on the diagonal. As $\mathbf{o}$ is a compact metric space, $|\Delta^r f|$ is continuous on $\mathbf{o}\times\mathbf{o}$ if and only if it is uniformly so. In particular on the diagonal, this says that for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\tilde{\Delta}^r f(\delta) := \sup_{a\in\mathbf{o}} \sup_{x,y\in \mathrm{B}_{\leq\delta}(a)} |\Delta^r f|(x,y) < \varepsilon$$

where $\mathrm{B}_{\leq\delta}(a)$ is the ball of radius δ around a inside $\mathbf{o}$. We have to prove $\tilde{\Delta}^r f(\delta) = \tilde{\boldsymbol{\Delta}}^r f(\delta)$, that is,

$$\sup_{a\in\mathbf{o}} \sup_{x,y\in \mathrm{B}_{\leq\delta}(a)} |\mathrm{R}^{\nu} f(x,y)|/|x-y|^r = \sup_{0<\gamma\leq\delta} \sup_{x\in\mathbf{o}} \sup_{y \text{ with } |y-x|\leq\gamma} |\mathrm{R}^{\nu} f(x,y)|/\gamma^r. \qquad (*)$$

We note that for any $x, y \in \mathbf{o}$ and $\delta > 0$ we have $|x - y| \leq \delta$ if and only if there is $x_0 \in \mathbf{o}$ such that $\max\{|x - x_0|, |y - x_0|\} \leq \delta$ by the strong triangle inequality. Thus the left-hand side of $(*)$ equals

$$\sup_{x \in \mathbf{o}} \sup_{y \text{ with } |y-x| \leq \delta} |\mathrm{R}^\nu f(x,y)|/|x-y|^r. \qquad (**)$$

Furthermore we note that for any $x, y \in \mathbf{o}$ we have $x \neq y$ and $|x - y| \leq \delta$ if and only if $|x - y| = \gamma$ for some $0 < \gamma \leq \delta$. Thus

$$\sup_{x \in \mathbf{o}} \sup_{y \text{ with } |y-x| \leq \delta} |\mathrm{R}^\nu f(x,y)|/|x-y|^r = \sup_{x \in \mathbf{o}} \sup_{0<\gamma\leq\delta} \sup_{y \text{ with } |x-y|=\gamma} |\mathrm{R}^\nu f(x,y)|/\gamma^r$$

Now keeping x fixed,

$$\sup_{0<\gamma\leq\delta} \sup_{y\in\mathbf{o} \text{ with } |x-y|=\gamma} |\mathrm{R}^\nu f(x,y)|/\gamma^r = \sup_{0<\gamma\leq\delta} \sup_{y\in\mathbf{o} \text{ with } |x-y|\leq\gamma} |\mathrm{R}^\nu f(x,y)|/\gamma^r$$

as $\mathrm{R}^\nu f(x,y) = 0$ for any $y = x$. But then

$$\sup_{x \in \mathbf{o}} \sup_{y \text{ with } |y-x| \leq \delta} |\mathrm{R}^\nu f(x,y)|/|x-y|^r = \sup_{x \in \mathbf{o}} \sup_{0<\gamma\leq\delta} \sup_{y \text{ with } |y-x|\leq\gamma} |\mathrm{R}^\nu f(x,y)|/\gamma^r.$$

This is the claimed Equality $(*)$ after substituting the left-hand side by $(**)$. □

We define accordingly the norm $\|\cdot\|_{\mathscr{C}^r_{\mathrm{CdI}}}$ on all $\mathscr{C}^r_{\mathrm{CdI}}$-functions from $\mathbf{o}$ to $\mathbf{E}$ to be that on $\mathscr{C}^r_{\mathrm{CdI}}(\mathrm{X}, \mathbf{E})$.

Origin from Representation Theory

The groups. We consider the case that $\mathbf{K}$ is $\mathbb{Q}_p$, and put $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_p)$. Let

- $\mathrm{T} = \left(\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right)$ be all diagonal matrices in G,
- $\mathrm{N} = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$ respectively $\mathrm{N}_0 = \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{smallmatrix}\right)$ all the unipotent respectively all integral unipotent matrices in G, and
- $\mathrm{P} = \mathrm{TN} = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$ resp. $\bar{\mathrm{P}} = \left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right)$ all upper respectively all lower triangular matrices in G.

The representation. One way to build a $\mathbf{E}$-linear action by GL_n on a vector space, that is, a module over $\mathbf{E}[\mathrm{GL}_n]$, is by *inducing* $\mathbf{E}$-linear actions by smaller $\mathrm{GL}_{n_1}, \ldots, \mathrm{GL}_{n_r}$ with $n_1 + \cdots + n_r = n$ on vector spaces $\mathrm{W}_1, \ldots, \mathrm{W}_r$, as follows:

Let $\mathrm{W} = \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_r$; that is, an $\mathbf{E}[\mathrm{M}]$-module for the subgroup $\mathrm{M} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ of GL_n. We extend its scalars to obtain the *induced* $\mathbf{E}[\mathrm{G}]$-module

$$\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} \mathrm{W} := \mathrm{W} \otimes_{\mathbf{E}[\mathrm{M}]} \mathbf{E}[\mathrm{G}].$$

In our case $n = 2 = 1 + 1$ the subgroup M consists of all diagonal matrices in G, that is, equals T, and T acts by a character.

The representation V of $\mathrm{GL}_2(\mathbb{Q}_p)$ that corresponds to crystalline representation of the absolute Galois group of $\mathbb{Q}_p$ on a 2-dimensional vector space is the *locally algebraic* induced representation of (locally algebraic) character of T.

The character $\chi \colon \mathrm{T} \to \mathbf{E}^*$ is the product $\chi = \psi \cdot \theta$ of

- the algebraic character $\psi = \psi_1 \otimes \psi_2 \colon \mathrm{T} \to \mathbf{E}^*$ given by the $\psi_1 = \cdot^{l+k}$ and $\psi_2 = \cdot^{l}$ on $\mathbb{Q}_p^*$ with $l + k \geq l \in \mathbb{Z}$, and
- the locally constant character $\theta = \theta_1 \otimes \theta_2 \colon \mathrm{T} \to \mathbf{E}^*$ satisfying $\theta_1(\mathbb{Z}_p^*) = \theta_2(\mathbb{Z}_p^*) = 1$ (and thus determined by their values on p).

Every character of T extends trivially to $\bar{\mathrm{P}}$ (and necessarily so, because its codomain $\mathbf{E}^*$ is abelian and T is the commutator of $\bar{\mathrm{P}}$). The induced G-representation $\mathrm{ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \chi$ is explicitly given by the $\mathbf{E}$-vector space

$$\mathrm{ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \chi = \{f \colon \mathrm{G} \to \mathbf{E} : f(\bar{p}g) = \chi(\bar{p}) \cdot f(g) \text{ for all } \bar{p} \in \bar{\mathrm{P}}, g \in \mathrm{G}\}$$

on which G acts by right translation, that is, $gf = f(\cdot g)$. We put

$$\mathrm{V} := \text{ all locally algebraic functions in } \mathrm{ind}_{\bar{\mathrm{P}}}^{\mathrm{G}} \chi.$$

The action by P *on* V. By the *Iwasawa decomposition* $\mathrm{G} = \mathrm{PK}$ for the compact subgroup $\mathrm{K} = \mathrm{GL}_2(\mathbb{Z}_\mathrm{P})$ and the upper triangular matrices P. Thence every norm $|\cdot|$ on V that is invariant under P gives by

$$\|v\| := \inf\{|kv| : k \in \mathrm{K}\}$$

rise to a norm that is invariant under G and so it suffices to find a P-invariant norm on V. For this we study the P-stable subspace

$$\mathrm{V(N)} := \{\text{all functions in V that vanish outside } \mathrm{N}\bar{\mathrm{P}}\}.$$

Proposition. *Let* $\mathscr{C}^{\mathrm{lp}\le k}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})$ *be all functions* $f\colon \mathrm{N}\to\mathbf{E}$ *that are locally polynomial of degree* $\le k$ *and of compact support. The restriction map* $f\mapsto f\restriction_{\mathrm{N}}$ *is an isomorphism of* $\mathbf{E}$*-vector spaces*

$$\mathrm{V}(\mathrm{N}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{lp}\le k}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})$$

and the group P *acts on* $\mathscr{C}^{\mathrm{lp}\le k}(\mathrm{N},\mathbf{E})$ *via the above isomorphism by*

- $tf=\chi(t)f({}^t\cdot)$ *for all* t *in* T *with* ${}^t\cdot$ *the right-conjugation by* t *on* N, *and*
- $nf=f(n\cdot)$ *for all* n *in* N.

The invariant norm on N. We are looking for a P-invariant norm on V. Because $\mathrm{P}=\mathrm{TN}$ we are looking for a norm $\|\cdot\|$ on $\mathscr{C}^{\mathrm{lp}\le k}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})$ that is

- invariant under translation, and
- fulfills $\|tf\|=\|f\|$ for all t in T.

The function $\|\cdot\|:\mathscr{C}^{\mathrm{lp}\le k}_{\mathrm{cpt}}(\mathrm{N},\mathbf{E})\to\mathbb{R}_{\ge 0}\cup\{\infty\}$ given by $\|f\|=\sup\{|pf(1)| : p \text{ in } \mathrm{P}\}$

- vanishes if and only if f vanishes,
- fulfills the strong triangle inequality, and
- because the support of f is compact, the supremum $\sup\{|f(n)| : n\in\mathrm{N}\}$ under all translates in N exists,

but is unbounded on the translates of f under T. The estimate

$$\|f_0\| = \|\mathbf{1}_{\mathrm{N}_0}\| = \|\sum_{n\in\mathrm{N}_0/\mathrm{N}_0^t} 1_{n\mathrm{N}_0^t}\| = \|\sum_{n\in\mathrm{N}_0/\mathrm{N}_0^t} 1/\chi(t)ntf_0\| \le 1/|\chi(t)|\|f_0\|$$

shows that a necessary condition for the existence of a norm invariant under translation by T is $|\chi(t)|\le 1$ for all t in T that stabilize N_0 under right-conjugation. The norm

$$\|f\| := \sup\{\|\mathrm{R}^{nt}f(1,\cdot)\|_{\mathrm{N}_0} : t\in\mathrm{T}, n\in\mathrm{N}\} \tag{4.1}$$

shows that this condition is sufficient (where $\|\mathrm{R}^{nt}f(1,\cdot)\|_{\mathrm{N}_0}$ is the supremum norm of the function $n\mapsto \mathrm{R}^{nt}f(1,n)$ on N_0). This supremum is bounded, because f locally equals its Taylor polynomial. (The term $\mathrm{R}f(n_0,n_0n)$ vanishes for n in a sufficiently small neighborhood ${}^t\mathrm{N}_0$ of 1 or equivalently when $|\chi(t)|$ is sufficiently big.) It is 0 if and only if f is 0 because the support of f is compact.

The invariant norm on $\mathbb{Q}_p$. Let us identify N with $\mathbb{Q}_p$ and see how, under this identification, the just defined norm $\|\cdot\|$ corresponds to the norm $\|\cdot\|_{\mathscr{C}^r_{\mathrm{CdI}}}$ by Colmez and de Ieso. First, we remark that the character $|\chi|\colon \mathrm{T} \to p^{\mathbb{Z}}$ is

- trivial on T_0 by definition, and
- trivial on the center Z of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the necessary condition that $|\chi(t)| \leq 1$ for all t in T that stabilize N_0 under right-conjugation.

Thus $|\chi|$ is determined by its values on $\mathrm{T}/\mathrm{T}_0\mathrm{Z}$. This group is isomorphic to $\mathbb{Z}$. Let t_0 in T, for example $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$, be a preimage of 1 in $\mathbb{Z}$ under this group isomorphism. Let $v_{\mathbf{E}}$ denote the valuation of $\mathbf{E}$ normalized by $v_{\mathbf{E}}(p) = 1$. We put

$$r := v_{\mathbf{E}}(\chi(t_0)) \geq 0,$$

and conclude that r determines $|\chi|\colon \mathrm{T} \to p^{\mathbb{Q}}$. Under the identification of N with $\mathbb{Q}_p$,

- $t = \begin{pmatrix} a & \\ & d \end{pmatrix}$ in T acts on N by $tx := d/ax$, and
- $n = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ in N acts on N by $nx := b + x$.

Every neighborhood of 1 in N is of the form ${}^{t_0^n}\mathrm{N}_0$ for some n in $\mathbb{N}$ and corresponds inside $\mathbb{Q}_p$ to the ball $\mathrm{B}_{\leq\delta}(0)$ around 0 of radius $\delta = p^{-n}$. We have $|\chi(t_0^n)| = 1/\delta^r$. We conclude that for t in T and n in N,

$$\|\mathrm{R}^{\,nt} f(1,\cdot)\|_{\mathrm{N}_0} = \|\chi(t)\mathrm{R}f(n,\cdot)\|_{\mathrm{N}_0^t} = \|\mathrm{R}f(n,\cdot)\|_{\mathrm{B}_{\leq\delta}}/\delta^r.$$

Hence the norm $\|\cdot\|$ on V(N) in Equation (4.1) identifies on $\mathscr{C}^{\mathrm{lp}\leq k}_{\mathrm{cpt}}(\mathbb{Q}_p, \mathbf{E})$ with the norm

$$\|f\| = \sup_{\delta>0} \sup_{x\in\mathbb{Q}_p} \sup_{|h|\leq\delta} |\mathrm{R}f(x,h)|/\delta^r.$$

This is the greatest norm on V(N) invariant under P. However, if we consider only the Taylor polynomial $\mathrm{R}^{\leq\nu} f$ of f up to degree ν, the resulting function

$$\|f\| = \sup_{\delta>0} \sup_{x\in\mathbb{Q}_p} \sup_{|h|\leq\delta} |\mathrm{R}^{\leq\nu} f(x,h)|\delta^r$$

is again a P-invariant norm on $\mathscr{C}^{\mathrm{lp}\leq k}_{\mathrm{cpt}}(\mathbb{Q}_p, \mathbf{E})$.

If we restrict this norm $\|\cdot\|$ to all functions in $\mathscr{C}^{\mathrm{lp}\leq k}_{\mathrm{cpt}}(\mathbb{Q}_p,\mathbf{E})$ that have support $\mathbb{Z}_p$, and identify these with $\mathscr{C}^{\mathrm{lp}\leq k}(\mathbb{Z}_p,\mathbf{E})$ then this norm is equivalent to the norm on $\mathscr{C}^{\mathrm{lp}\leq k}(\mathbb{Z}_p,\mathbf{E})$ given by

$$\|f\| = \max\{\|\mathrm{D}^0 f\|_{\sup},\ldots,\|\mathrm{D}^\nu f\|_{\sup}\} \cup \{\sup_{\delta>0}\sup_{x\in\mathbb{Z}_p}\sup_{|h|\leq\delta}|\mathrm{R}^{\leq k} f(x,h)|/\delta^r\}.$$

This is the norm whose completion of $\mathscr{C}^{\mathrm{lp}\leq k}(\mathbb{Z}_p,\mathbf{E})$ yields the $\mathbf{E}$-Banach space of $\mathscr{C}^r$-functions by Colmez and de Ieso.

To construct the whole Banach space $\widehat{\mathrm{V}}$ de Ieso regards the functions in V as defined on $\mathrm{G}/\bar{\mathrm{P}} = \mathbb{P}^1(\mathbf{K})$ covered by two disjoint copies of $\mathbf{o_K}$. He obtains at first the Banach subspace $\mathrm{V}^{\mathscr{C}^r}$ of $\mathscr{C}^r_{\mathrm{CdI}}(\mathbf{o_K},\mathbf{E}) \oplus \mathscr{C}^r_{\mathrm{CdI}}(\mathbf{o_K},\mathbf{E})$ whose derivatives vanish in higher degrees that are determined by the highest degrees of locally $\mathbb{Q}_p$-polynomial functions in V ([De 13b, Section 4.2]). However

- the norm of the first component of $\mathscr{C}^r_{\mathrm{CdI}}(\mathbf{o_K},\mathbf{E}) \oplus \mathscr{C}^r_{\mathrm{CdI}}(\mathbf{o_K},\mathbf{E})$ is only fixed by those matrices in T that preserve N_0 under right conjugation, whereas

- that of its second component only by those matrices in T that preserve $\bar{\mathrm{N}}_0$ under right conjugation.

To obtain a Banach space whose norm is invariant under all of T, all of T, he takes a quotient by a closed subrepresentation in which every element of the first component equals an element of the second component. Consequently every function is invariant under all matrices in T that preserve either N_0 or $\bar{\mathrm{N}}_0$ under right conjugation, that is, all of T.

Comparison

Let d be the degree of the field extension $\mathbf{K}$ over $\mathbb{Q}_p$. We explain that the differentiability condition by Colmez and de Ieso is a convergence condition on the multivariate Taylor-polynomial up to total degree $\lfloor r \rfloor$ on d copies of $\mathbb{Z}_p$.

Linear and polynomial maps over finite field extensions. Let $\sigma\colon \mathbf{K} \to \mathbf{E}$ be a field embedding. Let V be a $\mathbf{K}$-vector space and let W be a $\mathbf{E}$-vector space. A map $\mathrm{A}\colon \mathrm{V} \to \mathrm{W}$ is σ-*linear* if it is additive and for every scalar λ in $\mathbf{K}$ it satisfies $\mathrm{A}\lambda\cdot = \sigma(\lambda)\mathrm{A}\cdot$. Let $\sigma_1,\ldots,\sigma_{\mathrm{N}}\colon \mathbf{K} \to \mathbf{E}$ be field embeddings. A map $\mathrm{M}\colon \mathrm{V}\times\cdots\times\mathrm{V} \to \mathrm{W}$ of N variables is $(\sigma_1,\ldots,\sigma_{\mathrm{N}})$-*multilinear* if it is σ_1-linear in the first coordinate, …, σ_{N}-linear in the N-th coordinate.

Proposition 4.4. *We have a canonical bijection*

$$\bigoplus_{\sigma \in S} \mathrm{Hom}_\sigma(\mathbf{K}, \mathbf{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{K}, \mathbf{E}).$$

Proof: It is injective because by Dedekind's Theorem, cf. [Lan02, Theorem VI.4.1], all $\mathbb{Q}_p$-algebra homomorphisms $\sigma\colon \mathbf{K} \to \mathbf{E}$ are linearly independent over $\mathbf{E}$. It is a bijection because both sides have the same dimension over $\mathbf{E}$. □

Alternatively, we have

$$\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{K}, \mathbf{E}) = \mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{K}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{E} = \mathbf{K} \otimes_{\mathbb{Q}_p} \mathbf{E} = \prod_{\sigma \in S} \mathbf{E} = \bigoplus_{\sigma \in S} \mathrm{Hom}_\sigma(\mathbf{K}, \mathbf{E}).$$

Here we choose a basis of $\mathbf{K}$ over $\mathbb{Q}_p$ to identify $\mathbf{K}$ with its dual $\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{K}, \mathbb{Q}_p)$ and the splitting $\mathbf{K} \otimes_{\mathbb{Q}_p} \mathbf{E} = \prod_{\sigma \in S} \mathbf{E}$ comes from the Chinese remainder theorem.

Corollary 4.5. *We have a canonical isomorphism of finite-dimensional* $\mathbf{E}$*-vector spaces*

$$\sum_{\boldsymbol{\sigma} \in S^N} \mathrm{Mult}^N_{(\sigma_1, \ldots, \sigma_N)}(\mathbf{K} \times \cdots \times \mathbf{K}, \mathbf{E}) \xrightarrow{\sim} \mathrm{Mult}^N_{\mathbb{Q}_p}(\mathbf{K} \times \cdots \times \mathbf{K}, \mathbf{E}).$$

Proof: By the natural identification $\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{K}, \mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{K}, \mathbf{E})) = \mathbf{Bil}_{\mathbb{Q}_p}(\mathbf{K} \times \mathbf{K}, \mathbf{E})$ with the $\mathbb{Q}_p$-billinear maps on $\mathbf{K} \times \mathbf{K}$, the bijection in Proposition 4.4 inductively carries over to a bijection between

- all $\mathbb{Q}_p$-multilinear maps on $\mathbf{K} \times \cdots \times \mathbf{K}$ with values in $\mathbf{E}$, and
- all sums of all maps on $\mathbf{K} \times \cdots \times \mathbf{K}$ with values in $\mathbf{E}$ that are in every variable σ-linear for some σ in S. □

This bijection restricts to all symmetric maps. That is, we have a canonical isomorphism of finite-dimensional $\mathbf{E}$-vector spaces

$$\phi\colon \mathrm{Sym}\Big(\sum_{\sigma \in S^N} \mathrm{Mult}^N_{(\sigma_1, \ldots, \sigma_N)}(\mathbf{K} \times \cdots \times \mathbf{K}, \mathbf{E})\Big) \xrightarrow{\sim} \mathrm{Sym}^N_{\mathbb{Q}_p}(\mathbf{K} \times \cdots \times \mathbf{K}, \mathbf{E}).$$

Let us partition $\{1, \ldots, N\}$ into subsets n_σ for σ in S. The $\mathbf{K}$-vector space of all symmetric multilinear maps $M\colon \mathbf{K} \times \cdots \times \mathbf{K} \to \mathbf{E}$ of N variables that are σ-linear in n_σ variables for each σ in S has as basis the functions

$$x \mapsto x^{\boldsymbol{n}} := \prod_{\sigma \in S} \prod_{n \in n_\sigma} \sigma x_n.$$

By the Polarization Lemma 2.2, every multilinear symmetric map is determined by its restriction onto the diagonal. Hence the $\mathbf{K}$-vector space of all homogeneous $\mathbb{Q}_p$-polynomial maps of degree N on $\mathbf{K}$, that is, of all maps that are restrictions of $\mathbb{Q}_p$-multilinear maps $\mathrm{M}\colon \mathbf{K} \times \cdots \times \mathbf{K} \to \mathbf{E}$ of N variables, has as basis the monomial functions

$$x \mapsto x^{\boldsymbol{n}} := \prod_{\sigma \in \mathrm{S}} \sigma x^{n_\sigma}$$

for all $\boldsymbol{n}$ in $\mathbb{N}^{\mathrm{S}}$ whose sum of entries equals N. We conclude:

Corollary 4.6. *The $\mathbf{K}$-vector space of all $\mathbb{Q}_p$-polynomial functions on $\mathbf{K}$ has as basis the monomials*

$$x \mapsto x^{\boldsymbol{n}} := \prod_{\sigma \in \mathrm{S}} \sigma x^{n_\sigma} \quad \text{for } \boldsymbol{n} \text{ in } \mathbb{N}^{\mathrm{S}}.$$

Comparison of $\mathscr{C}^r$- and $\mathscr{C}^r_{\mathrm{CdI}}$-functions. We show that, after fixing an identification of $\mathbf{o}$ with $\mathbb{Z}_p^{\mathrm{S}}$, the differentiability condition over $\mathbf{o}$ of Definition 1.2 via iterated divided differences is equivalent to that of Definition 4.2 via Taylor polynomials.

Theorem 4.7. *Let $\mathbf{K}$ be a finite extension of $\mathbb{Q}_p$ and let $\mathbf{o}$ be its ring of integers that we regard as open subset of the $\mathbb{Q}_p$-vector space $\mathbf{K}$. Every r-times differentiable function from $\mathbf{o}$ to $\mathbf{E}$ is a $\mathscr{C}^r_{CdI}$-function and the inclusion map $\mathscr{C}^r(\mathbf{o},\mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathrm{CdI}}(\mathbf{o},\mathbf{E})$ is an isomorphism of normed $\mathbf{E}$-algebras.*

Proof: We recollect all preceding characterizations of r-fold differentiability:

1. Let us regard $\mathbf{o}$ as an open subset of the $\mathbb{Q}_p$-vector space $\mathbf{K}$. By Corollary 3.10, the inclusion of normed $\mathbf{E}$-algebras

   $$\mathscr{C}^r(\mathbf{o},\mathbf{E}) \hookrightarrow \mathscr{C}^r_{\mathrm{T}}(\mathbf{o},\mathbf{E}),$$

   that of $\mathscr{C}^r$-functions as in Definition A.1 via iterated divided differences included in that of $\mathscr{C}^r_{\mathrm{T}}$-functions as in Definition 2.1 via the convergence of their Taylor polynomial expansions, is an isomorphism.

2. Let $f\colon \mathbf{o} \to \mathbf{E}$. Definition 2.1 says that f is a $\mathscr{C}^r_{\mathrm{T}}$-function if there is a family of $\mathbb{Q}_p$-polynomial maps $(\mathrm{T}f_x : x \in \mathbf{o})$ such that for every a in $\mathbf{o}$ and every $\epsilon > 0$, there is a neighborhood U around a in $\mathbf{o}$ such that

   $$\|f(x+h) - \mathrm{T}f_x(h)\| \leq \varepsilon \|h\|^r$$

for all $x+h, x$ in U. The $\mathscr{C}^r_{\mathrm{T}}$-norm is defined as supremum of all these differences over all $x+h, x$ in $\mathbf{o}$ and the coefficients of $\mathrm{T}f_x$ on all x in $\mathbf{o}$.

Because every $\mathbb{Q}_p$-polynomial map is by Corollary 4.6 a linear combination of monomial functions $x \mapsto \prod_{\sigma\in \mathrm{S}} \sigma x^{n_\sigma}$ for $\boldsymbol{n}$ in $\mathbb{N}^{\mathrm{S}}$, it follows that f is a $\mathscr{C}^r_{\mathrm{T}}$-function as in Definition 2.1 if and only if f is a $\mathscr{C}^r_{\mathrm{CdI}}$-function as in Definition 4.2 and $\|\cdot\|_{\mathscr{C}^r_{\mathrm{T}}} = \|\cdot\|_{\mathscr{C}^r_{\mathrm{CdI}}}$.

3. By Proposition 4.3 every function $f\colon \mathbf{o} \to \mathbf{E}$ is a $\mathscr{C}^r_{\mathrm{CdI}}$-function if and only if it is a $\mathscr{C}^r_{\mathbf{CdI}}$-function. □

A. Schikhof's iterated divided differences

Let X be an accumulated subset of a finite dimensional $\mathbf{K}$-vector space V and let $\mathbf{E}$ be a $\mathbf{K}$-Banach space.

We compare the n-th divided difference of a function f as given in Section 1, denoted by $f^{[n]}$, with that by Schikhof, denoted by $f^{<n>}$. Let us recall Schikhof's definition as given

- in [Sch84, Section 29] when V is $\mathbf{K}$ and r in $\mathbb{N}$, and
- in [Nag11, Section 3.1] when V is a finite dimensional $\mathbf{K}$-vector space and $r \geq 0$ in $\mathbb{R}$.

Definition of Schikhof's iterated divided differences

The symmetry of the differential allowed Schikhof to reduce for increasing degree of differentiability ν the exponential growth in the number of variables of the total differential $f^{[\nu]}$ to a linear growth in the number of variables of certain partial differentials $f^{<\boldsymbol{n}>}$ with $\boldsymbol{n}$ in $\mathbb{N}^d$ such that $n_1 + \cdots + n_d = \nu$.

To define these partial differentials, we fix some notation. For a subset X of $\mathbf{K}$ and $n \in \mathbb{N}$, let $\mathrm{X}^{<n>} := \mathrm{X}^{\{0,\ldots,n\}}$ and

$$\mathrm{X}^{>n<} := \{(x^{<0>}, \ldots, x^{<n>}) \in \mathrm{X}^{<n>} : x^{<i>} = x^{<j>} \text{ only if } i = j\}.$$

For subsets $\mathrm{X}_1, \ldots, \mathrm{X}_d$ of $\mathbf{K}$ and $\boldsymbol{n} \in \mathbb{N}^d$, let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ and

$$\mathrm{X}^{<\boldsymbol{n}>} := \mathrm{X}_1^{<n_1>} \times \cdots \times \mathrm{X}_d^{<n_d>} \quad \text{and} \quad \mathrm{X}^{>\boldsymbol{n}<} := \mathrm{X}_1^{>n_1<} \times \cdots \times \mathrm{X}_d^{>n_d<}.$$

Given $\boldsymbol{x} \in \mathrm{X}^{<\boldsymbol{n}>}$, let $\boldsymbol{x} = (\boldsymbol{x}_1; \ldots; \boldsymbol{x}_d)$ represent its entries in $\mathrm{X}_1^{<n_1>}, \ldots, \mathrm{X}_d^{<n_d>}$.

Definition. Let $X_1,\dots,X_d$ be subsets of $\mathbf{K}$, let $X = X_1 \times \cdots \times X_d$ be their product and $f\colon X \to \mathbf{E}$. Let $\boldsymbol{n} \in \mathbb{N}^d$. We define $f^{>\boldsymbol{n}<}\colon X^{>\boldsymbol{n}<} \to \mathbf{E}$ by $f^{>\mathbf{0}<} = f$ and, by induction on $n = n_1 + \cdots + n_d$, if $(0,\dots,1,\dots,0)$ is the tuple that has a single nonzero entry 1 in the k-th coordinate then

$$
\begin{aligned}
&f^{>\boldsymbol{n}+(0,\dots,1,\dots,0)<}(..;x_k^{<0>},x_k^{<1>},x_k^{<2>},\dots,x_k^{<n_k+1>};..)\\
&:=\frac{f^{>\boldsymbol{n}<}(..;x_k^{<0>},x_k^{<2>},\dots,x_k^{<n_k+1>};..)-f^{>\boldsymbol{n}<}(..;x_k^{<1>},x_k^{<2>},\dots,x_k^{<n_k+1>};..)}{x_k^{<0>}-x_k^{<1>}}.
\end{aligned}
$$

Example. Let us consider a function of two variables $f\colon X\times Y \to \mathbf{E}$ for subsets X and Y of K. We have

$$
X^{>1,1<} = \{(x+u,x;y+v,y) : x+u,x \in X, y+v,y \in Y \text{ with } u,v \neq 0\}
$$

and $f^{>1,1<}$ is the first mixed partial difference quotient of f, that is,

$$
\begin{aligned}
&f^{>1,1<}(x+u,x;y+v,y)\\
&=\frac{[f(x+u,y+v)-f(x,y+v)]-[f(x+u,y)-f(x,y)]}{u\cdot v}.
\end{aligned}
$$

A subset X of V is *Cartesian* if it is of the form $X = X_1 \times \cdots \times X_d$ with $X_1,\dots,X_d$ subsets of $\mathbf{K}$ and *locally Cartesian* if every x in X has a Cartesian neighborhood.

Definition A.1. Let X be a subset of V and a in X.

- If X is Cartesian then the function $f\colon X \to \mathbf{E}$ is C^r at $a = (a_1,\dots,a_d)$ if $f^{>\boldsymbol{n}<}\colon X^{>\boldsymbol{n}<} \to \mathbf{E}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a_1,\dots,a_1;\dots;a_d,\dots,a_d)$ in $X^{<\boldsymbol{n}>}$ for all $\boldsymbol{n}$ in $\mathbb{N}^d$ such that $n_1 + \cdots + n_d = \nu$.
- If X is locally Cartesian then the function f is C^r at a in X if, after choosing a Cartesian neighborhood U of a included in X, the function $f\restriction_U : U \to \mathbf{E}$ is C^r at a.

The function f is a *C^r-function* if f is C^r at all a in X. Let $C^r(X,\mathbf{E})$ be the set of all C^r-functions $f\colon X \to \mathbf{E}$.

Let X be a locally Cartesian subset of V and let $f\colon X \to \mathbf{E}$ be a C^r-function. If X is accumulated, for example open, then all partial derivatives $f^{<\boldsymbol{n}>}$ of f are uniquely determined by f.

- Let a in X and $\boldsymbol{n}$ in $\mathbb{N}^d$ such that $n_1 + \cdots + n_d = \nu$. Let $\mathrm{D}^{\boldsymbol{n}} f(a)$ be the *$\boldsymbol{n}$-th partial derivative* of f at a, the unique value to which $f^{>\boldsymbol{n}<}$ extends at $\vec{a}$ as C^{ρ}-function. If $f^{(\boldsymbol{n})}$ is the Archimedean partial derivative of f taken n_1-times along $\boldsymbol{e}_1, \ldots, n_d$-times along $\boldsymbol{e}_d$, then $n_1! \cdots n_d!\, \mathrm{D}^{\boldsymbol{n}} f = f^{(\boldsymbol{n})}$ by a multivariate version of [Sch84, Theorem 29.5].

- Its partial differentials $f^{>\boldsymbol{n}<}$ extend by [Nag11, Proposition 3.8] to unique $\mathscr{C}^{\rho}$-functions

$$f^{<\boldsymbol{n}>} \colon \mathrm{X}^{<\boldsymbol{n}>} \to \mathbf{E}.$$

  Let

$$f^{<n>} := \prod_{\boldsymbol{n} \text{ in } \mathbb{N}^d \text{ with } n_1+\cdots+n_d=n} f^{<\boldsymbol{n}>}.$$

Definition. Let C be a compact Cartesian subset of V all of whose factors contain no isolated points. The *norm* $\|\cdot\|_{\mathscr{C}^r}$ on $\mathscr{C}^r(\mathrm{C}, \mathbf{E})$ is defined by

$$\begin{aligned}\|f\|_{\mathscr{C}^r} = \max\{&\|f^{<\boldsymbol{n}>}\|_{\sup} : \boldsymbol{n} \text{ in } \mathbb{N}^d \text{ such that } n_1 + \cdots + n_d < \nu\} \\ &\cup \{\|f^{<\boldsymbol{n}>}\|_{\mathscr{C}^{\rho}} : \boldsymbol{n} \text{ in } \mathbb{N}^d \text{ such that } n_1 + \cdots + n_d = \nu\}\end{aligned}$$

Equivalence between Schikhof's and our iterated divided differences

Let $\mathrm{V}_1, \ldots, \mathrm{V}_n$ be vector spaces. The normed $\mathbf{K}$-vector space

$$\mathrm{Hom}(\mathrm{V}_1, \mathrm{Hom}(\mathrm{V}_2, \ldots, \mathrm{Hom}(\mathrm{V}_n, \mathbf{E})) \cdots)$$

is canonically isomorphic to $\mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_n^* \otimes \mathbf{E}$. We identify $\mathbf{E}^{[\nu]}$ with $\mathrm{V}^{[\nu-1]^*} \otimes \cdots \otimes \mathrm{V}^* \otimes \mathbf{E}$ by this isomorphism (cf. Section 3).

Proposition A.2. *We have* $\mathrm{C}^r(\mathrm{X}, \mathbf{E}) = \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ *as sets. If* X *is a compact Cartesian subset of* V*, all of whose factors contain no isolated points, then the inclusion map* $\mathrm{C}^r(\mathrm{X}, \mathbf{E}) \hookrightarrow \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ *is an isomorphism of normed* $\mathbf{K}$*-vector spaces.*

Proof: Let $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_d$ where $\mathrm{X}_1, \ldots, \mathrm{X}_d$ are subsets of $\mathbf{K}$. Let us fix n in $\mathbb{N}$. We define mutually inverse homomorphisms between normed $\mathbf{K}$-vector spaces

$$\phi_n \colon \prod_{\boldsymbol{n} \in \mathbb{N}^d \text{ with } n_1+\cdots+n_d=n} \mathscr{C}^0(\mathrm{X}^{<\boldsymbol{n}>}, \mathbf{E}) \to \mathscr{C}^0(\mathrm{X}^{[n]}, \mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_n^* \otimes \mathbf{E})$$

and

$$\psi_n \colon \operatorname{im} \phi_n \to \prod_{\boldsymbol{n} \in \mathbb{N}^d \text{ with } n_1+\cdots+n_d=n} \mathscr{C}^0(\mathrm{X}^{<\boldsymbol{n}>}, \mathbf{E})$$

such that, if f in $C^r(X,\mathbf{E}) \cap \mathscr{C}^r(X,\mathbf{E})$ then

$$\phi_n(f^{<n>}) = f^{[n]} \quad \text{and} \quad \psi_n(f^{[n]}) = f^{<n>}.$$

We define ϕ_n by

$$F \mapsto \sum_{\boldsymbol{n}\in\mathbb{N}^d \text{ with } n_1+\cdots+n_d=n} \iota_{\boldsymbol{n}} \circ F \circ p_{\boldsymbol{n}}$$

where

$$p_{\boldsymbol{n}} \colon X^{[n]} \to X^{<\boldsymbol{n}>} \quad \text{and} \quad \iota_{\boldsymbol{n}} \colon \mathbf{E} \to V^{[n-1]^*} \otimes \cdots \otimes V^{[1]^*} \otimes V^* \otimes \mathbf{E}$$

are defined as follows: For n in $\mathbb{N}$, let p_0 and p_1 be the projections from the product $V^{[n+1]} = V^{[n]} \times V^{[n]}$ to its left and right factor. For $\boldsymbol{i}$ in $\{0,1\}^n$ let $p_{\boldsymbol{i}}$ denote the function

$$p_{i_1} \circ \cdots \circ p_{i_n} \colon V^{[n]} \to V.$$

Let $p_1, \ldots, p_d$ be the projections from the d-fold product $V = \mathbf{K} \times \cdots \times \mathbf{K}$ onto its first, …, last factor.

- Let us establish $p_{\boldsymbol{n}}$. Let $\boldsymbol{e}_0 = (0,\ldots,0)$ in $\{0,1\}^n$ and for $i = 1,\ldots,n$, let $\boldsymbol{e}_i = (\ldots,0,1,0,\ldots)$ be the tuple in $\{0,1\}^n$ whose sole nonzero entry is 1 at the i-th place. Let $\{N_1,\ldots,N_d\}$ be a partition of $\{1,\ldots,n\}$ into d families of n_1, …, n_d elements.

  We let $p_{\boldsymbol{n}}$ be the restriction of the function

  $$((p_1 \circ p_{\boldsymbol{e}_n})_{n\in N_1\cup\{0\}}; \ldots; (p_d \circ p_{\boldsymbol{e}_n})_{n\in N_d\cup\{0\}})$$

  to the domain $X^{[n]}$ and codomain $X^{<\boldsymbol{n}>}$.

- Let us establish $\iota_{\boldsymbol{n}}$. For n in $\mathbb{N}$ and $\boldsymbol{i}$ in $\{0,1\}^n$, let $\iota_{\boldsymbol{i}} \colon V^* \to V^{[n]}$ be the dual of $p_{\boldsymbol{i}}$. Let $\iota_1, \ldots, \iota_d \colon \mathbf{K}^* \to V^*$ be the duals of $p_1, \ldots, p_d$. Let $\mathbf{N}$ denote all tuples in $\{1,\ldots,d\}^{\{0,\ldots,n-1\}}$ that have n_1 entries equal to 1, …, n_d entries equal to d.

  We identify $\mathbf{K}^* \otimes \cdots \otimes \mathbf{K}^* \otimes \mathbf{E}$, where $\mathbf{K}$ appears n times, with $\mathbf{E}$ and define

  $$\iota_{\boldsymbol{n}} = \Big(\sum_{N\in\mathbf{N}} \sum_{\substack{\boldsymbol{b}_{n-1}\in\{0,1\}^{n-1} \\ \vdots \\ \boldsymbol{b}_0\in\{0,1\}^0}} \iota_{\boldsymbol{b}_{n-1}} \circ \iota_{N_{n-1}} \otimes \cdots \otimes \iota_{\boldsymbol{b}_0} \circ \iota_{N_0} \Big) \otimes \mathrm{id}_{\mathbf{E}}$$

We define ψ_n as follows. For n in $\mathbb{N}$, let υ_0 and υ_1 be the inclusions of the left and right summand of $V^{[n+1]} = V^{[n]} \oplus V^{[n]}$. For $\boldsymbol{i}$ in $\{0,1\}^n$ let $\upsilon_{\boldsymbol{i}}$ denote

$$\upsilon_{i_1} \circ \cdots \circ \upsilon_{i_n} \colon V \to V^{[n]}.$$

Let $\upsilon_1, \ldots, \upsilon_d$ be the inclusions of the first, ..., last summand of $V = \mathbf{K} \oplus \cdots \oplus \mathbf{K}$. For n in $\mathbb{N}$ and $\boldsymbol{i}$ in $\{0,1\}^n$, let $\pi_{\boldsymbol{i}} \colon V^{[n]^*} \to V^*$ be the dual of $\upsilon_{\boldsymbol{i}}$. Let $\pi_1, \ldots, \pi_d \colon V^* \to \mathbf{K}^*$ be the duals of $\upsilon_1, \ldots, \upsilon_d$.

We fix a tuple N in $\{1, \ldots, d\}^{\{0,\ldots,n-1\}}$ that has n_1 entries equal to 1, ..., n_d entries equal to d and let us fix n tuples $\boldsymbol{b}_{n-1}, \ldots, \boldsymbol{b}_0$ in $\{0,1\}^n, \ldots, \{0,1\}^0$.

Let

$$\pi_{\boldsymbol{n}} \colon V^{[n-1]^*} \otimes \cdots \otimes V^{[1]^*} \otimes V^* \otimes \mathbf{E} \to \mathbf{E}$$

denote the function

$$\left(\pi_{N_{n-1}} \circ \pi_{\boldsymbol{b}_{n-1}} \otimes \cdots \otimes \pi_{N_0} \circ \pi_{\boldsymbol{b}_0}\right) \otimes \mathrm{id}_{\mathbf{E}}$$

where we identify $(\mathbf{K}^* \otimes \cdots \otimes \mathbf{K}^*) \otimes \mathbf{E}$ with $\mathbf{E}$. If F in $\operatorname{im} \psi_n$ then the function $\pi_{\boldsymbol{n}} \circ F$ factorizes over $p_{\boldsymbol{n}} \colon X^{[n]} \to X^{<\boldsymbol{n}>}$. We define this factorization as $\psi_n(F)_{\boldsymbol{n}}$ and put

$$\psi_n(F) = \left(\psi_n(F)_{\boldsymbol{n}} \text{ for all } \boldsymbol{n} \text{ in } \mathbb{N}^d \text{ with } n_1 + \cdots + n_d = n\right).$$

Let $f \colon X \to \mathbf{E}$. We prove by induction on ν that f is a C^ν-function if and only if it is a $\mathscr{C}^\nu$-function.

For $\nu = 0$ this holds because by definition $f^{<\nu>} = f = f^{[\nu]}$.

Let ν in $\mathbb{N}$. We assume by induction that f is a $C^{\nu-1}$-function if and only if it is a $\mathscr{C}^{\nu-1}$-function. By definition f is a C^ν-function if for every ν in $\mathbb{N}^d$ with $\nu_1 + \cdots + \nu_d = \nu$, the function $f^{>\nu<}$ extends to a $\mathscr{C}^\rho$-function $f^{<\nu>}$ and by definition f is a $\mathscr{C}^\nu$-function if, with $F = f^{[\nu-1]}$, the function $F^{]1[}$ extends to a $\mathscr{C}^\rho$-function on $X^{[\nu]}$. Thence f is C^ν-function if and only if f is a $\mathscr{C}^\nu$-function because

$$\phi_\nu(f^{<\nu>}) = f^{[\nu]} \quad \text{and} \quad \psi_\nu(f^{[\nu]}) = f^{<\nu>}. \tag{$*$}$$

We conclude $C^r(X, \mathbf{E}) = \mathscr{C}^r(X, \mathbf{E})$ as sets.

Let X be an accumulated subset of V. Then X is the disjoint union $\coprod \mathfrak{U}$ of Cartesian subsets of V, so

$$C^r\left(\coprod \mathfrak{U}, \mathbf{E}\right) = \prod_{U \in \mathfrak{U}} C^r(U, \mathbf{E}) \quad \text{and} \quad \mathscr{C}^r\left(\coprod \mathfrak{U}, \mathbf{E}\right) = \prod_{U \in \mathfrak{U}} \mathscr{C}^r(U, \mathbf{E}).$$

Because $C^r(U,\mathbf{E}) = \mathscr{C}^r(U,\mathbf{E})$ as sets for each U in $\mathfrak{U}$ we conclude $C^r(X,\mathbf{E}) = \mathscr{C}^r(X,\mathbf{E})$ as sets as well.

Let X be compact Cartesian, that is, there are compact subsets $X_1,\ldots,X_d$ of $\mathbf{K}$ such that $X = X_1\times\cdots\times X_d$. By definition of the norms of $C^r(X,\mathbf{E})$ and $\mathscr{C}^r(X,\mathbf{E})$, the identity map of sets $\mathrm{id}\colon C^r(X,\mathbf{E}) \to \mathscr{C}^r(X,\mathbf{E})$ is an isometric isomorphism between normed spaces because for every $n = 0,\ldots,\nu$ the maps $\phi_0,\ldots,\phi_{\nu-1}$ and $\psi_0,\ldots,\psi_{\nu-1}$ respectively ϕ_ν and ψ_ν are isometries for the $\mathscr{C}^0$-norm respectively $\mathscr{C}^\rho$-norm and satisfy Equation $(*)$. □

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