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Fractional p-adic Differentiability under the Amice transform

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Abstract. One might wonder why the Mahler polynomials $\binom{*}{n}$ for $n \in \mathbb{N}$ constitute simultaneously an orthogonal basis of the continuous, differentiable and locally analytic functions on $\mathbb{Z}_p$, albeit with respect to well different convergence properties.

To respond to this question, we will firstly give a brief resume on fractional differentiability over non-Archimedean fields. Then we show that this coincidence can be discovered by comparing the operator norms of continuous linear forms on the common dense subset of locally polynomial functions with respect to the continuous, differentiable and locally analytic topology.

A certain uniformity of the Amice transforms on locally analytic distributions with respect to their radius of analyticity yields that the Mahler polynomials constitute an orthogonal basis of the Banach space of $\mathscr{C}^r$-functions for any $r \geq 0$, as well as a convergence condition on a function's Mahler coefficients for being r-times differentiable.

Contents

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Introduction

Let $\mathbb{Z}_p$ denote the p-adic integers, $\mathbb{Q}_p$ the p-adic numbers and $\mathbf{K} \supseteq \mathbb{Q}_p$ a complete non-Archimedeanly nontrivially valued field with ring of integers $\mathbf{o_K}$. Let $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ be the $\mathbf{K}$-Banach space of continuous functions $f : \mathbb{Z}_p \to \mathbf{K}$ and $\mathscr{D}^0(\mathbb{Z}_p, \mathbf{K}) = \mathrm{Hom}(\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K}), \mathbf{K})$ its continuous dual. Every continuous function $f : \mathbb{Z}_p \to \mathbf{K}$ can be uniformly approximated by locally constant functions $f_n \in \mathbf{K}[\mathbb{Z}_p/p^n\mathbb{Z}_p]$ for $n \in \mathbb{N}$ and so we obtain dually an isomorphism of $\mathbf{K}$-algebras

$$\mathbf{K} \otimes \mathbf{o_K}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathscr{D}^0(\mathbb{Z}_p, \mathbf{K})$$

where the left-hand side is the completed group algebra and the right-hand side is equipped with the convolution product. By the Iwasawa isomorphism of topological $\mathbf{K}$-algebras

$$\begin{aligned} \mathbf{o_K}[[\mathbb{Z}_p]] &\xrightarrow{\sim} \mathbf{o_K}[[\mathrm{X}]] \\ \mathbf{1} + 1 &\mapsto \mathrm{X} \end{aligned}$$

which holds with respect to the topologies of coordinate-wise convergence, we obtain by a duality argument, see Section 8, the orthogonal basis of $\mathscr{C}^0(\mathbb{Z}_p, \mathbf{K})$ given by the Mahler polynomials $\binom{x}{n}$ for $n \in \mathbb{N}$.

Correspondingly, in [**Ami64**] it was shown that the Mahler polynomials form an orthogonal basis of the locally analytic function, yielding by duality the *Amice isomorphism* which identifies $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{K})$, the continuous linear forms on the locally analytic functions on $\mathbb{Z}_p$, with the power series converging on the open unit disc.

Now let $r \geq 0$ be a real number and let us study this question for r-times differentiable functions instead: To this end, we firstly give a description of the topology of the *distribution space* $\mathscr{D}^r(\mathbb{Z}_p, \mathbf{K})$, the continuous dual of the $\mathscr{C}^r$-functions on $\mathbb{Z}_p$, by restricting such a distribution to all those functions which are analytic on a progressively smaller radius $\rho > 0$. This way, we will see that the Mahler polynomials constitute as well an orthogonal basis of the $\mathscr{C}^r$-function space. A certain "uniformity" of the Amice isomorphism with respect to the radius of analyticity ρ finally provides their characterizing convergence property via their basis coefficients.

First off, we give in Part 1 some motivation on the non-Archimedean concept of differentiability and recall the notion of fractional differentiability in one variable: Given a function $f: X \to \mathbf{K}$ on a subset $X \subseteq \mathbf{K}$ without isolated points, we present the point-wise definition of r-fold differentiability for $r \in \mathbb{R}_{\geq 0}$ through iterated difference quotients (as established in [**Nag12**] and residing upon [**Sch84**]).

In Part 2 we compare the locally analytic and $\mathscr{C}^r$-topologies and choose the locally polynomial functions as a common subspace. We noted above that the locally constant functions approximate the continuous ones uniformly. Correspondingly, given a d-times differentiable function f, we find a locally polynomial function of degree d whose d-th locally constant derivative approximates the d-th derivative $f^{(d)}$ uniformly. Along this observation, we obtain the locally polynomial functions of degree $d \leq r$ to be dense inside the $\mathscr{C}^r$-functions and finally single out certain indicator functions as an orthogonal basis. This is the van der Put basis, on which the $\mathscr{C}^r$-norm is easily determined. At the same time, the family of van der Put-functions rests orthogonal with respect to the topologies of locally analytic functions and their norms are directly determined, all of this by their very definitions. We then compare the norms on this orthogonal basis with respect to the $\mathscr{C}^r$- and locally analytic topologies, inferring that the dual of the $\mathscr{C}^r$-functions is an inverse limit of spaces $D(n)$ given by the duals on functions which are analytic on every neighborhood of radius $r(n)$.

The Amice transform translates this observation, together with a duality argument, to the fact that the Mahler polynomials constitute an orthogonal basis of the $\mathscr{C}^r$-function space. In the final Part 3, we make use of a certain uniformity of this Amice transform to obtain by the technical Lemma 7.2, solely resting on elementary calculus, the arising convergence condition with respect to the Mahler basis.

We remark that there is another approach to fractional differentiability in one variable via Taylor polynomials as successively introduced by Schikhof in [**Sch78**], Colmez in [**Col10**] and de Ieso in [**DI13**]. It known to be equivalent to the one presented here via difference quotients by [**Sch78**] and [**Nag12**]. Under this regard, the final comparison result via the Amice transform is implicitly contained in [**Col10**]. In contrast to the work of Colmez and de Ieso though, the comparison results in Part 2 are formulated for functions locally analytic over a general finite extension $\mathbf{K}$ of $\mathbb{Q}_p$ and yet unpublished. They will serve us in the upcoming work [**Nag13a**].

Part 1. A resume of non-Archimedean r-fold differentiability

We want to define r-fold differentiability for a real number $r \geq 0$. To this end, we will decompose $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ into its *integral part* $\nu \in \mathbb{N}$ and *fractional part* $\rho \in [0, 1[$, either one of these giving rise to different regularity conditions.

1. $\mathscr{C}^\rho$-functions

Let us begin with the fractional part. Given $\rho \in [0, 1[$, we introduce the $\mathscr{C}^\rho$-condition of a function at a point a.

As the notation suggests, this is a regularity condition resting somewhere in between continuity and differentiability of a function, and in fact amounts to a tightened Hölder condition at a point a, where we demand the difference quotient to asymptotically vanish when approaching a instead of the usual boundedness condition around a.

Definition. Let X be a metric space, Y a complete metric space, $f\colon A \to Y$ a mapping defined on a dense subset $A \subseteq X$ and a some point in X; we will say that f is $\mathscr{C}^\rho$ **at** a, if for every $\varepsilon > 0$ there is a neighborhood $U \ni a$ in X such that

$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x, y)^\rho \quad \text{for all } x, y \in U \cap A.$$

Then f is a $\mathscr{C}^\rho$**-function** if f is $\mathscr{C}^\rho$ at all points $a \in A$, where we note that this notion is independent of the ambient space X. We will denote the set of all $\mathscr{C}^\rho$-functions $f\colon A \to Y$ by $\mathscr{C}^\rho(A, Y)$.

Let X be a metric space, $\mathbf{E}$ a Banach space and $f\colon X \to \mathbf{E}$. We define $|f^{]\rho[}|$ on all pairs $(x, y) \in X \times X$ with distinct entries by

$$|f^{]\rho[}|(x, y) = \frac{\|f(x) - f(y)\|}{\mathrm{d}(x, y)^\rho} \in \mathbb{R}_{\geq 0}.$$

The mapping $f\colon X \to \mathbf{E}$ is $\mathscr{C}^\rho$ if and only if the function $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}|\colon X \times X \to \mathbb{R}_{\geq 0}$ vanishing on all diagonal entries (x, x) for $x \in X$. Let henceforth $\|\cdot\|_{\sup}$ denote the supremum norm of a bounded function with values in a normed space. We may define:

Definition. Let X be a compact metric space. We endow $\mathscr{C}^\rho(X, \mathbf{E})$ with the norm $\|\cdot\|_{\mathscr{C}^\rho}$ on $\mathscr{C}^\rho(X, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^\rho} = \max\{\|f\|_{\sup}, \| \, |f^{[\rho]}| \, \|_{\sup}\}.$$

In this way $\mathscr{C}^\rho(X, \mathbf{E})$ becomes a Banach space.

2. $\mathscr{C}^\nu$-functions or iterated non-Archimedean differentiability

Let us firstly see how the classic notion of differentiability over the real numbers compares to the one over non-Archimedeanly valued fields. Thereafter, we introduce the classical approach to iterated differentiability over non-Archimedean vector spaces due to Schikhof.

The Archimedean setting. Let $\mathrm{X} \subseteq \mathbb{R}$ be an open interval and $f\colon \mathrm{X} \to \mathbb{R}$ a function. We recall that f is said to be **differentiable** at $a \in \mathrm{X}$ if

$$f' = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Then f is a $\mathscr{C}^1$-function or **continuously differentiable** if f is differentiable at all $a \in \mathrm{X}$ and f' is continuous.

In general, a lot of fundamental facts in differential calculus over $\mathbb{R}$ depend on the connectedness of $\mathbb{R}$ in the form of the intermediate value theorem. For example, in many variables, the fact that a function of many variables is totally continuously differentiable if and only if it is partially continuously differentiable.

We want to point out the following observation: Another direct consequence of the intermediate value theorem is the fundamental theorem of calculus. This in turn is commonly used to prove the completeness of $\mathscr{C}^1(\mathrm{X}, \mathbb{R})$ with respect to its natural supremum norm $\|f\|_{\mathscr{C}^1} = \|f\|_{\sup} + \|f'\|_{\sup}$. Here, we want to give a different proof, giving a clue how to generalize adequately to the non-Archimedean setting.

Proposition 2.1. *A function $f\colon \mathrm{X} \to \mathbb{R}$ is continuously differentiable if and only if*

$$f^{]1[}(x, y) := \frac{f(x) - f(y)}{x - y} \quad \textit{with } x, y \in \mathrm{X} \textit{ distinct}$$

extends to a continuous function $f^{[1]}\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$.

Proof: This is essentially a consequence of the mean value theorem and we only want to remark on where it is employed. The key point is to see that if f is $\mathscr{C}^1$, then the difference quotient $f^{]1[}$ extends to a continuous function on all of $\mathrm{X} \times \mathrm{X}$. To this end, we choose a sequence $(x_n, y_n) \to (a, a)$ with distinct $x_n, y_n \in \mathrm{X}$, say $x_n < y_n$, and notice that

$$f^{]1[}(x_n, y_n) = f'(\xi) \to f'(a) = f^{[1]}(a, a) \quad \text{with } \xi \in [x_n, y_n].$$

Here the first equality we obtain by the mean value theorem, and the second one thanks to the continuity of f. □

Corollary. *Let X be compact. The $\mathbb{R}$-vector space of $\mathscr{C}^1$-functions $f\colon \mathrm{X} \to \mathbb{R}$ is complete with respect to the norm $\|f\|_{\mathscr{C}^1} = \|f\|_{\sup} + \|f'\|_{\sup}$.*

Proof: By the proof of the preceding Proposition 2.1, we see the norm $\|f\|_{\mathscr{C}^1}$ to be equivalent to the norm $\|f\| = \|f\|_{\sup} + \|f^{[1]}\|_{\sup}$. The completeness with respect to this latter norm is now completely formal. □

To compensate for the lack of the intermediate value theorem over complete non-Archimedean fields — which in fact renders for example the completeness property with the common Archimedean definition of differentiability false — we turn the premise of Proposition 2.1 into a definition.

The non-Archimedean setting. We let from now on $\mathbf{K}$ be a *non-Archimedean* field, that is, it is endowed with a non-Archimedean multiplicative valuation $|\cdot|$ which is nontrivial and turns it into a complete topological field. We will denote by $\mathbf{E}$ be a *non-Archimedean* Banach space over $\mathbf{K}$, that is, it is endowed with a norm $\|\cdot\|$ which satisfies the strong triangle inequality $\|x+y\| \leq \max\{\|x\|, \|y\|\}$.

We now put forward a definition of non-Archimedean differentiability in the simplest case, that one of one-fold differentiability in one variable, and see how to proceed from there towards a more general notion.

Definition. Let X be a subset in $\mathbf{K}$ without isolated points. A function $f\colon \mathrm{X} \to \mathbf{E}$ is a $\mathscr{C}^1$**-function** if

$$f^{[1]}(x,y) = \frac{f(x)-f(y)}{x-y} \quad \text{with } x, y \in \mathrm{X} \text{ distinct}$$

extends to a continuous function $f^{[1]}\colon \mathrm{X}\times\mathrm{X} \to \mathbf{K}$.

The property that X is free of any isolated points (in most occasions even open in $\mathbf{K}$) assures the continuous extension $f^{[1]}$ to be unique. Among other natural properties, the space $\mathscr{C}^1(\mathrm{X}, \mathbf{E})$ of continuously differentiable functions on compact X can be endowed with the above norm

$$\|f\|_{\mathscr{C}^1} = \max\{\|f\|_{\sup}, \|f^{[1]}\|_{\sup}\}$$

for which completeness, as noted above, holds by formal verification.

We are confronted with the problem that already the first differential quotient $f^{[1]}$ of a one-variable function is a function of two variables. (In contrast to the situation over an Archimedean field.) Therefore, already to obtain a notion of higher differentiability for one-variable functions, we must have established the definition of differentiability for a function of many variables in order to iterate this definition.

Let us recall the common notion of differentiability over non-Archimedean vector spaces: Start with a finite dimensional $\mathbf{K}$-vector spaces V (with its canonical $\mathbf{K}$-Banach space topology). Let $f\colon \mathrm{U} \to \mathbf{E}$ be some mapping defined on an open subset $\mathrm{U} \subseteq \mathrm{V}$. Then f is called *differentiable* or $\mathscr{C}^1$ in the point $a \in \mathrm{U}$ if there is a linear map $\mathrm{A} = \mathrm{D}f(a)\colon \mathrm{V} \to \mathbf{E}$ such that for every $\varepsilon > 0$ there is a neighborhood $\mathrm{U}_\varepsilon \ni a$ in U with

$$\|f(x+h) - f(x) - \mathrm{A}\cdot h\| \leq \varepsilon\|h\| \quad \text{for all } x+h, x \in \mathrm{U}_\varepsilon.$$

Now to iterate this differentiability notion, we opt for a choice of coordinates on the function's domain. We therefore assume $\mathrm{V} = \mathbf{K}^d$ and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be its canonical basis vectors. Then given any two points $x+h, x \in \mathrm{U}$ with $h \in \mathbf{K}^{*d}$, we can define $\mathrm{A} = f^{]1[}(x+h, h) \in \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ by the partial difference quotients

$$\mathrm{A}(h_k\cdot\boldsymbol{e}_k) = f(x+h_1\cdot\boldsymbol{e}_1+\cdots+h_k\cdot\boldsymbol{e}_k) - f(x+h_1\cdot\boldsymbol{e}_1+\cdots+h_{k-1}\cdot\boldsymbol{e}_{k-1}) \quad \text{for } k = 1, \ldots, d.$$

Then the mapping $f^{]1[}\colon \mathrm{U}^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ defined in this manner extends to a continuous function $f^{[1]}\colon \mathrm{U}^{[1]} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{V}, \mathbf{E})$ with $\mathrm{U}^{[1]} = \mathrm{U} \times \mathrm{U}$ if and only if f is $\mathscr{C}^1$ at every point of a. This function's domain lies again in the $\mathbf{K}$-vector space $\mathrm{V} \times \mathrm{V}$ inheriting a natural choice of coordinates, its range is again a $\mathbf{K}$-vector space, and so we can define f to be *twice continuously differentiable* if

$$f^{]2[} = (f^{[1]})^{]1[}\colon (\mathrm{X}^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}}(\mathrm{Hom}_{\mathbf{K}}(\mathrm{V} \times \mathrm{V}, \mathbf{E}), \mathbf{E})$$

extends to a continuous function $f^{[2]}$ on all of $\mathrm{X}^{[2]} = \mathrm{X}^{[1]} \times \mathrm{X}^{[1]}$, and we can continue in this manner to arrive at our notion of ν-fold differentiability for any $\nu \in \mathbb{N}$.

This definition can be given more concisely (and point-wise) by taking into account the symmetry properties of the difference quotients as observed by Schikhof. When testing for total differentiability, for a *symmetric* function such as the difference quotient $f^{]1[}$ of a function f, we are brought down to checking partial differentiability solely in its first coordinate. This reduces an exponential growth of parameters along the degree of differentiability to a linear one. Following [**Sch84**, Section 29 ff.], we recall the notion of the iterated difference quotient of a function on a non-Archimedeanly valued domain.

Definition. Let X be a subset of $\mathbf{K}$ and $f\colon \mathrm{X} \to \mathbf{E}$. For $\nu \in \mathbb{N}$ put

$$\mathrm{X}^{[\nu]} = \mathrm{X}^{\{0,\dots,\nu\}} \quad \text{and} \quad \mathrm{X}^{]\nu[} := \{(x_0, \dots, x_\nu) \in \mathrm{X}^{[\nu]} : x_i = x_j \text{ only if } i = j\}.$$

The ν**-th difference quotient** $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ of a function $f\colon \mathrm{X} \to \mathbf{E}$ is inductively given by $f^{]0[} = f$ and for $n \in \mathbb{N}$ and $(x_0, \dots, x_\nu) \in \mathrm{X}^{]\nu[}$ by

$$f^{]\nu[}(x_0, \dots, x_\nu) = \frac{f^{]\nu-1[}(x_0, x_2, \dots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \dots, x_\nu)}{x_0 - x_1}.$$

3. $\mathscr{C}^r$-functions

Having already defined $\mathscr{C}^\rho$-functions for $\rho \in [0, 1[$, we add up our definitions to obtain our notion of fractional differentiability over (non-Archimedeanly valued) complete fields.

We keep from now on a real number $r \geq 0$ fixed, together with its decomposition $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ into its *integral part* $\nu \in \mathbb{N}$ and *fractional part* $\rho \in [0, 1[$.

Definition 3.1. Let X be a subset of $\mathbf{K}$ without isolated points and $f\colon \mathrm{X} \to \mathbf{E}$ a function. We say that f is $\mathscr{C}^r$ (or r**-times continuously differentiable**) at a point $a \in \mathrm{X}$ if $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ is $\mathscr{C}^\rho$ at $\vec{a} = (a, \dots, a) \in \mathrm{X}^{[\nu]}$.

Then f is a $\mathscr{C}^r$**-function** (or an r**-times differentiable function**) if f is $\mathscr{C}^r$ at all points $a \in \mathrm{X}$. The set of all $\mathscr{C}^r$-functions $f\colon \mathrm{X} \to \mathbf{E}$ will be denoted by $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

Let f be ν-times differentiable at a. We define $\mathrm{D}_\nu f(a) = \lim_{x \to \vec{a}} f^{]\nu[}(x)$ for $x \in \mathrm{X}^{]\nu[}$. We remark that $\nu!\, \mathrm{D}_\nu f(a) = f^{(\nu)}(a)$ with $f^{(\nu)}$ denoting the usual Archimedean ν-th derivative of f (by [**Sch84**, Theorem 29.5]).

As this differentiability notion is bulkier than the usual Archimedean one, natural properties, evident by definition in the Archimedean case, have to be verified. For the following Definition 3.2, we note in particular that

- the $\mathscr{C}^r$-condition becomes stronger with rising degree of differentiability $r \geq 0$. If f is $\mathscr{C}^r$ at a, then f will be $\mathscr{C}^s$ at a for every $s \leq r$ as well (by [**Nag11**, Lemma 2.3]), and that
- the point-wise definition of differentiability can be characterized globally: Under the assumption that X is free of isolated points, we find $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$ if and only if $f^{]\nu[}\colon \mathrm{X}^{]\nu[} \to \mathbf{E}$ extends to a *unique* $\mathscr{C}^\rho$-function $f^{[\nu]}\colon \mathrm{X}^{[\nu]} \to \mathbf{E}$. (See [**Nag11**, Proposition 2.5].)

Henceforth, we denote by X a compact subset of $\mathbf{K}$ which is free of isolated points.

Definition 3.2. Let $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$. Then the aforesaid implies $f^{]0[}, \dots, f^{]\nu-1[}$ and $f^{]\nu[}$ to extend to continuous functions $f^{[0]}, \dots, f^{[\nu-1]}$ and a $\mathscr{C}^\rho$-function $f^{[\nu]}$. We thus may define a norm $\|\cdot\|_{\mathscr{C}^r}$ on $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$ by

$$\|f\|_{\mathscr{C}^r} = \|f^{[0]}\|_{\sup} \vee \cdots \vee \|f^{[\nu-1]}\|_{\sup} \vee \|f^{[\nu]}\|_{\mathscr{C}^\rho}.$$

Part 2. Comparison of the fractionally differentiable and locally analytic topologies

4. Density of locally polynomial functions

One incarnation of the total disconnectedness of a non-Archimedeanly valued domain is the density of the locally constant functions inside all continuous functions with respect to uniform convergence. In the case of an r-times differentiable function f, the function's i-fold derivatives $\mathrm{D}_i f$ for $i \leq r$ intervene and we have to ensure that (working inductively) for each differentiability degree i we find a locally polynomial function g_i such that the locally constant function $\mathrm{D}_i g_i$ and $\mathrm{D}_i f$ are close to each other.

We outline here a proof for integral order of differentiability $r = \nu \in \mathbb{N}$. The general case is treated in detail at [**Nag13b**, Section 2].

Let us call $f\colon \mathrm{X} \to \mathbf{E}$ a **locally polynomial function of degree at most** $d \in \mathbb{N}$, if for every point $a \in \mathrm{X}$, there is a neighborhood $\mathrm{U} \ni a$ such that $f_{|\mathrm{U}}$ is a polynomial function of degree at most d.

Let us denote the identity function on $\mathbf{K}$ by $*$ and, for $i \in \mathbb{N}$, accordingly $*^i\colon \mathrm{X} \to \mathbf{K}$ the monomial function $x \mapsto x^i$. We assume for notational convenience our $\mathbf{K}$-Banach space $\mathbf{E}$ to be employed with an action of $\mathbf{K}$ by the *right.*

Proposition 4.1. *The locally polynomial functions of degree at most* ν *are dense in* $\mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

Proof: We sketch a proof in the case $r = \nu \in \mathbb{N}$. Fix $\varepsilon > 0$ and $f \in \mathscr{C}^r(\mathrm{X}, \mathbf{E})$.

By downward induction on $n = \nu, \dots, 0$, we will inductively construct a sequence of constant functions $g_\nu, \dots, g_0 \colon \mathrm{X} \to \mathbf{E}$ such that $f_n = f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \dots - g_n *^n$ satisfies

$$|f_n^{[n]}(x_0, \dots, x_n)| \leq \varepsilon\delta^{\nu-n} \quad \text{if } \operatorname{dia}\{x_0, \dots, x_n\} \leq \delta.$$

Let $n = \nu$. By compactness of X, there is a $\delta > 0$ such that for all $(x_0, \dots, x_{\nu+1})$ and $a \in \mathrm{X}$,

$$|f^{[n]}(x_0, \dots, x_n) - f^{[n]}(\vec{a})| \leq \varepsilon\delta^\rho \quad \text{if } \|(x_0, \dots, x_n) - \vec{a}\| \leq \delta,$$

where we denote by $\operatorname{dia} \mathrm{A} := \sup\{|x - y| : x, y \in \mathrm{A}\}$ the diameter of a subset A. We find, by total disconnectedness of X, a δ-*constant* function g_n (meaning constant on any neighborhood of radius δ) such that $\|\mathrm{D}_\nu f - g_\nu *^\nu\|_{\sup} \leq \varepsilon$. We fix δ for the rest of the proof.

Let $n < \nu$ and assume we already constructed δ-constant functions $g_\nu, \dots, g_n \colon \mathrm{X} \to \mathbf{E}$ such that $f_n = f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \dots - g_n *^n$ satisfies

$$|f_n^{[n]}(x_0, \dots, x_n)| \leq \varepsilon\delta^{\nu-n} \quad \text{for all } (x_0, \dots, x_n) \text{ with } \operatorname{dia}\{x_0, \dots, x_n\} \leq \delta.$$

Then, by definition of the n-th difference quotient $f^{[n]}$, for all $(x_0, \dots, x_{n-1}), \vec{a} \in \mathrm{X}^n$ with $\|(x_0, \dots, x_{n-1}) - \vec{a}\| \leq \delta$ we have

$$|f_n^{[n-1]}(x_0, \dots, x_{n-1}) - f_n^{[n-1]}(\vec{a})| \leq \varepsilon\delta^{\nu-n} \cdot \delta = \varepsilon \cdot \delta^{\nu-(n-1)},$$

and just as before, we find δ-constant $g_{n-1} \colon \mathrm{X} \to \mathbf{E}$ such that $f_{n-1} = f_n - g_{n-1} *^{n-1}$ satisfies

$$|f_{n-1}^{[n-1]}(x_0, \dots, x_{n-1})| \leq \varepsilon\delta^{r-(n-1)} \quad \text{for all } (x_0, \dots, x_n) \text{ with } \operatorname{dia}\{x_0, \dots, x_\nu\} \leq \delta.$$

This finishes the construction of $g_0, \dots, g_\nu$. To conclude the proof, it rests to prove (by induction on $n = 0, \dots, \nu$) that $\|f_0^{[n]}\|_{\sup} \leq \varepsilon\delta^{\nu-n}$.

If $n = 0$, then $\operatorname{dia}\{x_0\} = 0 \leq \delta$ for all $x_0 \in \mathrm{X}$ and thus trivially $|f_0^{[0]}(x_0)| \leq \varepsilon\delta^\nu$ for all $x_0 \in \mathrm{X}$, that is, $\|f_0^{[0]}\|_{\sup} \leq \varepsilon\delta^\nu$. Let $n + 1 > 0$. Then we split up the domain of $f^{[n+1]}$ into two subsets on which we compute the supremum, that is

$$\begin{aligned}\|f_0^{[n+1]}\|_{\sup} = \max\{&\|f_0^{[n+1]}\|_{\{(x_0,\dots,x_{n+1}):\operatorname{dia}\{x_0,\dots,x_{n+1}\}\leq\delta\}},\\ &\|f_0^{[n+1]}\|_{\{(x_0,\dots,x_{n+1}):\|x_k - x_l\|>\delta \text{ for some } k,l\}}\}.\end{aligned}$$

We firstly show $\|f_0^{[n+1]}\|_{\{(x_0,\dots,x_{n+1}):\operatorname{dia}\{x_0,\dots,x_{n+1}\}\leq\delta\}} \leq \varepsilon\delta^{\nu-(n+1)}$. Let $i = 0, \dots, n$. Since g_i is δ-constant, the function $(g_i *^i)^{[i]}$ is constant on all $(x_0, \dots, x_i)$ with $\operatorname{dia}\{x_0, \dots, x_i\} \leq \delta$. As $i < n + 1$, the function $(g_i *^i)^{[n+1]}$ thus vanishes on all $(x_0, \dots, x_{n+1})$ with $\operatorname{dia}\{x_0, \dots, x_n + 1\} \leq \delta$. Therefore, restricting to all those $(x_0, \dots, x_{n+1})$ with $\operatorname{dia}\{x_0, \dots, x_{n+1}\} \leq \delta$, it obtains

$$\begin{aligned}f_0^{[n+1]} &= (f - g_\nu *^\nu - g_{\nu-1} *^{\nu-1} - \dots - g_0)^{[n+1]}\\ &= (f - g_\nu *^\nu - \dots - g_{n+1} *^{n+1})^{[n+1]} = f_{n+1}^{[n+1]}.\end{aligned}$$

We recall that, by construction of $g_\nu, \dots, g_0 \colon X \to \mathbf{E}$, we have

$$|f_{n+1}^{[n+1]}(x_0, \dots, x_{n+1})| \leq \varepsilon\delta^{r-(n+1)} \quad \text{if } \operatorname{dia}\{x_0, \dots, x_{n+1}\} \leq \delta$$

and we infer by the preceding that

$$\begin{aligned} &\|f_0^{[n+1]}\|_{\{(x_0,\dots,x_{n+1}):\operatorname{dia}\{x_0,\dots,x_{n+1}\}\leq\delta\}} \\ =&\|f_{n+1}^{[n+1]}\|_{\{(x_0,\dots,x_{n+1}):\operatorname{dia}\{x_0,\dots,x_{n+1}\}\leq\delta\}} \leq \varepsilon\delta^{\nu-(n+1)}. \end{aligned}$$

We turn to the inequality $\|f_0^{[n+1]}\|_{\{(x_0,\dots,x_{n+1}):\|x_k-x_l\|>\delta \text{ for some } k,l\}} \leq \varepsilon\delta^{\nu-(n+1)}$. Let $(x_0, \dots, x_{n+1})$ with $\|x_k - x_l\| > \delta$ for some coordinates k, l. By symmetry of $f_0^{[n+1]}$, we are actually reduced to the case $\|x_0 - x_1\| > \delta$. Then

$$\begin{aligned} &|f_0^{[n+1]}(x_0, x_1, \dots, x_{n+1})| \\ =&|f_0^{[n]}(x_0, x_2, \dots, x_{n+1}) - f_0^{[n]}(x_1, x_2, \dots, x_{n+1})|/|x_0 - x_1| \\ <&\|f_0^{[n]}\|/\delta \leq \varepsilon\delta^{r-n}/\delta = \varepsilon\delta^{r-(n+1)}; \end{aligned}$$

the last inequality by the induction hypothesis. This concludes the proof of the inequality $\|f_0^{[n]}\|_{\sup} \leq \varepsilon\delta^{\nu-n}$ for $n = 0, \dots, \nu$.

We may conclude as follows: Let us put $g = g_0 + g_1 *^1 + \dots + g_\nu *^\nu$, so that g is a locally polynomial function of degree at most ν and $f_0 = f - g$. Then $\|f - g\|_{\mathscr{C}^r} = \max\{\|f_0^{[n]}\|_{\sup} : n = 0, \dots, \nu\} \leq \varepsilon$. □

5. The van der Put-basis

As a convenient mean to compare the $\mathscr{C}^r$- with the locally analytic topology on the locally polynomial functions, we will establish an orthogonal basis of indicator functions, the so called van der Put-basis.

From now onwards, we assume $\mathbf{K}$ to be moreover a *local* non-Archimedean field, that is, non-Archimedeanly valued and locally compact. Equivalently, its valuation is discrete and its residue field finite (of characteristic p say). We let $\mathbf{o}$ be its compact valuation ring, v its additive valuation and $\pi \in \mathbf{o}$ be a uniformizer (that is, $v(\pi)$ generates $v(\mathbf{K}^*)$). We define its multiplicative valuation by $|\cdot| = p^{-v(\cdot)}$ and agree v to be normalized by $v(\mathbf{K}) \ni 1$ with $v(p) = 1$ if $\mathbf{K}$ is of characteristic 0, that is, a finite extension of $\mathbb{Q}_p$.

Let us denote the space of all locally polynomial functions $f \colon \mathbf{o} \to \mathbf{E}$ of highest degree d by $\mathscr{C}^{\mathrm{lp}\leq d}(\mathbf{o}, \mathbf{E})$. For an open subset $U \subseteq \mathbf{o}$, we let $\mathbf{1}_U \colon \mathbf{o} \to \mathbf{E}$ be the **indicator function** of U which takes value 1 on U and vanishes otherwise.

Definition 5.1.

(i) We choose for $i \in \mathbb{N}$ an *increasing* (finite) family of systems of representatives $(S_{\leq i})$ of $\mathbf{o}/\pi^i\mathbf{o}$, that is, $S_{\leq i} \subseteq S_{\leq j}$ for $i \leq j$ and put

$$S = \bigcup_{i \in \mathbb{N}} S_{\leq i} \subseteq \mathbf{o}.$$

(ii) We have a natural notion of level for the elements in S, namely we put

$$\ell(s) = \min\{i \in \mathbb{N} : s \in \mathrm{S}_{\leq i}\}.$$

We let $\mathrm{S}_i := \mathrm{S}_{\leq i} - \cup_{j<i}\mathrm{S}_j$ be the set of elements $s \in \mathrm{S}$ of level $\ell(s) = i$.

(iii) We are all set to define our generalized van der Put-basis $\{e_s : s \in \mathrm{S}\} \subseteq \mathscr{C}^{\mathrm{lc}}(\mathbf{o}, \mathbf{E})$ by the collection of indicator functions $\{e_s : s \in \mathrm{S}\}$ defined by

$$e_s = \mathbf{1}_{s+\pi^{\ell(s)}\mathbf{o}}.$$

To give an example, over $\mathbf{o} = \mathbb{Z}_p$, the set $\mathrm{S} = \mathbb{N}$ serves, with $\ell(i)$ being the length of its p-adic expansion, that is, $\ell(0) = 0$ and $\ell(i) = \lfloor \log_p(i) \rfloor + 1$ if $i > 0$.

Definition 5.2.

(i) Let $s \in \mathrm{S}$ with $i = \ell(s) \geq 1$. Then its **preceding** element $s^- \in \mathrm{S}$ is defined as the unique element in $\mathrm{S}_{\leq i^-}$ such that $s^- \equiv s \bmod \pi^{i^-}\mathbf{o}$ with $i^- = i - 1$. We denote $\delta(s) = s - s^-$.

(ii) Let $s, t \in \mathrm{S}$. We write $s \leq t$ if $\ell(s) \leq \ell(t)$ and $t \equiv s \bmod \pi^{\ell(s)}\mathbf{o}$. (That is, if and only if $e_s(t) = 1$.)

Coming back to the above example, if $n \in \mathbb{N}$, we find $n^- = a_0 + \cdots + a_{i-1}p^{i-1}$ to be the truncation of $n = a_0 + \cdots + a_{i-1}p^{i-1} + a_i p^i$ with respect to its p-adic expansion and $n \leq m$ if the p-adic expansion of m extends the one of n, that is, $n \leq m$ in the usual sense.

LEMMA. *The family* $\{e_s : s \in \mathrm{S}\}$ *is a basis of the* $\mathbf{K}$*-vector space* $\mathscr{C}^{\mathrm{lc}}(\mathbf{o}, \mathbf{E})$.

PROOF: For this it suffices to see that $\{e_s : s \in \mathrm{S}_{\leq \mathrm{I}}\}$ spans $\mathscr{C}(\mathbf{o}/\mathbf{o}_{\leq \mathrm{I}}, \mathbf{E})$ for $\mathrm{I} \in \mathbb{N}$. We make the following observation: Let $s_0 \in \mathrm{S}_{\leq \mathrm{I}}$ and $\ell(s_0) = i_0 = \mathrm{I} - 1$. Then

$$\mathbf{1}_{s_0+\mathbf{o}_{\leq i_0}} = e_{s_0} - \sum_{s \in \mathrm{S}_{\leq \mathrm{I}} \text{ with } s \equiv s_0 \bmod \mathbf{o}_{i_0}} e_s.$$

Thence we deduce inductively that given $s_0 \in \mathrm{S}_{\leq \mathrm{I}}$ and $\ell(s_0) = i_0 < \mathrm{I}$, we find

$$\mathbf{1}_{s_0+\mathbf{o}_{\leq \mathrm{I}}} = e_{s_0} - [\sum_{i=i_0+1,\ldots,\mathrm{I}} \sum_{s \in \mathrm{S}_i \text{ with } s \equiv s_0 \bmod \mathbf{o}_{\leq i-1}} e_s].$$

Therefore the canonical basis $\{\mathbf{1}_{s+\mathbf{o}_{\leq \mathrm{I}}} : s \in \mathrm{S}\}$ of $\mathscr{C}^{\mathrm{lc}}(\mathbf{o}/\pi^i\mathbf{o}, \mathbf{E})$ lies in the span of $\{e_s : s \in \mathrm{S}_{\leq \mathrm{I}}\}$. □

COROLLARY. *The family* $\{e_{s,i} = e_s(* - s)^i : s \in \mathrm{S}, i \in \{0, \ldots, d\}\}$ *is a basis of the* $\mathbf{K}$*-vector space* $\mathscr{C}^{\mathrm{lp} \leq d}(\mathbf{o}, \mathbf{E})$.

We want to show that this basis is in fact orthogonal with respect to the $\mathscr{C}^r$-topology. This is, given $f = \sum \lambda_{s,i} e_{s,i}$, it holds $\|f\|_{\mathscr{C}^r} \geq \max\{|\lambda_{s,i}| \|e_{s,i}\|_{\mathscr{C}^r}\}$. We firstly compute the norms $\|e_{s,i}\|_{\mathscr{C}^r}$ of these indicator functions, and then show that the coefficients $\lambda_{s,i}$ can be expressed via values of the differentials $f^{[0]}, \ldots, f^{[\nu]}$ of f multiplied by a scalar of valuation $\|e_{s,i}\|_{\mathscr{C}^r}$, yielding the above inequality.

Lemma 5.3. *We have*

- *for* $d \in \mathbb{N}$ *and* $n = 0$ *that* $\|\mathbf{1}_{\mathbf{o}} *^d\|_{\mathscr{C}^r} = 1$,
- *for* $d \leq r$ *and* $n \geq 1$ *that* $\|\mathbf{1}_{\pi^n \mathbf{o}} *^d\|_{\mathscr{C}^r} = |1/\pi|^{(n-1)(r-d)}$, *and*
- *for* $d \geq \lceil r \rceil$ *and* $n \geq 1$ *that* $\|\mathbf{1}_{\pi^n \mathbf{o}} *^d\|_{\mathscr{C}^r} = |1/\pi|^{n(r-d)}$.

Proof: By a straightforward computation. See [**Nag13b**, Lemma 3.1] for the first two assertions. The last one is proved in the same veins. □

Corollary 5.4. *Given* $d \leq r$, *it holds* $\|e_{0,i}\|_{\mathscr{C}^r} = 1$ *and, for nonzero* $s \in \mathrm{S}$, *that* $\|e_{s,i}\|_{\mathscr{C}^r} = |1/\pi|^{(\ell(s)-1)(r-k)}$.

Proof: By Lemma 5.3 and the translation invariance of $\|\cdot\|_{\mathscr{C}^r}$. □

It turns out that the coefficients $\lambda_{s,i}$ can be reobtained by an evaluation of the Taylor polynomial at s with expansion point s^-. In the case of a locally constant function $f = \sum \lambda_s e_s$, this says $\lambda_s = f(s) - f(s^-)$ and follows by s being by definition the unique element in S which is equivalent to s^- modulo π^ℓ if and only if $\ell \leq \ell(s)$.

Proposition 5.5. *Let* $f = f_0 + \cdots + f_d \in \mathscr{C}^{\mathrm{lp} \leq d}(\mathbf{o}, \mathbf{E})$ *with* $f_i = \sum_{s \in \mathrm{S}} \lambda_{s,i} e_s (* - s)^i$ *for* $i = 0, \ldots, d$. *We have* $\lambda_{0,i} = \mathrm{D}_i f(0)$ *and*

$$\lambda_{s,i} = \mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathrm{D}_{i+j} f(s^-)]$$

for all nonzero $s \in \mathrm{S}$.

Proof: See [**Nag13b**, Corollary 3.3]. □

Proposition 5.6. *The van der Put-basis* $\{e_s(* - s)^i : s \in \mathrm{S}, i = 0, \ldots, \nu\}$ *is an orthogonal basis of* $\mathscr{C}^r(\mathbf{o}, \mathbf{E})$.

Proof: Since $\{e_s : s \in \mathrm{S}\}$ is a basis of the $\mathbf{K}$-vector space $\mathscr{C}^{\mathrm{lc}}(\mathbf{o}, \mathbf{E})$, given $f \in \mathscr{C}^{\mathrm{lp} \leq \nu}(\mathbf{o}, \mathbf{E})$, there exists unique $\lambda_{s,i} \in \mathbf{E}$ such that

$$f = \sum_{i=0,\ldots,\nu} \sum_{s \in \mathrm{S}} \lambda_{s,i} e_s (* - s)^i.$$

Therefore $\|f\|_{\mathscr{C}^r} \leq \max\{|\lambda_{s,i}| \|e_s(* - s)^i\|_{\mathscr{C}^r} : s \in \mathrm{S}, i = 0, \ldots, \nu\}$. It rests to prove that

$$|\lambda_{s,i}| \|e_s(* - s)^i\|_{\mathscr{C}^r} \leq \|f\|_{\mathscr{C}^r} \quad \text{for } i \in \{0, \ldots, \nu\} \text{ and } s \in \mathrm{S}.$$

We computed in Corollary 5.4 above that $\|e_s(* - s)^i\|_{\mathscr{C}^r} = |1/\pi|^{(r-i)(\ell(s)-1)}$. By the above Proposition 5.5 we have

$$\lambda_{s,i} = \mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\ldots,d-i} (s - s^-)^j \binom{i+j}{i} \mathrm{D}_{i+j} f(s^-)$$

By [**Sch84**, Theorem 29.4 and Lemma 78.1] we can express these Taylor polynomials through values of the difference quotients $f^{[\nu+1]}$, obtaining

$$\begin{aligned}
&|\lambda_{s,i}| \|e_s(* - s)^i\|_{\mathscr{C}^r} \\
=&|\mathrm{D}_i f(s) - \mathrm{D}_i f(s^-) - [\sum_{j=1,\dots,d-i} (s-s^-)^j \binom{i+j}{i} \mathrm{D}_{i+j} f(s^-)||\pi|^{(r-i)(\ell(s)-1)} \\
=&|\mathrm{D}_i f^{[\nu+1-i]}(s,s^-,\dots,s^-)||\pi|^{(1-\rho)(\ell(s)-1)} \\
=&|\sum_{i=1,\dots,\nu+1} f^{[\nu+1]}(\underbrace{s,\dots,s}_{i\text{-times}},s^-,\dots,s^-)||s-s^-|^{1-\rho} \\
\leq&|f^{[r]}|(s,s^-,\dots,s^-) \leq \|f\|_{\mathscr{C}^r}. \qquad \square
\end{aligned}$$

6. Comparison between Differentiability and Analyticity

We say that a function $f \colon \mathbf{o} \to \mathbf{E}$ is n-**analytic**, if f is ($\mathbf{K}$-)analytic on any ball of radius p^{-n}. That is, its values are obtained by evaluation of a power series in one variable which converges on $\mathbf{o}$. It is **locally analytic** if it is n-analytic for some $n \in \mathbb{N}$. We endow the $\mathbf{K}$-vector space $\mathscr{C}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$ of n-analytic functions with the unique Banach space norm which restricts, on any ball of radius p^{-n}, to the norm $\|\cdot\|_n$ defined on all power series converging on the ball of radius p^{-n} given by $\|\sum_{i\in\mathbb{N}} a_i x^i\|_n = \max\{|a_i|p^{-ni} : i \in \mathbb{N}\}$. The locally analytic functions

$$\mathscr{C}^{\mathrm{la}}(\mathbf{o},\mathbf{E}) = \bigcup_{n\in\mathbb{N}} \mathscr{C}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$$

then inherit a natural Fréchet topology.

Let $\mathscr{D}^r(\mathbf{o},\mathbf{E})$ and $\mathscr{D}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$ be the continuous duals of the $\mathscr{C}^r$ and n-analytic functions. These are $\mathbf{K}$-Banach spaces with respect to their operator norm.

Let $\mathscr{D}^{\mathrm{la}}(\mathbf{o},\mathbf{E})$ be the dual of the locally analytic functions. We saw in Proposition 4.1 above, that the set $\mathscr{C}^{\mathrm{lp}\leq r}(\mathbf{o},\mathbf{E})$ of all locally $\mathbf{K}$-polynomial functions $f \colon \mathbf{o} \to \mathbf{E}$ of degree $d \leq r$ is dense in $\mathscr{C}^r(\mathbf{o},\mathbf{E})$, and thus obtain an injection $\mathscr{D}^r(\mathbf{o},\mathbf{E}) \hookrightarrow \mathscr{D}^{\mathrm{la}}(\mathbf{o},\mathbf{E})$ of the continuous duals.

Given $a \in \mathbf{o}, i \in \mathbb{N}$ and $d \in \mathbb{N}$ let us put $e_{a,i,d} = \mathbf{1}_{a+\pi^i\mathbf{o}}(x-a)^d$. The set $\{e_{a,i,d}\}$ of all these functions contains, as asserted in Proposition 5.6 above, an orthogonal basis of $\mathscr{C}^r(\mathbf{o},\mathbf{E})$, the van der Put-basis. Likewise, by definition of $\|\cdot\|_{\mathscr{C}^{n-\mathrm{an}}}$, the family $\{e_{a,i,d} : a \in \mathbf{o}/\mathbf{o}_{\leq p^{-n}}, m \leq n, d \in \mathbb{N}\}$ with $\mathbf{o}_{\leq p^{-n}} = \{x \in \mathbf{o} : |x| \leq p^{-n}\}$ constitutes an orthogonal basis of $\mathscr{C}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$. We can therefore start comparing these operator norms of a continuous linear form $\mu \in \mathscr{D}^r(\mathbf{o},\mathbf{E})$, by noting that

$$\mathscr{D}^r(\mathbf{o},\mathbf{E}) \subseteq \mathscr{D}^{\mathrm{la}}(\mathbf{o},\mathbf{E}) = \bigcap_{n\in\mathbb{N}} \mathscr{D}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E}),$$

on these translates of monomial indicator functions.

Corollary 6.1. *The operator norm* $\|\cdot\|_{\mathscr{D}^r}$ *on* $\mathscr{D}^r(\mathbf{o},\mathbf{E}) \hookrightarrow \mathscr{D}^{\mathrm{la}}(\mathbf{o},\mathbf{E})$ *is equivalent to*

$$\|\cdot\| = \sup\{p^{-nr}\|\cdot\|_{\mathscr{D}^{n-\mathrm{an}}} : n \in \mathbb{N}\}.$$

Proof: We have to show that there are constants $0 < c \leq 1 \leq C$ such that, on $\mathscr{D}^r(\mathbf{o},\mathbf{E})$,

$$c\|\cdot\| \leq \|\cdot\|_{\mathscr{D}^r} \leq C\|\cdot\|.$$

We firstly show $\|\cdot\|_{\mathscr{D}^r} \geq \|\cdot\|$, that is, $\|\cdot\|_{\mathscr{D}^r} \geq p^{-nr}\|\cdot\|_{\mathscr{D}^{n-\mathrm{an}}}$ for all $n \in \mathbb{N}$. As noted above, the functions $e_{a,i,d}$ with $i \in \mathbf{o}/\mathbf{o}_{\leq p^{-n}}$ contain an orthogonal basis of $\mathscr{C}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$, which we can reformulate by

$$\|\mu\|_{\mathscr{D}^{n-\mathrm{an}}} = \sup\{|\mu(e_{a,i,d})|/\|e_{a,i,d})\|_{\mathscr{C}^{n-\mathrm{an}}} : a \in \mathbf{o}, i \in \mathbf{o}/\mathbf{o}_{\leq p^{-n}}, d \in \mathbb{N}\}$$

for any $\mu \in \mathscr{D}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$. As a direct consequence of Lemma 5.3 above, we have

$$\|\mathbf{1}_{\mathbf{o}_{\leq p^{-n}}} *^d\|_{\mathscr{C}^r} \leq p^{n(r-d)} = p^{nr} \cdot \|\mathbf{1}_{\mathbf{o}_{\leq p^{-n}}} *^d\|_{\mathscr{C}^{n-\mathrm{an}}},$$

and by translation invariance of either of these norms, this holds for all $e_{a,i,d} = 1_{a+p^i\mathbf{o}}(x-a)^d$ as well. Let $\mu \in \mathscr{D}^r(\mathbf{o},\mathbf{E})$. We obtain

$$\begin{aligned}\|\mu\|_{\mathscr{D}^r} &\geq \sup\{|\mu(e_{a,i,d})|/\|e_{a,i,d})\|_{\mathscr{C}^r} : a \in \mathbf{o}, i \in \mathbf{o}/\mathbf{o}_{\leq p^{-n}}, d \in \mathbb{N}\} \\ &\geq p^{-nr}\sup\{|\mu(e_{a,i,d})|/\|e_{a,i,d})\|_{\mathscr{C}^{n-\mathrm{an}}} : a \in \mathbf{o}, i \in \mathbf{o}/\mathbf{o}_{\leq p^{-n}}, d \in \mathbb{N}\} \\ &= p^{-nr}\|\mu\|_{\mathscr{D}^{n-\mathrm{an}}}.\end{aligned}$$

We turn to the inverse inequality. We would like to see that there is a constant $C \geq 1$ such that

$$\|*\|_{\mathscr{D}^r} \leq C \cdot \sup\{p^{-nr}\|*\|_{\mathscr{D}^{n-\mathrm{an}}} : n \in \mathbb{N}\}.$$

As a direct consequence of Lemma 5.3 above, we have

$$\|\mathbf{1}_{\mathbf{o}_{\leq p^{-n}}} *^d\|_{\mathscr{C}^r} \geq c \cdot p^{n(r-d)} \geq c \cdot p^{nr} \cdot \|\mathbf{1}_{\mathbf{o}_{\leq p^{-n}}} *^d\|_{\mathscr{C}^{n-\mathrm{an}}} \quad \text{with } c = p^{-r}$$

and by translation invariance of either of these norms, this holds in particular for all $e_{s,i}$ in the van der Put-basis as well. We compute

$$\begin{aligned}\|\mu\|_{\mathscr{D}^r} &= \sup\{|\mu(e_{s,i})|/\|e_{s,i})\|_{\mathscr{C}^r} : s \in S, i \leq r\} \\ &\leq C \cdot \sup\{p^{-nr}\sup\{|\mu(e_{s,i})|/\|e_{s,i})\|_{\mathscr{C}^{n-\mathrm{an}}} : s \in \mathbf{o}/\mathbf{o}_{\leq p^{-n}}, i \leq r\} : n \in \mathbb{N}\} \\ &\leq C \cdot \sup\{p^{-nr}\|\mu\|_{\mathscr{D}^{n-\mathrm{an}}} : n \in \mathbb{N}\}\end{aligned}$$

with $C = 1/c = p^r > 0$. □

Corollary. *Let* $\{\epsilon_i : i \in I\}$ *be an orthogonal family of* $\mathscr{C}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$ *for all* $n \in \mathbb{N}$. *Then* $\{\epsilon_i\}$ *is an orthogonal family of* $\mathscr{C}^r(\mathbf{o},\mathbf{E})$ *as well.*

Proof: By the ultrametric Hahn-Banach theorem for spaces of countable type, such as $\mathscr{C}^r(\mathbf{o},\mathbf{E})$ or $\mathscr{C}^{n-\mathrm{an}}(\mathbf{o},\mathbf{E})$ as seen above by the density of locally polynomial functions (cf. [**PGS10**, Theorem 4.2.4]), the norm can be computed by their double dual, that is

$$\|f\|_{\mathscr{C}^r} = \sup\{|\mu(f)|/\|\mu\|_{\mathscr{D}^r} : \mu \in \mathscr{D}^r(\mathbf{o},\mathbf{E})\}$$

and likewise for $\|\cdot\|_{\mathscr{C}^{n-\mathrm{an}}}$. Thence a family $\{\epsilon_i : i \in \mathrm{I}\}$ is orthogonal if and only if $\|\mu\|_{\mathscr{D}^r} = \max\{|\mu(\epsilon_i)|/\|\epsilon_i\|_{\mathscr{C}^r} : i \in \mathrm{I}\}$ and likewise for $\|\cdot\|_{\mathscr{C}^{n-\mathrm{an}}}$. The conclusion is therefore a consequence of the above Corollary 6.1. □

Part 3. The Amice transform

In this final Part 3, we assume $\mathbf{E}$ to be a closed subfield of a completed algebraic closure $\mathbb{C}_p$ of the p-adic numbers. We recall briefly the Amice transform, giving a handy description of the dual of the locally analytic functions via their evaluation on the Mahler polynomials. The locally analytic characters $\chi\colon \mathbb{Z}_p \to \mathbf{E}^*$ generate the $\mathbf{E}$-vector space of locally analytic functions, and are parametrized by the open unit disc $\mathrm{B}_{<1}$ of $\mathbf{E}$ via the assignment

$$z \mapsto \chi_z : x \mapsto (1+z)^x = \sum_{n\geq 0} z^n \binom{x}{n}.$$

We thence obtain an injection $\mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{E}) \hookrightarrow \{\text{ all functions } f\colon \mathrm{B}_{<1} \to \mathbf{E}\}$, and the theorem of Amice asserts that its image can be described as follows.

Theorem 6.2 ([**Ami64**, Théoréme 10.3]). *We have an isomorphism of* $\mathbf{E}$-*Fréchet algebras*

$$\mathrm{T}\colon \mathscr{D}^{\mathrm{la}}(\mathbb{Z}_p, \mathbf{E}) \xrightarrow{\sim} \mathscr{O}(\mathrm{B}_{<1}) = \{f(x) = \sum_{n\geq 0} a_n \mathrm{X}^n : f(x) \textit{ exists for } x \in \mathrm{B}_{<1}\}$$

given by $\mu \mapsto \sum a_n \mathrm{X}^n$ *with* $a_n = \mu\binom{*}{n}$ *for* $n \in \mathbb{N}$.

Here, as explained in detail below, the Fréchet topology on the right-hand side stems from a family of supremum norms over closed balls with radii $0 < \rho_n < 1$ where $\rho_n \nearrow 1$. The multiplication on the left-hand side is given by the convolution product. The above isomorphism T is commonly labeled the *Amice transform*. We proceed studying its finer properties.

7. Local analyticity via the Amice transform

Let $v_n = 1/(p-1)p^n$ for $n \in \mathbb{N}$ and put $0 < \rho_n = p^{-v_n}$, so that $\rho_n \nearrow 1$ for $n \to \infty$. We denote by $\|\cdot\|_{\rho_n}$ the norm on $\mathscr{O}(\mathrm{B}_{<1})$ given by the supremum norm on the closed ball of radius ρ_n, that is

$$\|f\|_{\rho_n} = \sup\{f(x) : x \in \mathrm{B}_{\leq\rho_n}\}.$$

We now want to compare these norms with the n-analytic ones. The key point will be that the inequalities of norms below are given by uniform constants $0 < c \leq 1 \leq \mathrm{C}$, independent of n.

Lemma 7.1. *Let* $\mu \in \mathscr{D}^{\mathrm{la}}(\mathbf{o}, \mathbf{E})$. *There are constants* $0 < c \leq 1 \leq \mathrm{C}$ *such that, for all* $n \in \mathbb{N}$,

$$c \cdot \|\mathrm{T}(\mu)\|_{\rho_n} \leq \|\mu\|_{\mathscr{D}^{n-\mathrm{an}}} \leq \mathrm{C} \cdot \|\mathrm{T}(\mu)\|_{\rho_{n+1}}.$$

PROOF: We refer to [**Ami64**] for the results below and cite in particular [**Sch99**, Lecture 2] for a concise account of those in our setting.

The Mahler polynomials have n-analytic norm $\|\binom{*}{i}\|_{n-\mathrm{an}} = 1/|[i/p^n]!|$. Let us regard the first inequality. Let us fix $n \in \mathbb{N}$. We have, by a standard p-adic estimate of the factorial, see [**Sch84**, Theorem 25.5] for example,

$$|[i/p^n]!| > \rho_n^i \quad \text{for all } i \in \mathbb{N},$$

from which the first inequality follows for $c = 1$ independent of n.

Let us turn to the second inequality. One firstly checks

$$|[i/p^n]!| < \rho_{n+1}^i \quad \text{for } i \geq p^{n+2}.$$

Let us fix $n \in \mathbb{N}$. We have $|[i/p^n]!| \leq 1$ for all i in $\mathbb{N}$. The sequence $(\rho_{n+1}^i : i \in \mathbb{N})$ is monotonously decreasing with its minimum at $i = p^{n+2}$ of value $p^{-p/(p-1)}$. Putting $\mathrm{C} = p^{p/(p-1)}$, we therefore obtain

$$|[i/p^n]!| \leq \mathrm{C} \cdot \rho_{n+1}^i \quad \text{for all } i \in \mathbb{N}$$

from which the second inequality follows for $\mathrm{C} \geq 1$ independent of $n \in \mathbb{N}$. □

Given a vector space V, an **unbounded norm** on V is a map $\|\cdot\|\colon \mathrm{V} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies all axioms of a norm with the natural conventions for extending $+, \cdot$ and $\leq$ from $\mathbb{R}$ onto $\mathbb{R} \cup \{\infty\}$. We will call two unbounded norms $\|\cdot\|'$ and $\|\cdot\|''$ on V **equivalent** if $\|v\|' < \infty$ if and only if $\|v\|'' < \infty$ for all $v \in \mathrm{V}$ and its induced topologies on $\{v \in \mathrm{V} : \|v\|' < \infty\}$ coincide.

Lemma 7.2. *Let $f(\mathrm{X}) = \sum a_n \mathrm{X}^n$ in $\mathcal{O}(\mathrm{B}_{<1})$. The unbounded norms*

$$\|f\|' = \sup(\{|a_0|\} \cup \{|a_n|/n^r : n \in \mathbb{N}\}) \quad \textit{and} \quad \|f\|'' = \sup\{p^{-nr}\|f\|_{\rho_n} : n \in \mathbb{N}\}$$

are equivalent.

PROOF: This is proved by elementary calculus in [**Col03**, Lemme V.3.19]. □

THEOREM 7.3. *The Amice transform induces by restriction an isomorphism of topological* **E***-vector spaces*

$$\mathcal{D}^r(\mathbf{o}, \mathbf{E}) \xrightarrow{\sim} \{f = \sum_{n \in \mathbb{N}} a_n \mathrm{X}^n \in \mathbf{E}[[\mathrm{X}]] : \{|a_n|/n^r : n \geq 1\} \textit{ bounded }\}$$

where we endow the right-hand side with the natural norm $\|f\| = \sup(\{|a_0|\} \cup \{|a_n|/n^r : n \in \mathbb{N}\})$ for $f = \sum_{n \in \mathbb{N}} a_n \mathrm{X}^n$.

PROOF: For this, it solely rests to recollect the prior obtained equivalences of (unbounded) norms.

- The preceding Lemma 7.2 shows, on the above right hand side, the norms $\|f\| = \|f'\| = \sup(\{|a_0|\} \cup \{|a_n|/n^r : n \in \mathbb{N}\})$ and $\|f\|'' = \sup\{p^{-nr}\|f\|_{\rho_n} : n \in \mathbb{N}\}$ to be equivalent.

- By the uniformity of the equivalences of norms asserted in Lemma 7.1, we find that the unbounded norms $\|T\cdot\|_r = \sup\{p^{-nr}\|T\cdot\|_{\rho_n} : n \in \mathbb{N}\}$ and $\sup\{p^{-nr}\|\cdot\|_{\mathscr{D}^{n-\mathrm{an}}} : n \in \mathbb{N}\}$ are equivalent.
- By Corollary 6.1 the latter norm is in turn equivalent to the operator norm $\|\cdot\|_{\mathscr{D}^r}$.

We can therefore conclude by restricting these unbounded norms to the normed $\mathbf{E}$-vector subspaces of bounded elements. □

8. Duality

We want to translate our findings obtained on the duals of $\mathscr{C}^r$-functions to the actual underlying function spaces. Therefore a duality argument is needed, which we quickly introduce here.

Schikhof Duality. We give a brief account on Schikhof duality. See [**Sch95**] for the original general formulation and its proof.

Let $\mathbf{K}$ be a non-Archimedean field and V be a non-Archimedean $\mathbf{K}$-Banach space. We denote by V^* its dual $\mathbf{K}$-Banach space of continuous $\mathbf{K}$-linear forms endowed with the supremum norm. The natural duality map

$$\begin{aligned} V &\to (V^*)^* \\ v &\mapsto \mathrm{ev}_v : [V^* \ni v^* \mapsto v^*(v) \in \mathbf{K}] \end{aligned}$$

is surjective only if V is finite dimensional. To see this, note that (for example, by [**Sch02**, Section 10]) any $\mathbf{K}$-Banach space has an orthogonal basis, that is, there is an index set X such that it is equivalent to

$$\begin{aligned} c_0(X) = \{f\colon X \to \mathbf{K} :\ & \text{Given any } \varepsilon > 0, \text{ it holds } |f(x)| \le \varepsilon \\ & \text{for all but finitely many } x \in X\}. \end{aligned}$$

This is the completion of the $\mathbf{K}$-vector space with basis indexed by X, explicitly

$$\mathbf{K}^{\oplus X} = \{f\colon X \to \mathbf{K} : \text{ It holds } f(x) = 0 \text{ for all but finitely many } x \in X\}.$$

The continuous dual of $c_0(X)$ is given by the bounded functions $c_b(X)$ on X, that is,

$$c_b(X) = \mathbf{o}^X \otimes_{\mathbf{o}} \mathbf{K}$$

with its supremum norm. We see that it does not contain any dense $\mathbf{E}$-vector subspace of cardinality X with respect to the topology of uniform convergence.

To establish the desired duality, we note that nevertheless the subspace $\mathbf{K}^{\oplus X}$ is dense in $c_b(X)$ with respect to the weaker topology of *point-wise convergence*. Then the usual dual of bounded linear forms on this topological $\mathbf{K}$-vector space will identify with $c_0(X)$ again.

We obtain therefore an equivalence of categories by the duality pairing if we endow the continuous dual with the topology of point-wise instead of uniform convergence:

Theorem (Schikhof duality). *There is an anti-equivalence between the categories*

- *of* $\mathbf{K}$*-Banach spaces* V *with continuous maps, and*
- *of torsionfree* bounded *topological* $\mathbf{o}$*-modules (tensor products of* $\mathbf{K}$ *with a torsionfree compact topological* $\mathbf{o}$*-module) with continuous linear morphisms.*

It is given by the quasi-inverse functors defined

- *by* $\mathrm{V} \mapsto \mathrm{V}^* = \{$ *all uniformly continuous linear* $f\colon \mathrm{V} \to \mathbf{K}\}$ *with the topology of point-wise convergence, and*
- *by* $\mathrm{M} \mapsto \mathrm{M}' = \{$ *all point-wise continuous linear* $f\colon \mathrm{M} \to \mathbf{K}\}$ *with the topology of uniform convergence.*

Proof: See [Sch95] for the general formulation. We refer to [ST02, Section 1] for a complete account of the situation under consideration here. □

The $\mathscr{C}^r$-function space. Let us denote by $c_0^r(\mathbb{N}_0, \mathbf{E})$ the $\mathbf{E}$-vector space of zero sequences $c_0^r(\mathbb{N}_0, \mathbf{E}) = \{(a_n)_{n\in\mathbb{N}}\colon |a_n| n^r \to 0\}$ and endow it with the natural supremum norm $\|(a_n)_{n\in\mathbb{N}}\| = \sup\{|a_n| n^r : n \in \mathbb{N}\}$.

Theorem. *We have an isomorphism of topological* $\mathbf{E}$*-vector spaces*

$$c_0^r(\mathbb{N}_0, \mathbf{E}) \xrightarrow{\sim} \mathscr{C}^r(\mathbb{Z}_p, \mathbf{E})$$

by $(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n \binom{*}{n}$.

Proof: By Theorem 7.3 we have an isomorphism of $\mathbf{E}$-Banach spaces

$$\mathscr{D}^r(\mathbb{Z}_p, \mathbf{E}) \xrightarrow{\sim} c_b^r(\mathbb{N}_0, \mathbf{E}) = \{(a_n)_{n\in\mathbb{N}} : \{|a_n|/n^r : n \in \mathbb{N}\} \text{ bounded } \}.$$

This holds in particular with respect to their weaker topologies of bounded point-wise convergence on either side. We can therefore apply the above duality Section 8 and obtain, noting that $c_b^r(\mathbb{N}_0, \mathbf{E})$ is the continuous dual of $c_0^r(\mathbb{N}_0, \mathbf{E})$, the above claimed isomorphism of topological $\mathbf{E}$-vector spaces. □

Corollary. *Let* $f \in \mathscr{C}^r(\mathbb{Z}_p, \mathbf{E})$ *and expand it as* $f = \sum_{n\in\mathbb{N}} a_n \binom{*}{n}$. *Then* f *is* r*-times differentiable if and only if* $|a_n| n^r \to 0$ *as* $n \to \infty$.

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